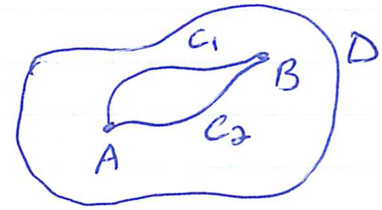
14.4 Independence of Path

Definition: A line integral $\int F \cdot dr$ is said to be independent of path in a domain D if for each pair of points $A, B \in D$, the value of

$$\int_C F \cdot dr$$

is the same for all piecewise-smooth paths $C \subseteq D$ from A to B.

ie. $$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$$



- In such a situation, $\int F \cdot dr$ only depends on the points A, B.
- Such vector fields seem very special, and are the kind we saw from 14.1
- Motivation: To be dependent on only A and B sounds like $\int_C F \cdot dr$ can be evaluated by some function at those points (like the FTC).

  Then $F = \nabla f$ is a good guess of obtaining such a function.

Theorem: Suppose $P(x,y,z)$, $Q(x,y,z)$ and $R(x,y,z)$ are continuous on some domain D. The line integral

$$\int F \cdot dr = \int P\,dx + Q\,dy + R\,dz$$

is independent of path in D iff there exists a function $f(x,y,z)$ defined in D such that $\nabla f = F$.

← if and only if

Corollary: When a line integral is independent of path in a domain D, and $A, B \in D$, then

$$\int_C F \cdot dr = f(x_B, y_B, z_B) - f(x_A, y_A, z_A)$$

where $\nabla f = F$, for every piecewise-smooth curve C in D from A to B.

Proof: If $F = \nabla f$, then $P = f_x$, $Q = f_y$, $R = f_z$

and $$\int_C f_x\,dx + f_y\,dy + f_z\,dz = \int_C df = f(B) - f(A)$$

Example 1: Evaluate $\int_C 2xy\,dx + x^2\,dy + 2z\,dz$, where C is the first octant intersection of $x^2+y^2=1$ and $z = 2x+4$ from $(0,1,4)$ to $(1,0,6)$.

- Since $\nabla(x^2y + z^2) = 2xy\,\hat{i} + x^2\,\hat{j} + 2z\,\hat{k}$,

  $\int F \cdot dr$ is independent of path (everywhere).

- $$\int_C 2xy\,dx + x^2\,dy + 2z\,dz = \left\{x^2y + z^2\right\}_{(0,1,4)}^{(1,0,6)} = 36 - 16 = 20.$$

Notice that $\int_{-C} F \cdot dr = -20$, so orientation still matters.

Proof of Theorem:

($\Leftarrow$) Assume that there exists a function $f(x,y,z)$ defined on D such that $\nabla f = F = P\hat{i} + Q\hat{j} + R\hat{k}$.

Let $C \subset D$ going from A to B be defined by

$$r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}, \quad \alpha \le t \le \beta.$$

Note: • $\nabla f = F \Rightarrow P(x,y,z) = \frac{\partial f}{\partial x}$ etc.

• $x = x(t) \Rightarrow dx = \left(\frac{dx}{dt}\right) dt$

Then $\int_C F \cdot dr = \int_C P\,dx + Q\,dy + R\,dz$

$$= \int_\alpha^\beta \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$$

$$= \int_\alpha^\beta \frac{d\, f(r(t))}{dt}\, dt$$ by the chain rule applied to $f(x(t), y(t), z(t))$

$$= f(r(\beta)) - f(r(\alpha))$$ by the FTC

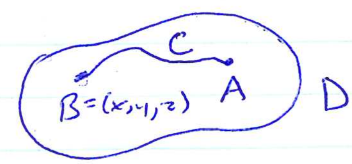
$$= f(B) - f(A)$$

$\therefore$ The integral only depends on the points A, B, so is independent of path.

($\Rightarrow$) Suppose instead that the line integral

$$\int F \cdot dr = \int P\,dx + Q\,dy + R\,dz$$

is independent of path in D. Let $A \in D$ be some fixed point, and $B = (x, y, z)$ some other point in D. Choose any piecewise-smooth curve $C \subset D$ from A to B.



