14.9 - The Divergence Theorem

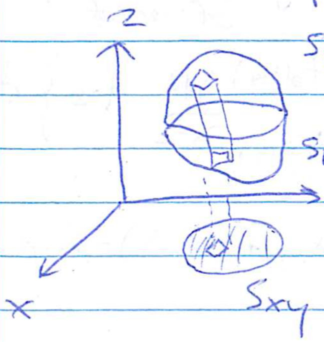
Theorem: Let S be a piecewise-smooth surface enclosing a region V. Let $F(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$ be a vector field whose components $P, Q, R \in C^1(D)$ where D is a domain containing S and V. If $\hat{n}$ is the unit outer normal to S, then

$$\oiint_S F \cdot \hat{n}\, dS = \iiint_V \nabla \cdot F\, dV$$

or

$$\oiint_S (P\hat{i} + Q\hat{j} + R\hat{k}) \cdot \hat{n}\, dS = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV.$$

Proof: Assume S can be divided into an upper and lower portion S_2 and S_1, both of which have the same projection S_{xy}



$S_2: z = f_2(x,y)$

$S_1: z = f_1(x,y)$ Then

$$\oiint_S R\hat{k} \cdot \hat{n}\, dS = \iint_{S_1} R\hat{k} \cdot \hat{n}\, dS + \iint_{S_2} R\hat{k} \cdot \hat{n}\, dS$$

On S_1, $\hat{n} = \dfrac{\nabla(f_1(x,y) - z)}{|\nabla(f_1 - z)|} = \dfrac{\left(\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}, -1\right)}{\sqrt{1 + \left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2}}$

We use $f_1 - z$ instead of $z - f_1$ to ensure the normal vector is facing outwards

On S_2, $\hat{n} = \dfrac{\nabla(z - f_2)}{|\nabla(z - f_2)|} = \dfrac{\left(-\frac{\partial f_2}{\partial x}, -\frac{\partial f_2}{\partial y}, 1\right)}{\sqrt{1 + \left(\frac{\partial f_2}{\partial x}\right)^2 + \left(\frac{\partial f_2}{\partial y}\right)^2}}$

Then $\iint_{S_1} R\hat{k}\cdot\hat{n}\,dS = \iint_{S_1} \dfrac{-R}{\sqrt{1+\left(\frac{\partial f_1}{\partial x}\right)^2+\left(\frac{\partial f_1}{\partial y}\right)^2}}\,dS$

$$= \iint_{S_{xy}} \frac{-R(x,y,f_1(x,y))}{\sqrt{1+\left(\frac{\partial f_1}{\partial x}\right)^2+\left(\frac{\partial f_1}{\partial y}\right)^2}}\sqrt{1+\left(\frac{\partial f_1}{\partial x}\right)^2+\left(\frac{\partial f_1}{\partial y}\right)^2}\,dA$$

$$= \iint_{S_{xy}} -R(x,y,f_1(x,y))\,dA$$

Similarly for S_2, so that

$$\oiint_S R\hat{k}\cdot\hat{n}\,dS = \iint_{S_{xy}} R(x,y,f_2(x,y)) - R(x,y,f_1(x,y))\,dA$$

On the other hand — see diagram at beginning of proof

$$\iiint_V \frac{\partial R}{\partial z}\,dV = \iint_{S_{xy}} \left\{ \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial R}{\partial z}\,dz \right\} dA$$

$$= \iint_{S_{xy}} R(x,y,f_2(x,y)) - R(x,y,f_1(x,y))\,dA$$

by the FTC.

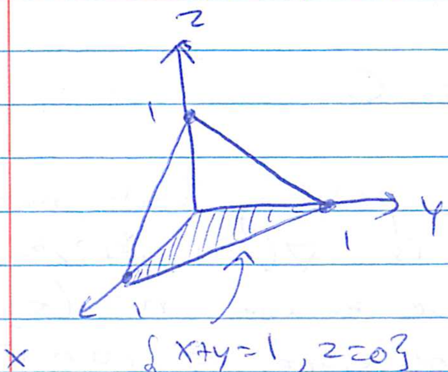
$$\therefore \oiint_S R\hat{k}\cdot\hat{n}\,dS = \iiint_V \frac{\partial R}{\partial z}\,dV.$$

- Projecting S to the xz- and yz-planes shows the other equalities

- In general, subdivide a surface S into pieces where the above proof works. The most general result is proven in more advanced texts.

Example 1: Evaluate $\iint_S F \cdot \hat{n}\, dS$, where

$F = x^2\hat{i} + yz\hat{j} + x\hat{k}$ and $\hat{n}$ is the inward pointing normal to the surface S that encloses a volume V bounded by $x+y+z=1$, $x=0$, $y=0$, $z=0$



- Notice that we would have to divide S into 4 subsurfaces to evaluate the integral.

