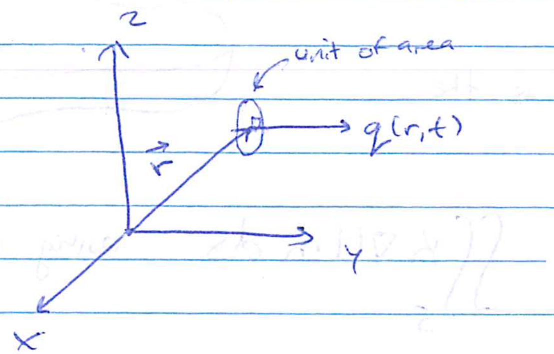
Heat Conduction

We wish to model heat flow in various mediums. There are 3 types of heat flow, namely by radiation, convection or conduction. We will consider only the latter type.

Define the <u>heat flux vector</u> $\underset{\sim}{q}(\vec{r}, t)$ as a vector function depending on position $\vec{r}$ and time t. The direction of $\underset{\sim}{q}$ corresponds to the direction of heat flow at position $\vec{r}$ and time t, and its magnitude is the amount of heat per unit time crossing unit area which is normal to the direction of $\underset{\sim}{q}$.



- $\underset{\sim}{q}(r, t)$ is measured in W/m^2 and is defined at points which are not sources or sinks of heat

- If a medium has no preferred direction for heat flow, it is said to be <u>isotropic</u> (heat spreads out equally in all directions. Experiments have shown that heat flows in the direction in which temperature decreases most rapidly.

- Recall from Math 1, the direction where a function f increases most rapidly is ∇f, with magnitude $|\nabla f|$.

- Let $U(r, t)$ be the temperature distribution in the medium (so it is a scaler function telling you the temperature at a specific point at a certain time). Then ∇U is a vector field as described above (note, the rapid increase here is in physical directions so ∇ is being applied for the $\vec{r}$ coordinates, not t).

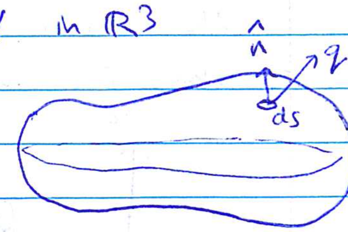
- Fourier's law of heat conduction states that heat flows in the direction of most rapid decrease in temperature, and the amount is proportional to the rate of change in that direction.

In other words the heat flux vector $q(r,t)$ is proportional to $-\nabla U$ (the $-$ for most rapid decrease instead of increase).

$$\Rightarrow \quad q(\vec{r},t) = -K\nabla U(\vec{r},t) \qquad (*)$$

- Here K is the thermal conductivity of the medium (look at Table 20.1 for examples of this constant when K is independent of position).

To derive a PDE governing temperature, let S be a surface bounding some region V in $\mathbb{R}^3$



- Heat added via conduction (flowing into V through S) is the flux integral

$$\iint_S q\cdot(-\hat{n})\,dS = \iint_S K\nabla U\cdot\hat{n}\,dS \quad \text{using } (*)$$

inward pointing unit normal

- There is also heat conduction from internal heat sources or sinks, and the total heat generation here is

$$\iiint_V g(r,t)\,dV$$

where $g(r,t)$ is the amount of heat generated (or removed) per unit time per unit volume.

- The rate of change of the temperature at a point is $\frac{\partial U}{\partial t}$.

Then $\iiint_V \frac{\partial U}{\partial t} s\rho\,dV$ measures the total temperature change over V, where ρ is the density of the medium and s is the specific heat of the medium (this is a constant which measures the amount of heat required to produce a unit temperature change in unit mass).

- Energy balance now requires

$$\iiint_V \frac{\partial u}{\partial t} s\rho \, dV = \iiint_V g(r,t)\, dV + \iint_S K \nabla u \cdot \hat{n} \, dS.$$

by the divergence theorem, $\iint_S K\nabla u \cdot \hat{n}\, dS = \iiint_V \nabla \cdot (K \nabla u)\, dV$

$$\Rightarrow \quad \iiint_V \left[\rho s \frac{\partial u}{\partial t} - g(r,t) - \nabla \cdot (K\nabla u)\right] dV = 0$$

For this to be true for any arbitrarily small volume V, we must have

$$\rho s \frac{\partial u}{\partial t} - g(r,t) - \nabla \cdot (K \nabla u) = 0$$

(this requires continuity of the integrand, which we will assume here).

