14.5 Energy and Conservative Force Fields

A force field $F(x,y,z)$, defined in a domain D, is said to be conservative in D if the line integral $\int F \cdot dr$ is independent of path in D.

Recall that $\int_C F \cdot dr$ is the total work done by F along C. Then

means that work done from A to B is the same along any path

F is conservative $\iff$ F is ind. of path $\iff$ $\oint_C F \cdot dr = 0$ for all closed curve C

$\iff \exists f$ such that $\nabla f = F$ $\implies \text{curl } F = 0$.

- We can associate a potential energy function $U(x,y,z)$ to a conservative force field F, which assigns potential energy at each point so that the difference in potential energy $U(A) - U(B)$ between A and B is the work done by F.

$$\int_C F \cdot dr = U(A) - U(B).$$

Then $\int_C F \cdot dr > 0 \implies$ Potential energy at $A >$ " " B etc.

- Since F is conservative, $\exists f$ such that $\nabla f = F$.

$$\implies \int_C F \cdot dr = f(B) - f(A)$$

$$\implies U(A) - U(B) = f(B) - f(A)$$

Since A, B are arbitrary, $U(x,y,z) + f(x,y,z)$ are the same at every point

$$\Rightarrow \quad U(x,y,z) + f(x,y,z) = k$$
$$\Rightarrow \quad U(x,y,z) = -f(x,y,z) + k.$$

and $-\nabla U = F$.

- We could also show that $\text{Work} = K(B) - K(A)$ where $K(x,y,z)$ is the kinetic energy.

$$\Rightarrow \quad U(A) - U(B) = K(B) - K(A)$$
$$\Rightarrow \quad U(A) + K(A) = K(B) + U(A)$$

so $E(A) = E(B)$ where E is total energy.

This is the law of conservation of energy for a conservative force field.

Example 1: Determine whether the following force field is conservative.

$$F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

- If we try to guess a function f such that $\nabla f = F$, we would need

$$f(x,y) = \int \frac{-y}{x^2+y^2}\, dx$$

which will involve arctan. Before committing to this, let's check the curl.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

where $R = 0$

$$\frac{\partial P}{\partial z} = \frac{\partial Q}{\partial z} = 0$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

Then $\nabla \times F = 0$ iff $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$.

We check that
$$\frac{\partial Q}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

and
$$\frac{\partial P}{\partial y} = \frac{-1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

$$\Rightarrow \quad \nabla \times F = 0.$$

Can we conclude that F is conservative?

F is defined on $\mathbb{R}^2 \setminus \{0\}$ which is not simply connected! Therefore we cannot conclude that F is a conservative force field.

(However, if $\nabla \times F \neq 0$ then definitly F is not conservative since F ind. of path $\Rightarrow \nabla \times F = 0$ is equivalent to the contrapositive $\nabla \times F \neq 0 \Rightarrow F$ not ind. of path)

- We could show that there is no f such that $\nabla f = F$. This seems a bit tedious.

- Instead recall that F conservative $\Leftrightarrow \oint_C F \cdot dr = 0$ for all C.

So it's enough to find one curve where this doesn't happen.

If we choose a curve that does not go around the origin then

$$\oint_C F \cdot dr = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

by Green's theorem in the next section.

We computed that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ away from the origin.

So $\oint_C F \cdot dr = 0$.

- It's probably no surprise that we should choose a curve which encloses the origin.

Choose the unit circle so that $x(t) = \cos t$, $y(t) = \sin t$, $0 \le t \le$

Then $\oint_C F \cdot dr = \int_\alpha^\beta F(r(t)) \cdot r'(t)\, dt$

This is the same as using $= \int_\alpha^\beta P\,dx + Q\,dy$

$$= \int_0^{2\pi} \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t} \right) \cdot \left(-\sin(t), \cos(t) \right) dt$$

$$= \int_0^{2\pi} \sin^2(t) + \cos^2(t)\, dt \quad = 2\pi$$

Since $\oint_C F \cdot dr \neq 0$, F is not a conservative force field.

• What if you did try to solve for a potential function?

$$-y\int \frac{dx}{x^2+y^2} = \frac{-1}{y}\int \frac{dx}{\left(\frac{x}{y}\right)^2+1}$$

$$= \frac{-1/y \arctan(x/y)}{1/y} = -\arctan\left(\frac{x}{y}\right).$$

and we can check that $\frac{\partial(-\arctan(\frac{x}{y}))}{\partial y} = \frac{-1}{\frac{x^2}{y^2}+1}\cdot\left(\frac{-x}{y^2}\right)$

$$= \frac{x}{x^2+y^2}$$

so $\nabla\left(-\arctan\left(\frac{x}{y}\right)\right) = F$

• In fact you can check that $\nabla\left(\arctan\left(\frac{y}{x}\right)\right) = F$.

Does this contradict previous results?

$-\arctan\left(\frac{x}{y}\right)$ only defined for $y \neq 0$ and $\arctan\left(\frac{y}{x}\right)$ for $x \neq 0$.

