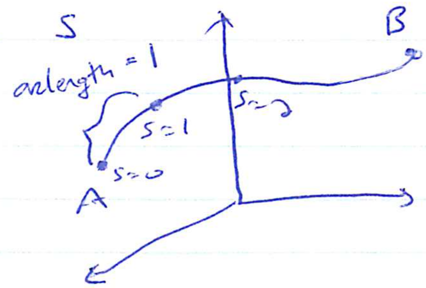
14.3 - Line Integrals Involving Vector Functions

Suppose that a curve $C \subseteq \mathbb{R}^3$ is parameterized by arclength. Then a point on C is described by

$$r(s) = x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k}$$

where $r(0) = A$ the initial point, and s measures length along the curve



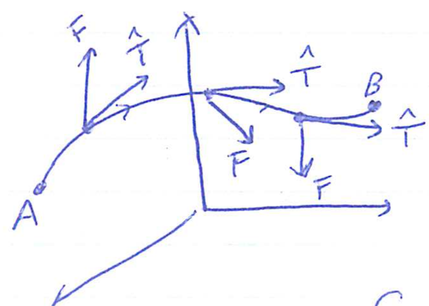
- Recall from chapter 11 that $\hat{T} = \dfrac{dr}{ds}$ is a unit tangent vector.

- Also recall that the component of a vector $\vec{v}$ in the direction of a vector $\vec{u}$ is

$$\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|} \quad \text{or} \quad \vec{v} \cdot \hat{u} \quad \text{where } \hat{u} = \frac{\vec{u}}{|\vec{u}|}$$

- We want to restrict to line integrals $\int_C f(x,y,z)\,ds$ where given a vector field defined along C, we have $f(x,y,z) = F \cdot \hat{T}$

- This is the tangential component of $F(x,y,z)$ along C.



Be sure to use the unit tangent vector in the direction of the curve.

We can write such line integrals as.

$$\int_C f(x,y,z)\,ds = \int_C F \cdot \hat{T}\,ds = \int_C F \cdot \frac{dr}{ds}\,ds = \int_C F \cdot dr$$

Since finding parameterizations by arclength can be difficult, we want to be able to compute these using any parameterization.

Suppose C is parameterized by $x = x(t)$, $y = y(t)$, $z = z(t)$, and $F(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$.

Then
$$F \cdot \vec{T}\, ds = (P, Q, R) \cdot \frac{(x'(t), y'(t), z'(t))}{|(x', y', z')|} ds$$
$$= \frac{(Px' + Qy' + Rz')}{\sqrt{(x')^2 + (y')^2 + (z')^2}} ds$$

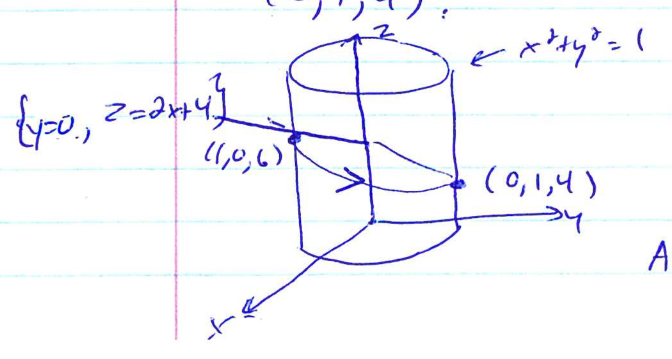
From 14.2
$$= \frac{(Px' + Qy' + Rz')}{\sqrt{(x')^2 + (y')^2 + (z')^2}} \sqrt{(x')^2 + (y')^2 + (z')^2}\, dt$$
$$= (Px' + Qy' + Rz')\, dt$$

But $dx = x'(t)\, dt$ etc

so
$$\int_C F \cdot dr = \int_C P\,dx + Q\,dy + R\,dz$$

Example 1: Evaluate $\int_C \frac{z}{y} dx + (x^2 + y^2 + z^2) dz$,

where C is the first octant part of the intersection of $x^2 + y^2 = 1$ and $z = 2x + 4$ joining $(1,0,6)$ to $(0,1,4)$.



Use C:
$$\left.\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}\right\} \text{cylinder}$$
$$z = 2\cos t + 4 \quad \big\} \text{ coming from plane.}$$

At $t = 0$ we have $(x,y,z) = (1,0,6)$
$t = \pi/2$ we have $(x,y,z) = (0,1,4)$.

Then $\displaystyle\int_C \frac{z}{y}\,dx + (x^2+y^2+z^2)\,dz$

$$= \int_0^{\pi/2} \frac{(2\cos t+4)}{\sin(t)}\,d(\cos t) + \left(\cos^2(t)+\sin^2(t) + (2\cos t+4)^2\right)d(2\cos t$$

$$= \int_0^{\pi/2} -2\cos t - 4 + (1 + 4\cos^2 t + 16\cos t + 16)(-2\sin t)\,dt$$

$$= -2\int_0^{\pi/2} 2 + \cos t + 17\sin t + 16\cos t\sin t + 4\cos^2 t\sin t\,dt$$

$$= -2\left(2t + \sin t - 17\cos t - \frac{16\cos^2(t)}{2} - \frac{4\cos^3(t)}{3}\right)\Bigg|_0^{\pi/2}$$

$$= -2\left(\pi + 1 + 17 + \frac{16}{2} + 4/3\right) = -2\pi - \frac{164}{3}$$

Note: If we used $x = \cos t$, $y = -\sin t$, $z = 2\cos t + 4$

$-\pi/2 \le t \le 0$.

