Here $\frac{Q(x)}{P(x)} = \frac{2x^2-3x+1}{x^2-1} = \frac{(2x-1)(x-1)}{(x+1)(x-1)}$ ← removable singularity at $x=1$ since $\lim_{x\to 1} Q/P$ exists.

$$\Rightarrow \quad Q/P = \frac{2x-1}{x+1}$$

and $\frac{R(x)}{P(x)} = \frac{4x^2-3x}{x^2-1} = \frac{x(4x-3)}{(x-1)(x+1)}$

- From $\frac{Q(x)}{P(x)}$, we see that $x=-1$ is a singular point.

- From $\frac{R(x)}{P(x)}$, we see that $x=-1$ and $x=1$ are singular points (indeed R/P is asymptotic at these points).

Note: It's not enough that there are no discontinuities. We need the Taylor series to exist. $g(x) = \begin{cases} e^{-1/x} & x \neq 0 \\ 0 & = 0 \end{cases}$ is ∞-differentiable at $x=0$, but not analytic there, so there doesn't exist a Taylor series at $x=0$; $\Longrightarrow$ $x=0$ is a singular point if $g(x)$ was a coefficient as above.

Example 3: Find all singular points for

$$(x^2-1)y'' + (2x^2-3x+1)y' + (4x^2-4x)y = 0$$

Here $\frac{Q(x)}{P(x)} = \frac{2x^2-3x+1}{x^2-1} = \frac{(2x-1)(x-1)}{(x+1)(x-1)}$

and $\frac{R(x)}{P(x)} = \frac{4x^2-4x}{x^2-1} = \frac{4x(x-1)}{x(x-1)}$

- We can cancel out $x-1$ from both quotients

- The remaining Q/P is analytic except at $x=-1$

• The remaining R/p is analytic except at $x=-1$

• The only singular point is $x=-1$

In this case we could have written

$$(x-1)\left[\underbrace{(x+1)y'' + (2x-1)y' + 4xy}\right] = 0$$

and considered only this ODE.

Example 4: Find all singular points for

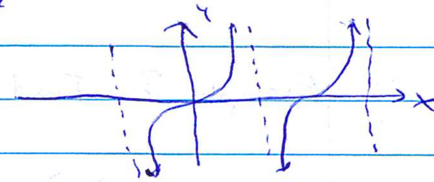
$$xy'' + (x+1)y' + (\tan x)y = 0.$$

Here $\frac{Q(x)}{P(x)} = \frac{x+1}{x} = 1 + \frac{1}{x}$ and $\frac{R(x)}{P(x)} = \frac{\tan x}{x}$

• Here $\frac{1}{x}$ is asymptotic at $x=0$, so $x=0$ is a singular point

• $\tan x$ is not defined at $x = \pi/2 + n\pi$ for $n \in \mathbb{Z}$

What about $\frac{\tan x}{x}$?



We see that $\lim_{x\to 0} \frac{\tan x}{x} \overset{LH}{=} \lim_{x\to 0} \frac{\sec^2 x}{1} = 1$

This can only happen if the Taylor series for $\tan x$ has 0 constant term so that x cancels

Therefore $\frac{\tan x}{x}$ is analytic everywhere except at $x = \pi/2 + n\pi$ for $n \in \mathbb{Z}$

Theorem: When c is an ordinary point of $P(x)y'' + Q(x)y' + R(x)y = 0$, there exists two linearly independent solutions $y_1(x)$ and $y_2(x)$ with power series which converge in some interval $|x-c| < \rho$. A general solution in this interval is $C_1 y_1(x) + C_2 y_2(x)$.

Example 5: Is $x=0$ an ordinary point of

$$xy'' + 2xy' + y\sin x = 0 \,?$$

We compute that $\overline{\frac{Q(x)}{P(x)}} = \overline{\left(\frac{2x}{x}\right)} = 2$

and $\overline{\frac{R(x)}{P(x)}} = \overline{\left(\frac{\sin x}{x}\right)} = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(Recall that $\lim_{x\to 0} \frac{\sin x}{x} = 1$)

- The Maclaurin series for 2 is just 2.
- The Maclaurin series for $\overline{\left(\frac{\sin x}{x}\right)}$ is

$$\frac{\sin x}{x} = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

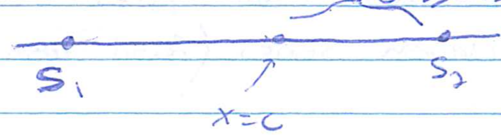
(Note: If cancellation did not occur, then there would be a term of $\frac{1}{x}$ in the series which wouldn't be valid at $x=0$)

$\therefore$ $x=0$ is an ordinary point.

