Note: Green's theorem does not work when P, Q are not defined at all points of R.
For example, P, Q need to be defined over some simply connected domain (or need to be restricted to such a domain). See exercise #30 for a way around this.

- What about the reverse orientation?

Recall that

$$\oint_{-C} F \cdot dr = -\oint_C F \cdot dr = -\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dA$$

- Note, we can recover the area of the region R as a line integral.

$$\text{area}(R) = \iint_R 1\, dA = \oint_C P\,dx + Q\,dy$$

Either $\frac{\partial Q}{\partial x} = 1$ and $\frac{\partial P}{\partial y} = 0$

or $\frac{\partial Q}{\partial x} = 0$ and $\frac{\partial P}{\partial y} = -1$

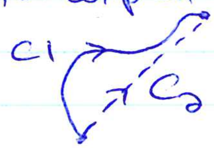
(or more generally, any $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$)

like $P = \frac{-y}{2}$
$Q = x/2$

So $\text{area}(R) = \oint x\,dy = \oint -y\,dx$

Finally, can we apply Green's theorem when C is not a closed curve?

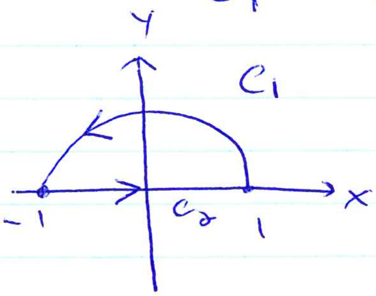
Idea is to complete curve to a closed curve



and use $C = C_1 - C_2$

Example 2: Compute $\displaystyle\int_{C_1} -y^3\,dx + x^3\,dy$

where C_1 is the upper half unit circle



Add in the line C_2 so that $C = C_1 + C_2$ is a closed curve.

By Green's theorem, $$\oint_C F\cdot dr = \iint_R \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{3x^2 - (-3y^2) = 3(x^2+y^2)} dA$$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

$$\iint_R 3(x^2+y^2)\,dA = 3\int_0^{\pi}\int_0^1 r^2\cdot r\,dr\,d\theta$$

$$= 3\int_0^{\pi} \frac{r^4}{4}\Big|_0^1 d\theta = \frac{3}{4}\pi$$

On the other hand $$\int_{C_2} -y^3\,dx + x^3\,dy = \int_{-1}^{1} -(0)^3\,dx + x^3\,d(0) = 0$$

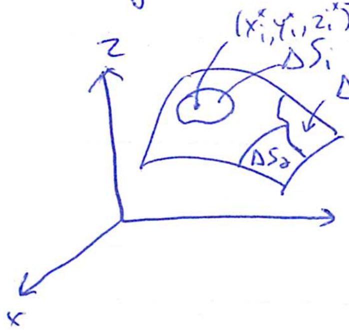
Then $$\oint_C F\cdot dr = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr$$

$$\frac{3}{4}\pi = \int_{C_1} F\cdot dr + 0$$

14.7 Surface Integrals

In 14.2 we introduced line integrals which generalized the integral of $f(x)$ over an interval $[a,b]$. What if we do the same for surfaces?

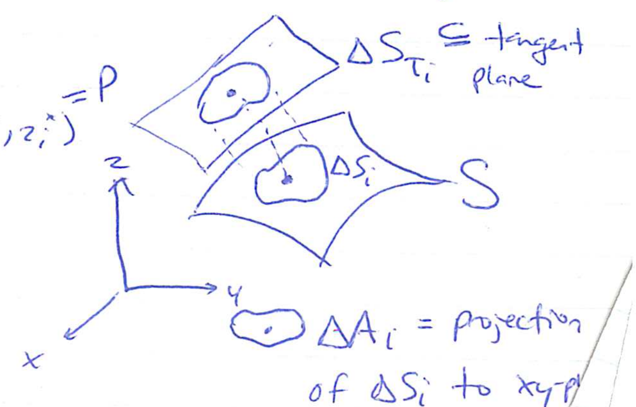
Let $S \subseteq \mathbb{R}^3$ be a surface, and suppose the function $f(x,y,z)$ is defined on S. What is the integral of f over S (again this is taking place in $\mathbb{R}^4$).



- Divide S into finitely many subsurfaces with area ΔS_i.
- Choose a test point in each S_i.