- To avoid this, we use the divergence theorem.

$$\iint_S F \cdot \hat{n}\, dS = -\iint_S F \cdot (-\hat{n})\, dS$$ to ensure the normal vector is outward pointing.

$$= -\iiint_V \nabla \cdot F\, dV$$

$$= -\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (2x+z)\, dz\, dy\, dx$$

$$= -\int_0^1 \int_0^{1-x} 2x(1-x-y) + \frac{(1-x-y)^2}{2}\, dy\, dx$$

$$= -1/8$$

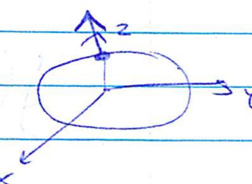
- Note that the statement of the divergence theorem used the outward pointing normal, which is why we adjusted our integral to utilize the theorem.

Example 2: Let $g(x,y,z)$ be a differentiable function, and $F(x,y,z) = x\hat{i} + y\hat{j} + 3z\hat{k}$. Let S be the ellipsoid $x^2 + y^2 + 3z^2 = 6$. Compute $\iint_S (gF)\cdot\hat{n}\, dS$ for:

(i) $g(x,y,z) = (x^2 + y^2 + 9z^2)^{-1/2}$

(ii) $g(x,y,z) =$ constant C

- We will need the outward pointing unit normal to S. An outward normal vector is $\nabla(x^2 + 2y^2 + 3z^2 - 6) = (2x, 2y, 6z)$ but we can use $n = (x, y, 3z)$. We know that it is outward since at $(0,0,\sqrt{2})$, the vector is $n = (0,0,3\sqrt{2})$ which is in an outward direction.



Then $\hat{n} = \dfrac{(x, y, 3z)}{\sqrt{x^2 + y^2 + 9z^2}}$

(i) Now $(gF)\cdot\hat{n} = \big(g(x,y,z)(x, y, 3z)\big)\big(g(x,y,z)(x, y, 3z)\big)$

$$= g^2\left[x^2 + y^2 + 9z^2\right] = 1$$

Then $\iint_S (gF)\cdot\hat{n}\, dS = \iint_S dS$ = surface area of S

There was no need for the divergence theorem. In fact, computing $\text{div}\left(\dfrac{x}{\sqrt{x^2+y^2+9z^2}}, \dfrac{y}{\sqrt{x^2+y^2+9z^2}}, \dfrac{3z}{\sqrt{x^2+y^2+9z^2}}\right)$ seems messy.

ii) If g is a constant C, then

$$(gF)\cdot\hat{n} = C\sqrt{x^2+y^2+9z^2}$$

and $\iint_S C\sqrt{x^2+y^2+9z^2}\,dS$ doesn't look as easy to compute.

By the divergence theorem, it is equal to

$$\iiint_V \operatorname{div} F\,dV = C\iiint_V 1+1+3\,dV$$

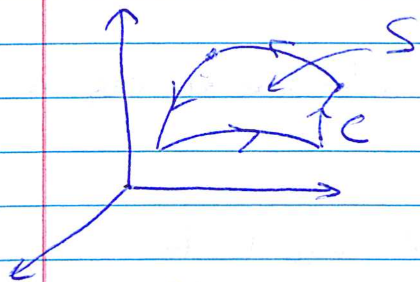
$$= 5C(\operatorname{vol}(V))$$

- Even if $g(x,y,z)$ was any polynomial function, $\operatorname{div} F$ would be some polynomial and $\iiint_V \operatorname{div} F\,dV$ would be reasonable to compute.

In short, the divergence theorem provides one more tool to evaluate surface integrals, when convenient to do so. It requires that we work with closed surfaces however. What if S is not closed?

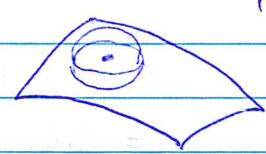
14.10 Stokes's Theorem

Let S be a surface with C as its boundary.

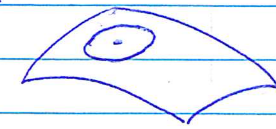


- Note this is in the sense of a "manifold with boundary" rather than the topological definition from 14.1. Here for example, $\text{int}(S) = \emptyset$ and $\text{bdry}(S) = S$ when S is viewed $\subseteq \mathbb{R}^3$.

- However, if we were to try the above with "disks" rather than "balls", then C would be the boundary, but we denote it by $\partial S = C$.



vs.



Theorem: Let C be a closed, piecewise-smooth curve that does not intersect itself, and let S be a piecewise-smooth, orientable surface with C as boundary (in the above sense). Let $F(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$ be a vector field where $P, Q, R \in C^1(D)$ with D a domain containing S and C. Then

$$\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot \hat{n}\, dS, \text{ or}$$

$$\oint_C P\,dx + Q\,dy + R\,dz = \iint_S \left[(R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{k}\right] \cdot \hat{n}\, dS$$

where $\hat{n}$ is chosen so that if when moving along C, the surface is on the left, then $\hat{n}$ must be chosen on that side of S. If on the right, use the opposite side.