By the theorem, we expect there to be 2 linearly independent solutions $y_1(x)$ and $y_2(x)$ which are analytic at the origin.

The theorem states that the general solution is valid on some interval $|x-c|<\delta$. We could find δ based on y_1 and y_2 if we have found them. We can also find a minimum δ based on the singular points.

$\delta \geq |s_2 - c|$



s_i are singular points.

Assuming real solutions

Example 6: What can be said about the radius of convergence for a power series solution to

$$(x^2+4)y'' + 3xy' + 2y = 0$$

about (a) $x=0$ and (b) $x=1$?

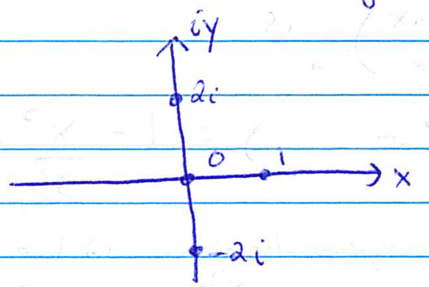
- $\dfrac{Q(x)}{P(x)} = \dfrac{3x}{x^2+4}$ and $\dfrac{R(x)}{P(x)} = \dfrac{2}{x^2+4}$

Allowing for complex solutions, we see that $x^2+4=0$

$\Rightarrow x^2=-4$

$\Rightarrow x=\pm 2i$.

and these are singular points.



distance between two complex numbers $a+bi$, $c+di$ is $\sqrt{(b-d)^2+(a-c)^2}$

- Minimum value for ρ is (a) $|2i| = \sqrt{(2-0)^2} = 2$

(b) $|2i-1| = \sqrt{(-1)^2+(2)^2} = \sqrt{5}$

- It turns out that if P, Q and R are polynomials, then the minimum value for ρ is actually the radius of convergence

- The above analysis is for a minimum since a radius of convergence can extend past singular points. For example $x=0$ is a singular point of $xy''-y'-4x^3y=0$ with solutions $y_1(x)=\cosh x^2$ and $y_2(x)=\sinh(x^2)$ which have a Maclaurin series $\left(\text{Recall } \cosh x = \dfrac{e^x+e^{-x}}{2}\right)$

- $y''+2xy'+(x^2+1)y=0$ has no singular points, so the distance to the nearest singular point is ∞, so radius of convergence $=\infty$

17.3 Frobenius Solutions of Differential Equations

- We saw that a power series solution may or may not exist at a singular point $x=c$.

- A general solution might not be found using 17.2 methods since solutions may not be analytic at $x=c$.

- We will try to find solutions of the form $(x-c)^r \sum_{n=0}^{\infty} a_n (x-c)^n$ for some r.

Consider $x^2y'' + 4xy' + (x^2+2)y = 0$. We see that $x=0$ is a singular point, and $y(x)=0$ is the only analytic solution.

Assume that $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ is a solution for some r.

Then $$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} 4(n+r)a_n x^{n+r} +$$
$$+ \sum_{n=0}^{\infty} a_n x^{n+r+2} + \sum_{n=0}^{\infty} 2a_n x^{n+r}$$

Since $\sum_{n=0}^{\infty} a_n x^{n+r+2} = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$ we get

$$0 = \sum_{n=2}^{\infty} \left[(n+r)(n+r-1)a_n + 4(n+r)a_n + a_{n-2} + 2a_n\right]x^{n+r}$$
$$+ \left[r(r-1)a_0 + 4ra_0 + 2a_0\right]x^r$$
$$+ \left[(r+1)r + 4(r+1) + 2\right]a_1 x^{r+1}$$

Note: We didn't have these terms outside of the sum $\sum_{n=2}^{\infty}$ before, since the $n=0$ and $n=1$ terms were 0. (in 17.1 and 17.2 for example)

$\Longrightarrow$

$$0 = x^r\left[(r^2+3r+2)a_0 + (r^2+5r+6)a_1 x + \sum_{n=2}^{\infty}\left[((n+r)(n+r+3)+2)a_n + a_{n-2}\right]x^n\right]$$

For the series to vanish for all x in an interval around $x=0$ $\Rightarrow$ every coefficient must vanish.