Then we are tracing out C from $(0,1,4)$ to $(1,0,6)$ instead.

Line integral $\displaystyle = \int_{-\pi/2}^{0} \frac{(2\cos t+4)}{-\sin(t)}(-\sin t\,dt) + \cdots$

$$= 2\pi + \frac{164}{3}$$

In general given a curve with initial point A to final point B, the reverse orientation (directed from B to A) will be denoted by $-C$.

- We can show then that

$$\int_{-C} F \cdot dr = -\int_{C} F \cdot dr$$

Speed along curve doesn't matter, only orientation. Therefore these line integrals have the same absolute value.

Comparing Line Integrals: In 14.2, we saw that $\int_C f(x,y,z)\,ds$ did not depend on orientation, therefore $\int_{-C} f\,ds = \int_C f\,ds$. This is for line integrals of scalar fields.

- Here we have $\int_C F \cdot dr$ which is a line integral of a vector field, It depends on the orientation of C.

- The case where F is a vector field with vectors a scalar multiple of tangent vectors gives the integrals from 14.2. That is if $F = f(x,y,z)\hat{T}$ along C

$$\int_C F \cdot \hat{T}\,ds = \int_C f(x,y,z)\,\hat{T} \cdot \hat{T}\,ds$$

$$= \int_C f(x,y,z)\,|\hat{T}|^2 ds$$

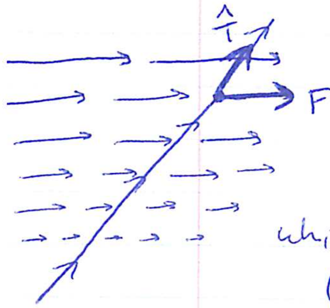
$$= \int_C f(x,y,z)\,ds.$$

Physics Motivation:

As a real life example, suppose our vector field F measures current speed and direction at a point, and C is a the path of a boat in the water assuming calm waters.

- We know that the more the boat goes against the current, the more work has to be done to move the boat, and vice versa.

- Measure of flow of water going with the path or against it is measured by



$$F(r(t)) \cdot r'(t) = |F(r(t))||r'(t)| \cos\theta$$

which measures how aligned the two vectors are.

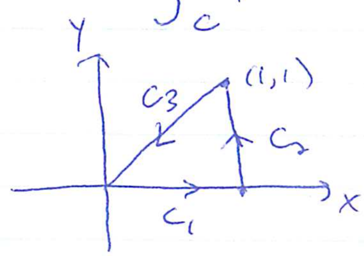
$\left(\begin{array}{l} > 0 \text{ means water pushing with the boat} \\ < 0 \text{ means water pushing against} \end{array} \right)$

↑
e see that
rientation matters

- Net accumulation of the current on the boat is total work

$$W = \int_{t_{\text{initial}}}^{t_{\text{final}}} F(r(t)) \cdot r'(t)\, dt = \int_C F \cdot dr$$

Example 2: Evaluate $\oint_C y^2\,dx + x^2\,dy$ where C is the closed curve



C_1: $y = 0$, $\quad 0 \leq x \leq 1$

C_2: $x = 1$, $\quad 0 \leq y \leq 1$

C_3: $y = x = 1 - t$ $\quad 0 \leq t \leq 1$

Then $\oint_C y^2 dx + x^2 dy = \int_{C_1} + \int_{C_2} + \int_{C_3}$

$$= \int_0^1 (0)^2 dx + x^2 \underbrace{dy}_{=0} + \int_0^1 y^2 \underbrace{dx}_{=0} + (1)^2 dy + \int_0^1 2(1-t)^2(-dt)$$

$$= \int_0^1 dy - 2\int_0^1 (1-t)^2 dt$$

$$= y\Big|_0^1 + \frac{2(1-t)^3}{3}\Big|_0^1 = 1 - 2/3 = 1/3$$

Example 3: Show that $\oint_C f(x)dx + g(y)dy + h(z)dz$

must have a value of zero when f, g, h are continuous on some domain containing C.

- $\oint_C f(x)dx + g(y)dy + h(z)dz = \oint_C f(x)dx + \oint_C g(y)dy + \oint_C h(z)d$

- Since f, g, h are continuous, antiderivative for $f(x(t))x'(t)$ exists, etc
- By the fundamental theorem of calculus

$$\oint_C f(x)dx = \int_\alpha^\beta f(x(t))x'(t)dt = F(\beta) - F(\alpha)$$

- C is closed so $x(\alpha) = x(\beta)$ and F is evaluated at the same points (initial point = terminal point).

- Therefore each $\oint_C f\,dx = 0$

$\oint_C g\,dy = 0$

$\oint_C h\,dz = 0$

In fact $\oint f(x)dx = F(\text{final}) - F(\text{initial})$ where F is an antiderivative for f.