Then
$$\iint_S f(x,y,z)\, dS = \lim_{\|\Delta S_i\| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*) \Delta S_i$$

How do we compute ΔS_i? Recall from 13.6, we did the following:



- We can approximate S near $(x_i^*, y_i^*, z_i^*) = P$ by the tangent plane S_{T_i} there
- Projection ΔA_i of ΔS_i provides a corresponding region in S_{T_i}.
- Area of $S = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta S_{T_i}$

and each $\Delta S_{T_i} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\Big|_P\right)^2 + \left(\frac{\partial z}{\partial y}\Big|_P\right)^2}\, \Delta A_i$

Then surface area of S is just

$$\iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Applying this to the original problem, we get the following result for the surface integral $\iint_S f(x,y,z)\,dS$.

Theorem: If a function $f(x,y,z)$ is continuous on a smooth surface S of finite area, then the surface integral of $f(x,y,z)$ over S exists.

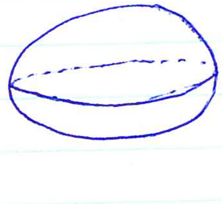
Suppose that S is defined by $z=g(x,y)$, and let S_{xy} be the projection of S to the xy-plane. Then

$$\iint_S f(x,y,z)\,dS = \iint_{S_{xy}} f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA.$$

- We can write $dS = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$.
- Similar results hold for projections to the yz or xz-planes with $x=g(y,z)$ or $y=g(x,z)$ respectively.

Note, if S is not defined by a single $z=g(x,y)$, then we need to break S up into pieces that are.

For example, if S is the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$,



then we can divide S into

S_1: $z = c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

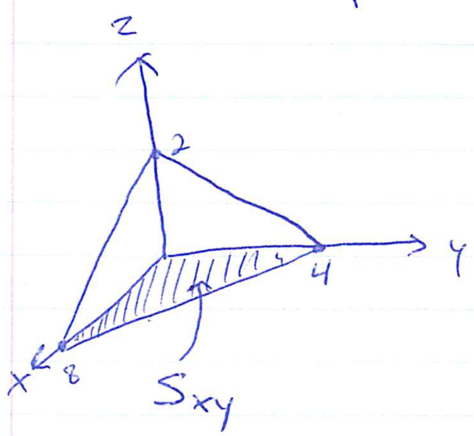
S_2: $z = -c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

Each projects to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z=0$

and $\iint_S f(x,y,z)\,dS = \iint_{S_1} f(x,y,z)\,dS + \iint_{S_2} f(x,y,z)\,dS$

Example 1: Evaluate $\iint_S (x+y+z)\,dS$, where

S is the positive octant part of $x+2y+4z=8$.



- S_{xy} can be described by the triangle bounded by

$$\{x+2y=8,\ z=0\}$$ in the first quadrant of the xy-plane.

- we can solve for z so that $z = 2 - \frac{y}{2} - \frac{x}{4}$

and $\frac{\partial z}{\partial x} = -1/4$ and $\frac{\partial z}{\partial y} = -1/2$

- $$\iint_S (x+y+z)\,dS = \iint_{S_{xy}} \left(x+y+2-\frac{y}{2}-\frac{x}{4}\right)\sqrt{1+(-1/4)^2+(-1/2)^2}\,dA$$

$$= \frac{1}{4}\cdot\frac{1}{4}\iint_{S_{xy}} (3x+2y+8)\sqrt{16+1+4}\,dA$$

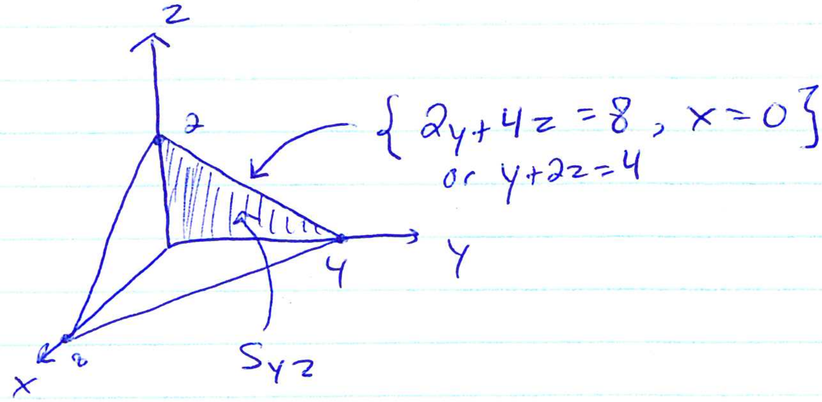
$$= \frac{\sqrt{21}}{8}\int_0^4\int_0^{8-2y} (3x+2y+8)\,dx\,dy$$