We will use the lowest power of x to determine r.

$$r^2+3r+2=0 \quad \Rightarrow \quad (r+2)(r+1)=0$$

This is called the indicial equation. Here $r=-1$ and $r=-2$ are two possibilities.

Next we need $(r^2+5r+6)a_1=0$.

- Satisfied for $r=-2$, but need $a_1=0$ for $r=-1$.

For the general series, we have

$$a_n = \frac{-a_{n-2}}{(n+r)(n+r+3)+2} \quad n \geq 2.$$

When $r=-2$, this becomes

$$a_n = \frac{-a_{n-2}}{(n-2)(n+1)+2} = \frac{-a_{n-2}}{n(n-1)}$$

$\Rightarrow$ n even $a_2 = \frac{-a_0}{2}$, $a_4 = \frac{a_0}{4!}$, $a_6 = -\frac{a_0}{6!}$

n odd $a_3 = \frac{-a_1}{3!}$, $a_5 = \frac{a_1}{5!}$, $a_7 = \frac{-a_1}{7!}$

Then $y(x) = x^{-2}\left[a_0\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots\right)+a_1\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right)\right]$

$$= \frac{a_0 \cos x + a_1 \sin x}{x^2}$$

which is valid everywhere away from $x=0$.

- Since this is a general solution away from $x=0$, there is no need to check the $r=-1$ case (which gives $\frac{a_0 \sin x}{x^2}$).

- Since $r=-2$ did not put any restriction on a_0 and a_1, it was the best choice for finding a general solution.

- We could have tried $\sum_{n=0}^{\infty} a_n (x-1)^n$ to find a solution analytic at $x=1$, but we would need to center the coefficients of
$$x^2 y'' + 4xy' + (x^2+2)y = 0 \quad \text{about } x=1$$
(for example $x^2 = (x-1+1)^2 = (x-1)^2 + 2(x-1) + 1$).
Furthermore, this could lead to a solution only valid for $x>0$, and it is not clear whether using $x=1$ is the best choice.

- In essence, we have noticed that $\frac{4x}{x^2}$ and $\frac{x^2+2}{x^2}$ have poles of order 1 and 2, and that multiplying our solution by x or x^2 gets rid of the pole

Definition: A singular point $x=c$ is said to be regular if
$$\frac{(x-c)\,Q(x)}{P(x)} \quad \text{and} \quad \frac{(x-c)^2 R(x)}{P(x)}$$
both have Taylor series expansions about $x=c$. Otherwise c is said to be an irregular singular point.

- Since $x\left(\frac{4x}{x^2}\right) = 4$ and $x^2\left(\frac{x^2+2}{x^2}\right) = x^2+2$, $x=0$ above is a regular singular point.

- On the other hand, $x=0$ is irregular for $x^3 y'' + y = 0$ since $x^2\left(\frac{1}{x^3}\right) = \frac{1}{x}$ does not have a Maclaurin series.

- $x^2y'' + y'\sin x - 3y = 0$ is singular for $x=0$ since

  $\frac{\sin x}{x^2}$ and $\frac{-3}{x^2}$ do not have Maclaurin series.

- $\overline{x\left(\frac{\sin x}{x^2}\right)} = \overline{\frac{\sin x}{x}}$ which is analytic at $x=0$

- $x^2\left(-\frac{3}{x^2}\right) = -3$ which is also analytic at $x=0$.

Theorem: The indicial equation for a Frobenius solution about a regular singular point is quadratic.

Theorem: Indicial roots can be used to find general solutions (the indicial roots are associated to a Frobenius solution about a regular singular point).
See Theorem 17.3 for full details.

- What we are really doing is finding Laurent series solutions to the ODE. A Laurent series allows terms $a_n x^n$ where $n \in \mathbb{Z}$ or, series of the form $\sum_{n=-\infty}^{\infty} a_n x^n$.

Summary: Given $P(x)y'' + Q(x)y' + R(x)y = 0$

Does a general power series solution exist at $x=c$?

ordinary point
Ans: Always

singular point

regular
Ans: Yes around $x=c$. Maybe at $x=c$.

irregular
Ans: Maybe around $x=c$.