- If K is homogeneous (the same at each point), we can write $k = K/s\rho$ (the thermal diffusivity of the medium)

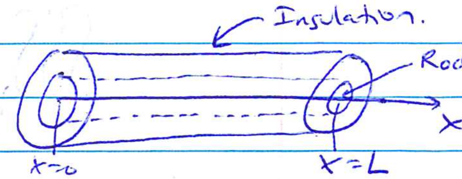
In this case we get

$$\frac{\partial u}{\partial t} = k\left[\nabla^2 u + \frac{g(r,t)}{K}\right]$$

We will now restrict to the one-dimensional case. Consider a circular rod of length L at some initial time (we will take $t=0$). Suppose the curved portion of the rod is perfectly insulated, and that the heat only flows in the x-direction. (Note, of course the rod is three-dimensional, but all cross-sections are identical with the boundary of those cross-sections being perfectly insulated, and the heat distribution is initially a function of x alone. Therefore heat flows only in the x direction)

In this case u is a function of x and t, and $g=0$ (no internal sources or sinks). Then u must satisfy

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$



This is the one-dimensional diffusion equation (diffusion of other quantities behaves in a similar way, so this equation appears in chemistry, probability theory and even financial mathematics).

Example 1: Formulate the initial boundary value problem for a cylindrical rod which has flat ends at $x=0$ and $x=L$ and insulated sides. At $t=0$, its temperature is $f(x)$ for $0 \leq x \leq L$. Assume that both ends are kept at $100°C$ for $t>0$.

It must satisfy the equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0<x<L, \quad t>0$$

- Since both ends are heated and kept at $100°C$ so

$$u(0,t) = 100 = u(L,t) \quad \text{for } t>0.$$

- Since the initial temperature of the rod is a function of x, we have

$$u(x,0) = f(x) \quad 0<x<L$$

To find solutions, consider the initial value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad , \quad 0<x<L \quad , \quad t>0$$

Neumann conditions : $\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t) \quad , \quad t>0$

$$u(x,0) = x \qquad 0<x<L.$$

Using separation of variables, we get

$$u(x,t) = X(x)T(t) \Rightarrow XT' = kX''T$$
$$\Rightarrow \frac{T'}{kT} = \frac{X''}{X}$$

The separation principle using $-\lambda$ implies

$$X'' + \lambda X = 0 \quad , \quad T' + \lambda kT = 0$$

The boundary conditions imply $X'(0) = 0 = X'(L) = 0.$

- Using the Sturm-Liouville computations, a solution to the first ODE is $X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$ corresponding to the eigenvalue $\lambda_n = \frac{n^2\pi^2}{L^2}$, $n \geq 0$.

- Now $T' + \lambda kT = T' + \frac{n^2\pi^2}{L^2}kT = 0$

has solution $T(t) = De^{-\frac{n^2\pi^2 kt}{L^2}}$

since the auxiliary equation is just $m + \frac{n^2\pi^2 k}{L^2} = 0.$

$\therefore \; u(x,t) = ae^{-\frac{n^2\pi^2 kt}{L^2}}\cos\left(\frac{n\pi x}{L}\right)$ for $n \geq 0$, $a \in \mathbb{R}$.

Satisfies the homogeneous system.

If we now require that $u(x,0) = x$, then

$$x = a\cos\left(\frac{n\pi x}{L}\right), \quad 0 < x < L$$

which is impossible. Just like in the wave equation, we can add up separated homogeneous solutions and take

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2 k t}{L^2}} \cos\left(\frac{n\pi x}{L}\right).$$

and now the initial conditions require

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad 0 < x < L.$$

Recognizing this as the Fourier cosine series for x, we require

$$a_0 = \frac{2}{L}\int_0^L x\,dx = \frac{2L^2}{2L} = L$$

and $$a_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2L\left[(-1)^n - 1\right]}{n^2\pi^2}, \quad n > 0.$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{4L}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

Then $$u(x,t) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} e^{-\frac{(2j-1)^2\pi^2 k t}{L^2}} \cos\left(\frac{(2j-1)\pi x}{L}\right)$$

Notice that $\lim_{t\to\infty} u(x,t) = L/2$, which means the temperature eventually stabilizes, as expected.

- Read through Section 20.1 for more details about the diffusion equation. Try exercises from that section as well as 21.2 Part A.