Therefore F is conservative on each of the simply connected domains $\{x>0\}$ or $\{x<0\}$ or $\{y>0\}$ or $\{y<0\}$

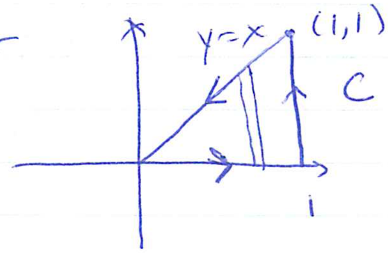
14.6 Green's Theorem

Theorem: Let C be a piecewise-smooth, closed curve in the xy-plane that does not intersect itself and that encloses a region R. If $P(x,y)$ and $Q(x,y)$ have continuous first partial derivatives in a domain D containing C and R, then

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Example 1: Evaluate $\oint_C y^2\,dx + x^2\,dy$

over



We can describe the region R using vertical strips

By Green's theorem $\oint_C y^2\,dx + x^2\,dy$

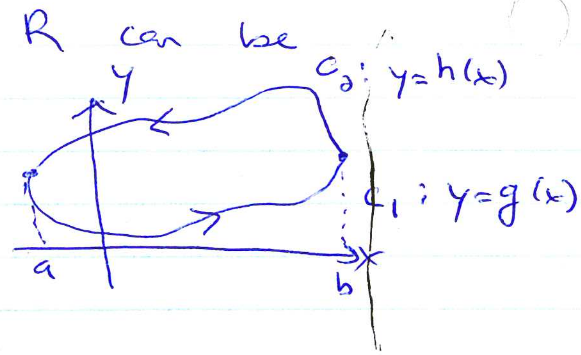
$$= \iint_R (2x - 2y)\,dA = 2\int_0^1 \int_0^x x - y\,dy\,dx$$

$$= 2\int_0^1 \left\{ xy - \frac{y^2}{2} \right\}\Big|_0^x dx$$

$$= 2\left(\frac{x^3}{3} - \frac{x^3}{6} \right)\Big|_0^1 = 1/3$$

- This is much quicker than solving $\oint_C F \cdot dr$ by breaking C into 3 subcurves.

Proof: Suppose our region R can be described by



First compute

$$\iint_R -\frac{\partial P}{\partial y}\, dA = \int_a^b \int_{g(x)}^{h(x)} -\frac{\partial P}{\partial y}\, dy\, dx$$

$$= \int_a^b \{-P\}\Big|_{g(x)}^{h(x)} dx = \int_a^b -P(x,h(x)) + P(x,g(x))\, d$$

On the other hand

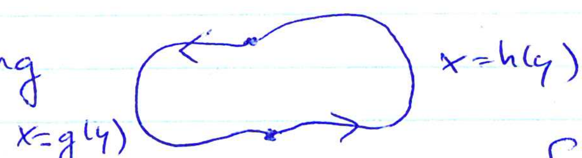
$$\oint_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx = \int_{C_1} P\,dx - \int_{-C_2} P\,dx$$

So C_1 and $-C_2$ go from a to b. Then

$$= \int_a^b P(x,g(x))\,dx - \int_a^b P(x,h(x))\,dx$$

$$\therefore \quad \oint_C P\,dx = \iint_R -\frac{\partial P}{\partial y}\, dA$$

Using



will show

$$\oint_C Q\,dy = \iint_R \frac{\partial Q}{\partial x}\, dA$$

More generally, subdivide regions into curves as above

and then add up Green's theorem applied to each region.

(More general curves require advanced techniques)

Note: Green's theorem does not work when P, Q are not defined at all points of R.
For example, P, Q need to be defined over some simply connected domain (or need to be restricted to such a domain). See exercise #30 for a way around this.

- What about the reverse orientation?

Recall that

$$\oint_{-C} F \cdot dr = -\oint_C F \cdot dr = -\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dA$$

- Note, we can recover the area of the region R as a line integral.

$$\text{area}(R) = \iint_R 1 \, dA = \oint_C P\,dx + Q\,dy$$

Either $\frac{\partial Q}{\partial x} = 1$ and $\frac{\partial P}{\partial y} = 0$

or $\frac{\partial Q}{\partial x} = 0$ and $\frac{\partial P}{\partial y} = -1$

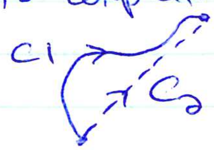
(or more generally, any $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$)

like $P = \frac{-y}{2}$, $Q = x/2$

So $\text{area}(R) = \oint x\,dy = \oint -y\,dx$

Finally, can we apply Green's theorem when C is not a closed curve?

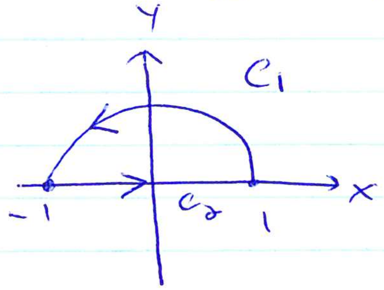
Idea is to complete curve to a closed curve



and use $C = C_1 - C_2$

Example 2: Compute $\oint_C -y^3\,dx + x^3\,dy$

where C is the upper half unit circle



Add in the line C_2 so that $C = C_1 + C_2$ is a closed curve.

By Green's theorem, $\oint_C F \cdot dr = \iint_R \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{3x^2 - (-3y^2) = 3(x^2+y^2)} dA$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

$$\iint_R 3(x^2+y^2)\,dA = 3\int_0^{\pi}\int_0^1 r^2 \cdot r\,dr\,d\theta$$

$$= 3\int_0^{\pi} \frac{r^4}{4}\Big|_0^1 d\theta = \frac{3}{4}\pi$$

On the other hand $\int_{C_2} -y^3\,dx + x^3\,dy = \int_{-1}^{1} -(0)^3\,dx + x^3\,d(0)$

$$= 0$$

Then $\oint_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr$

$$\frac{3}{4}\pi = \int_{C_1} F \cdot dr + 0$$