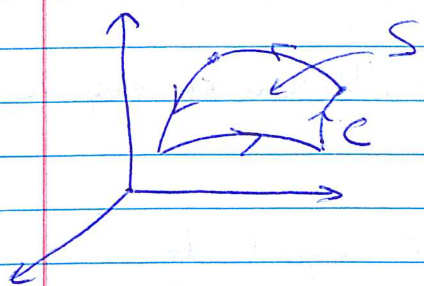
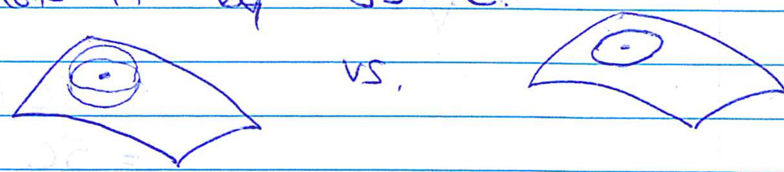
14.10 Stokes's Theorem

Let S be a surface with C as its boundary.



- Note this is in the sense of a "manifold with boundary" rather than the topological definition from 14.1. Here for example, $\text{int}(S) = \emptyset$ and $\text{bdry}(S) = S$ when S is viewed $\subseteq \mathbb{R}^3$.
- However, if we were to try the above with "disks" rather than "balls", then C would be the boundary, but we denote it by $\partial S = C$.



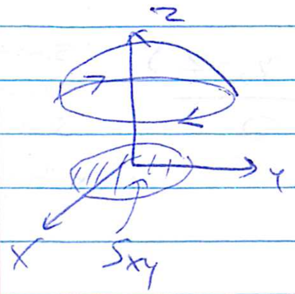
Theorem: Let C be a closed, piecewise-smooth curve that does not intersect itself, and let S be a piecewise-smooth, orientable surface with C as boundary (in the above sense). Let $F(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$ be a vector field where $P, Q, R \in C^1(D)$ with D a domain containing S and C. Then

$$\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot \hat{n}\, dS, \text{ or}$$

$$\oint_C P\,dx + Q\,dy + R\,dz = \iint_S \left[(R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{k} \right] \cdot \hat{n}\, dS$$

where $\hat{n}$ is chosen so that if when moving along C, the surface is on the left, then $\hat{n}$ must be chosen on that side of S. If on the right, use the opposite side.

Example 1: Verify Stokes's theorem if $F = x^2\hat{i} + x\hat{j} + xyz\hat{k}$ and S is that part of the sphere $x^2+y^2+z^2=4$ above the plane $z=1$.



- If we choose $\hat{n}$ as the downward normal to S, then $C = \partial S$ must be in the clockwise direction when viewed from the point $(0,0,4)$.

- we can describe C by $\{z=1,\ x^2+y^2=3\}$ or parameterize it as

$$C: \quad x = \sqrt{3}\cos t, \quad y = -\sqrt{3}\sin t, \quad z=1$$
$$0 \le t \le 2\pi$$

Then
$$\oint_C F\cdot dr = \oint_C x^2\,dx + x\,dy + xyz\,dz$$
$$= \int_0^{2\pi} 3\cos^2 t\,(-\sqrt{3}\sin t) + \sqrt{3}\cos t\,(-\sqrt{3}\cos t) + (-3\cos t\sin t)\,d(0)\ dt$$
$$= -3\sqrt{3}\int_0^{2\pi}\cos^2 t\sin t\,dt - \frac{3}{2}\int_0^{2\pi}(1+\cos 2t)\,dt$$
$$= -3\sqrt{3}\left(-\frac{\cos^3 t}{3}\right)\Big|_0^{2\pi} - \frac{3}{2}\left(t + \frac{\sin 2t}{2}\right)\Big|_0^{2\pi}$$
$$= -3\sqrt{3}\left(-\frac{1}{3} + \frac{1}{3}\right) - \frac{3}{2}(2\pi)$$
$$= -3\pi$$

Note: We will see in this example that the line integral was easier to compute than the surface integral. That is, the work done along the boundary is equal to the flux of the curl vector field over S, but the former in this case is easier to calculate.