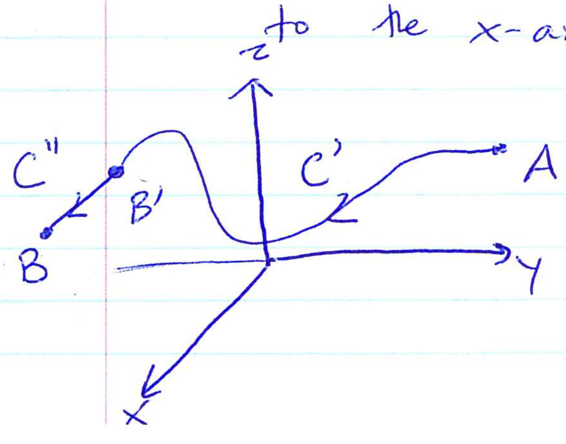
- We need a function f such that $\nabla f = F$. In single-variable calculus, we used $\int_a^x g(t)dt$. A good guess is

$$f(x,y,z) = \int_C F \cdot dr$$

- For a fixed B, $\int_C F \cdot dr$ is just a number, so $f : D \longmapsto \mathbb{R}$, and it is well-defined since $\int F \cdot dr$ is independent of path by assumption.

- We need to show $\nabla f = F$. This is not immediately clear (nor is the fact that f is differentiable).

- Let us choose a convenient curve C to work with: Choose C so that there is a fixed point $B' = (x', y, z)$ on C so that C is composed of the straight line C'' from B to B', and any piecewise-smooth curve C' from A to B'. Note that C'' is parallel to the x-axis.



Then

$$\int_C F \cdot dr = \int_{C'} F \cdot dr + \int_{C''} F \cdot dr$$

- A parametrization for C'' is

$$x(t) = t \quad , \quad y(t) = y \quad , \quad z(t) = z \qquad x' \leq t \leq x$$

so $$\int_{C''} F \cdot dr = \int_{x'}^{x} P(x(t), y(t), z(t))\, d(x(t)) + \underbrace{Q\,dy}_{=0} + \underbrace{R\,dz}_{=0}$$

$$= \int_{x'}^{x} P(t, x, y)\, dt$$

- To compute $\frac{\partial f}{\partial x}$, we fix y and z and vary x. With B' fixed, the "varying" in the definition of the derivative is occuring with the x-component of B.

- Then $\int_{C'} F \cdot dr$ is a constant since A and B' are fixed and $\int_{x'}^{x} P(t, x, y)\, dt$ is differentizble since P is continuous.

- This means $f(x,y,z) = \int_{C'} F \cdot dr + \int_{x'}^{x} P(t, y, z)\, dt$ can be differentiated in the x-direction.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_{C'} F \cdot dr + \frac{\partial}{\partial x} \int_{x'}^{x} P(t, y, z)\, dt$$

$$= 0 + P(x, y, z)$$

- Choose other curves C to get similar results for Q and R. Vary A and B as needed to show that for all points in D, $\nabla f = F$.

Corollary: The line integral $\int F \cdot dr$ is independent of path in a domain D iff

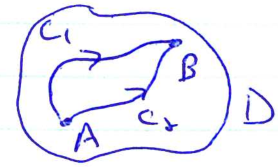
$$\oint_C F \cdot dr = 0 \quad \text{for every closed path in D.}$$

Proof: ($\Rightarrow$) If $\int F \cdot dr$ is independent of path then $\exists$ f such that $\nabla f = F$ and

$$\oint_C F \cdot dr = f(B) - f(A) = 0 \quad \text{since } A = B.$$

($\Leftarrow$) Suppose $\oint_C F \cdot dr = 0$ for every closed path in D.

Choose an $A, B \in D$ and let C_1, C_2 be two paths in D from A to B.



Then the curve C composed of C_1 and $-C_2$ is a closed curve.

$$0 = \oint_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{-C_2} F \cdot dr = \int_{C_1} F \cdot dr - \int_{C_2} F \cdot dr$$

$$\Rightarrow \int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$$

So $\int F \cdot dr$ is independent of path.

