We have shown that $y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n n!}$

Notice that $y(x)$ only contains one constant, where we expect $y(x)$ to be an arbitrary linear combination of two linearly independent functions. (since the ODE is 2^{nd}-order)

$\Rightarrow$ $y(x)$ is not a general solution.

$\Rightarrow$ any other solution independent of $y(x)$ will not be analytic at the origin.

valid on some open set away from the origin.

Let us look at two cases where the solution space is not what we might expect.

Example 5: Consider $x^2 y'' - 4xy' + 6y = 0$.

It's not hard to check that x^2 and x^3 are solutions. This ODE has order 2, so we expect a general solution to be $y(x) = c_1 x^2 + c_2 x^3$.

However, it turns out that $x^2|x|$ is also a solution. It is differentiable (for example $\lim_{h \to 0} \frac{h^2|h| - 0}{h} = 0$), and it is linearly independent of x^2 and x^3. What's going on?

The earlier statement is valid on intervals where $a_0(x) \neq 0$. The existence and uniqueness theorem for ODE's applies to the ODE

$$y'' - \frac{4y'}{x} + \frac{6y}{x^2} = 0$$

valid away from $x=0$. In fact, if $x \neq 0$, then $x^2|x| = \begin{cases} x^3 & \text{if } x > 0 \\ -x^3 & \text{if } x < 0 \end{cases}$

and therefore would not be linearly independent of $\{x^2, x^3\}$.

Example 6: To show what happens in the non-linear case, consider

$$y' = y\left(1 - \frac{y}{4}\right)$$

You can check that $y(x) = \frac{4}{1+e^{-x}}$ is a solution.

However, a multiple of this function is not a solution (you would get a C^2 on the RHS and C on the left).

A general solution is actually $\frac{4}{1+Ce^{-x}}$

(the equation is separable, so you can check this).

- No two are multiples of each other
- The $y=0$ solution is not even part of this family of solutions.

In short linear ODEs are special and well-behaved.

Example 7: Find a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ for

$$(x^2+2)y'' + 4xy' + 2y = 0$$

Then $\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} 4na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n$

$$= \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=-2}^{\infty} 2(n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} 4_n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[2(n+2)(n+1)a_{n+2} + (n^2+3n+2)a_n\right]x^n = 0$$

$$\Rightarrow \quad a_{n+2} = -\frac{(n^2+3n+2)a_n}{2(n+2)(n+1)} = \frac{-a_n}{2} \qquad n \geq 0$$

after factoring n^2+3n+2.

For even n, $a_2 = \frac{-a_0}{2}$, $a_4 = \frac{a_0}{2^2}$, $a_6 = \frac{-a_0}{2^3}$, ...

For odd n, $a_3 = \frac{-a_1}{2}$, $a_5 = \frac{a_1}{2^2}$, $a_7 = \frac{-a_1}{2^3}$

Therefore $y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{2n+1}$

Using the root test, since $\lim_{n\to\infty} \frac{1}{\sqrt[2n]{\underbrace{\left|\left(\frac{-1}{2^n}\right)\right|}_{=|a_{2n}|}}} = \sqrt{2}$, the radius of convergence in both cases is $R = \sqrt{2}$.

- We can also view these as geometric series

$$y_1(x) = \frac{2}{x^2+2} \quad \text{and} \quad y_2(x) = \frac{2x}{x^2+2} = x\left(\frac{2}{x^2+2}\right)$$

Note: If you use the ratio test, then you need to write

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n} = a_0 \sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n z^n}{2^n}}_{b_n} \quad \text{where } z = x^2$$

Then $\lim_{n\to\infty} \left|\frac{b_n}{b_{n+1}}\right| = \lim_{n\to\infty} \frac{\frac{1}{2^n}}{1/2^{n+1}}$

$= 2$

But this is the radius of convergence for z, so

$|z| < 2 \Rightarrow x^2 < 2 \Rightarrow |x| < \sqrt{2}$

17.2 Ordinary and Singular Points

- We will restrict to homogeneous, linear second-order differential equations for this section. Our goal is to determine when a series solution exists.
- Much of this analysis extends to higher-order ODE's, but would require a better understanding of the existence and uniqueness theorem for ODEs.

Definition: Consider the differential equation

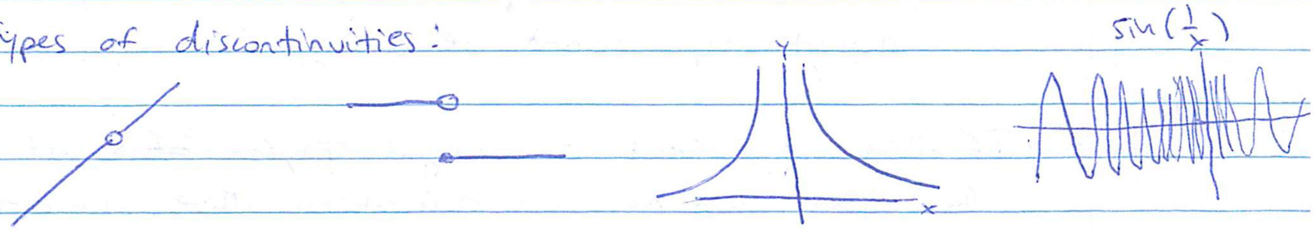
$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0.$$

When both $\left(\overline{\frac{Q(x)}{P(x)}}\right)$ and $\left(\overline{\frac{R(x)}{P(x)}}\right)$ have convergent Taylor series in an interval around a point $x=c$, then $x=c$ is called an <u>ordinary</u> point of the differential equation. Otherwise, c is called a <u>singular</u> point.

Here, $\overline{f(x)}$ means a function $f(x)$ with all removable discontinuities "filled in". That is, if $f(x)$ is not defined only at $x=c$, but $\lim_{x\to c} f(x) = L$ exists, then

$$\overline{f(x)} = \begin{cases} f(x) & x \neq c \\ L & x = c \end{cases}.$$

Types of discontinuities:



removable — jump — asymptotic or infinite ($x=0$ is sometimes called a pole) — essential (none of the previous 3)

- $\overline{f(x)}$ means fill-in all removable singularities.

Example 1: Determine whether $x=0$ and $x=5$ are ordinary or singular points of

$$xy'' - 2y' + e^x y = 0$$

Here $\frac{Q(x)}{P(x)} = \frac{-2}{x}$ and $\frac{R(x)}{P(x)} = \frac{e^x}{x}$

- At $x=5$, $\frac{-2}{x}$ and $\frac{e^x}{x}$ both have convergent Taylor series

- This follows since the quotient of analytic functions at a point is also analytic there — provided the denominator is nonzero at the point.

We could also notice, for example, that

$$\frac{-2}{x} = -2 \cdot \frac{1}{5-(5-x)} = -\frac{2}{5}\left(\frac{1}{1-(1-\frac{x}{5})}\right)$$

$$= -2/5 \sum_{n=0}^{\infty} \frac{(5-x)^n}{5^n}$$

- At $x=0$, neither $-\frac{2}{x}$ or $\frac{e^x}{x}$ have convergent Taylor series.

Firstly, as $x \to 0$, each goes to $\pm\infty$, so the discontinuity is not removable.

- If there did exist a convergent Taylor series at $x=0$, then both would be ∞ differentiable there, which they are not.

Example 2: Find all singular points for

$$(x^2-1)y'' + (2x^2-3x+1)y' + (4x^2-3x)y = 0$$