On the other hand, a lower normal for S is

$$n = \nabla(4-x^2-y^2-z^2) = (-2x,-2y,-2z)$$

So $\hat{n} = \dfrac{(-x,-y,-z)}{\sqrt{x^2+y^2+z^2}} = \dfrac{(-x,-y,-z)}{2}$ since a point on the sphere satisfies $x^2+y^2+z^2=4$

Similarly, $\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & x & xyz \end{vmatrix} = (xz, -yz, 1)$

Then $\iint_S (\nabla \times F)\cdot \hat{n}\, dS = \iint_S (xz,-yz,1)\cdot \dfrac{(-x,-y,-z)}{2}\, dS$

$$= -\frac{1}{2}\iint_{S_{xy}} z(x^2-y^2+1)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$z=\sqrt{4-x^2-y^2}$

$$= -\frac{1}{2}\iint_{S_{xy}} z(x^2-y^2+1)\sqrt{1+\left(\frac{-x}{z}\right)^2+\left(\frac{-y}{z}\right)^2}\, dA$$

$$= -\frac{1}{2}\iint_{S_{xy}} (x^2-y^2+1)\sqrt{x^2+y^2+z^2}\, dA$$

$$= -\iint_{S_{xy}} x^2-y^2+1\, dA$$

Here $S_{xy} = \{z=0,\ x^2+y^2 \le 3\}$, so using polar coordinates

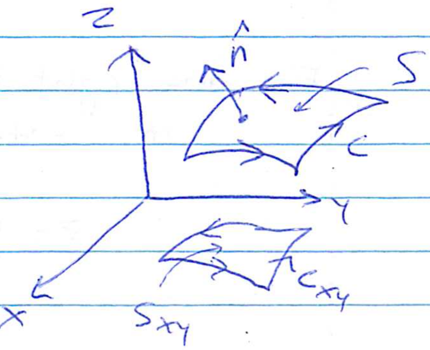
$$= -\int_0^{2\pi}\int_0^{\sqrt{3}} (r^2\cos^2\theta - r^2\sin^2\theta + 1)\, r\, dr\, d\theta$$

$$= -\int_0^{2\pi} \left\{\frac{r^4}{4}(\cos^2\theta-\sin^2\theta) + \frac{r^2}{2}\right\}\Big|_0^{\sqrt{3}} d\theta$$

$$= -\int_0^{2\pi} \frac{9}{4}(\cos 2\theta) + 3/2\, d\theta = -3\pi$$

Proof of Stokes's Theorem:

Assume S projects in a one-to-one fashion onto the xy-, yz- and xz-planes.



- We can write $S = \{ z = f(x,y) \}$

- Let us start by showing that

$$(*) \quad \oint_C P\,dx = \iint_S \left(\frac{\partial P}{\partial z}\hat{j} - \frac{\partial P}{\partial y}\hat{k} \right) \cdot \hat{n}\,dS$$

- $P(x,y,z)$ as a function over C gives the same values as $P(x,y,f(x,y))$ over C_{xy} since $z = f(x,y)$ for points on S.

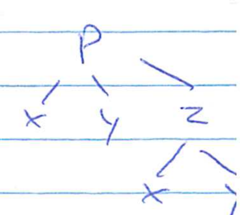
Thus $$\oint_C P\,dx = \oint_{C_{xy}} P(x,y,f(x,y))\,dx$$

and the benefit now is that the integral is viewed as a line integral in two variables. We can apply Green's theorem.

- Recall that Green's theorem states that

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

- C_{xy} is oriented counterclockwise as needed. It encloses the region S_{xy}.



- Here $Q = 0$ so $\frac{\partial Q}{\partial x} = 0$

- By the chain rule $\frac{\partial P(x,y,f(x,y))}{\partial y} = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y}$

$$\therefore \oint_{C_{xy}} P(x,y,f(x,y))\,dx = \iint_{S_{xy}} -\left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y}\right) dA$$

For the right hand side of $(*)$, start by computing an outward-pointing unit normal vector to S.

$$z = f(x,y) \quad \text{so} \quad \hat{n} = \frac{\nabla(z - f(x,y))}{|\nabla(z - f(x,y))|}$$

$$= \frac{\left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right)}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}$$

- Then $\left(\frac{\partial P}{\partial z}\hat{j} - \frac{\partial P}{\partial y}\hat{k}\right)\cdot\hat{n} = \dfrac{0 - \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} - \frac{\partial P}{\partial y}}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}$

- Then $\iint_S \left(\frac{\partial P}{\partial z}\hat{j} - \frac{\partial P}{\partial y}\hat{k}\right)\cdot\hat{n}\,dS$

$$= \iint_{S_{xy}} \frac{-\frac{\partial P}{\partial z}\frac{\partial z}{\partial y} - \frac{\partial P}{\partial y}}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \oint_{C_{xy}} P(x,y,f(x,y))\,d$$

- By projecting C and S to the yz- and xz-planes, prove a similar result for the remaining integrals.