While it is sometimes obvious if there is a function f such that $\nabla f = F$, we would like a sure way to check whether $\nabla f = F$.

- Recall in 14.1, we saw if $\exists f$ s.t. (such that) $\nabla f = F$, then $\nabla \times F = 0$.
Conversely, if D is simply connected and $\nabla \times F = 0$, then there exists an f where $\nabla f = F$.

By the previous theorem, there exists an f where $\nabla f = F \iff \int F \cdot dr$ is independent of path.

Therefore, if D is simply connected, then the following are equivalent (TFAE):

(1) $\int F \cdot dr$ is independent of path in D

(2) $F = \nabla f$ for some function $f(x,y,z)$ defined over D

(3) $\nabla \times F = 0$ in D

If D is not simply connected, (For example, $\underbrace{D \text{ is } \mathbb{R}^3 \setminus \{0\}}$)

needed when functions involved use $\frac{1}{x^2+y^2+z^2}$ for instance

then:

$$\{\int F \cdot dr \text{ is ind. of path}\}$$
$$\Updownarrow$$
$$\{F = \nabla f \text{ for some } f\}$$
$$\Downarrow$$
$$\{\nabla \times F = 0\}$$

In other words, if D is not simply connected and $\nabla \times F = 0$, then $\int F \cdot dr$ may or may not be independent of path.

- Finally, the equivalence is valid in $\mathbb{R}^2$, except $\nabla \times F = 0$ in the xy-plane simplifies to the condition $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

Example 2: Evaluate $\int_C 2xye^z dx + (x^2e^z + y)dy + (x^2ye^z - z)dz$

Note: We can take $D = \mathbb{R}^3$.

along the straight line C from $(0,1,2)$ to $(2,1,-8)$

Method 1: We can parameterize C by $0 \le t \le 1$

$$x = 0 + (2-0)t \qquad y = 1 + (1-1)t \qquad z = 2 + (-8-2)t$$
$$= 2t \qquad = 1 \qquad = 2 - 10t$$

Then $\int_C F \cdot dr = \int_0^1 4te^{2-10t}(2dt) + 0 + [4t^2e^{2-10t} - 2 + 10t](-10dt)$

$$= \int_0^1 8e^2(t - 5t^2)e^{-10t} + 20 - 100t \, dt$$

Using integration by parts ...

$$= 4e^{-8} - 30$$

Method 2: By inspection

$\nabla\left(x^2ye^z + \frac{y^2}{2} - \frac{z^2}{2}\right) = F$, and the integral is independent of path.

Thus $\int_C F \cdot dr = \left\{x^2ye^z + \frac{y^2}{2} - \frac{z^2}{2}\right\}_{(0,1,2)}^{(2,1,-8)}$

$$= 4e^{-8} + \tfrac{1}{2} - \frac{64}{2} - 0 - \tfrac{1}{2} + \tfrac{4}{2}$$
$$= 4e^{-8} - 30$$

Method 3: Compute $\nabla \times F$

which is $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xye^z & x^2e^z+y & x^2ye^z-z \end{vmatrix}$

$$= (x^2e^z - x^2e^z)\hat{i} - (2xye^z - 2xye^z)\hat{j}$$
$$+ (2xe^z - 2xe^z)\hat{k}$$

$$= 0$$

$\Rightarrow \int F \cdot dr$ is independent of path

$\Rightarrow$ there exists some $f(x,y,z)$ such that $\nabla f = F$

- $\frac{\partial f}{\partial x} = 2xye^z \Rightarrow f = x^2ye^z + g(y,z)$

- $\frac{\partial f}{\partial y} = x^2e^z + y \Rightarrow \frac{\partial g}{\partial y} = y \Rightarrow g(y,z) = \frac{y^2}{2} + h(z)$

- $\frac{\partial f}{\partial z} = x^2ye^z - z \Rightarrow \frac{\partial h}{\partial z} = -z \Rightarrow h(z) = -\frac{z^2}{2} + k$

constant $\in \mathbb{R}$

and then use Method 2.