- Again, if we cannot assume these conditions for S then break S into subsurfaces where we can. More general surfaces are handled in more advanced texts.

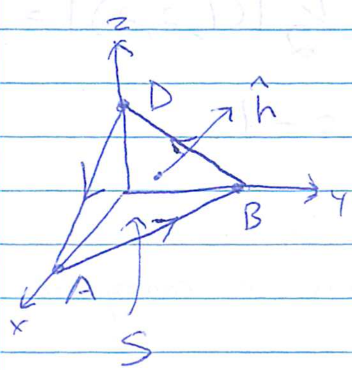
Example 2: Show that the flux of a curl field over a closed surface is 0.

- If S is a closed surface, then $\partial S = \emptyset$ (Think of the sphere for example).

- The flux is given by $\iint_S (\nabla \times F) \cdot \hat{n} \, dS$ and by Stokes's theorem, this is

$$\oint_{\partial S} F \cdot dr = 0 \qquad \text{since } \partial S = \emptyset$$

Example 3: Evaluate $\oint_C 2xy^3 \, dx + 3x^2y^2 \, dy + (2z+x) \, dz$, where C consists of line segments joining the points $A = (2,0,0)$ to $B = (0,1,0)$ to $D = (0,0,1)$ to A.



- Instead of writing the integral as a sum of 3 integrals over each line, let's apply Stokes's theorem to get

$$= \iint_S \nabla \times (2xy^3, 3x^2y^2, 2z+x) \cdot \hat{n} \, dS$$

- Here S can be any surface where $\partial S = C$ (Stokes's theorem applies in each case).

- We choose S to be the flat triangle bounded by C.

- To find a normal vector, use $\overrightarrow{BD} \times \overrightarrow{BA} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 1 \\ 2 & -1 & 0 \end{vmatrix}$

so $\hat{n} = \dfrac{(1,2,2)}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}(1,2,2)$ we can see it's on the correct side of S $\longrightarrow$ $= (1, 2, 2)$

- Next we compute the curl

$$\nabla \times (2xy^3, 3x^2y^2, 2z+x) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy^3 & 3x^2y^2 & 2z+x \end{vmatrix}$$

$$= 0\hat{i} - 1\hat{j} + \underbrace{(6xy^2 - 6xy^2)}_{=0}\hat{k}$$

So $(\nabla \times F)\cdot\hat{n} = -2/3 = -\hat{j}\cdot\frac{(1,2,2)}{3}$

By Stokes's theorem

$$\oint_C F\cdot dr = \iint_S (-2/3)\,dS = -2/3\ (\text{area of } S)$$

Recall that the area of triangle ABD can be found using $\frac{1}{2}|\vec{BD}\times\vec{BA}| = \frac{1}{2}|(1,2,2)| = 3/2$

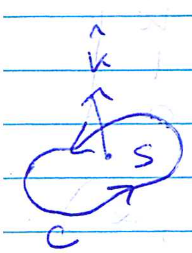
Then $\oint_C F\cdot dr = -2/3 \cdot 3/2 = -1$

Example 4: Show that Green's theorem is a consequence of Stokes's theorem.

If $F = P(x,y)\hat{i} + Q(x,y)\hat{j}$ then by Stokes's theorem

$$\oint P\,dx + Q\,dy = \iint_S \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\text{curl } F}\hat{k}\cdot\hat{n}\,dS$$

C is a curve on the xy-plane



for any choice of surface with $\partial S = C$. Choose S as the unique surface contained in the xy-plane. $\Rightarrow \hat{n} = \hat{k}$

So $\hat{k}\cdot\hat{n} = 1$, $dA = dS$ and $\oint P\,dx + Q\,dy = \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$