$$= \frac{56\sqrt{21}}{3}$$

- If we tried using S_{yz} (so that $x = 8-2y-4z$)

then $\frac{\partial x}{\partial y} = -2$ $\frac{\partial x}{\partial z} = -4$

and



Then
$$\iint_S (x+y+z)\,dS = \iint_{S_{yz}} (8-2y-4z+y+z)\sqrt{1+(-2)^2+(-4)^2}\,dA$$

$$= \sqrt{21}\int_0^2\int_0^{4-2z} (8-y-3z)\,dy\,dz$$

$$= \sqrt{21}\int_0^2 \left\{8y - \frac{y^2}{2} - 3yz\right\}\Big|_0^{4-2z} dz$$

$$= \sqrt{21}\int_0^2 32-16z-8+8z-2z^2-12z+6z^2\,d$$

$$= \sqrt{21}\int_0^2 24-20z+4z^2\,dz$$

$$= 4\sqrt{21}\left(6z - \frac{5z^2}{2} + \frac{z^3}{3}\right)\Big|_0^2$$

$$= 4\sqrt{21}\left(12-10+\frac{8}{3}\right)$$

$$= \frac{4\sqrt{21}}{3}(6+8) = \frac{56\sqrt{21}}{3}$$

You always have the choice of using a projection that is convenient computationally

Definition: A closed surface is a surface S which encloses some volume. The surface integral over a closed surface is denoted by

$$\oiint_S f(x,y,z)\,dS$$

Example 2: Evaluate $\iint_S z^2\, dS$, where S is the sphere $x^2+y^2+z^2=4$.

- Divide S into two hemispheres

$$S_1: y=\sqrt{4-x^2-z^2} \qquad S_2: y=-\sqrt{4-x^2-z^2}$$

- Then $\left(\frac{\partial y}{\partial x}\right)^2 = \left(\frac{\pm 2x}{2\sqrt{4-x^2-z^2}}\right)^2 = \frac{x^2}{4-x^2-z^2}$

and $\left(\frac{\partial y}{\partial z}\right) = \left(\frac{\pm 2z}{2\sqrt{4-x^2-z^2}}\right)^2 = \frac{z^2}{4-x^2-z^2}$

- The projection to the xz-plane is $\{x^2+z^2 \le 4,\ y=0\} = S_{xz}$

$$\iint_S z^2\, dS = \iint_{S_1} z^2\, dS + \iint_{S_2} z^2\, dS$$
$$= 2\iint_{S_{xz}} z^2 \sqrt{1+\frac{x^2}{4-x^2-z^2}+\frac{z^2}{4-x^2-z^2}}\; dA$$
$$= 4\iint_{S_{xz}} \frac{z^2}{\sqrt{4-x^2-z^2}}\, dA$$

Be careful about using symmetry arguments since integrals of other functions over the sphere would be different.

In polar coordinates $x = r\cos\theta$, $z = r\sin\theta$

$$= 4\int_0^{2\pi}\int_0^2 \frac{r^2\sin^2\theta}{\sqrt{4-r^2}}\, r\,dr\,d\theta$$

$$= 4\int_0^{2\pi} \underbrace{\sin^2\theta}_{=\frac{1-\cos 2\theta}{2}}\, d\theta \int_0^2 \frac{r^3}{\sqrt{4-r^2}}\, dr$$

$$= 4\pi \int_0^2 r^2 \cdot \frac{r}{\sqrt{4-r^2}}\, dr$$

$$= 4\pi \left(-r^2 \cdot (4-r^2)^{1/2} \Big|_0^2 - \int_0^2 -2r\,(4-r^2)^{1/2}\, dr \right)$$

$$= 4\pi \left(-4\,(0)^{1/2} + 0^2 \cdot (4)^{1/2} - \left\{ \frac{(4-r^2)^{3/2}}{3/2} \right\} \Big|_0^2 \right)$$

$$= 4\pi \left(0 + \frac{4^{3/2}}{3/2} \right) = 4\pi \left(\frac{2 \cdot 2^3}{3} \right) = \frac{64\pi}{3}$$

Note: The solution in the text uses the upper and lower hemispheres, which is easier to solve since z^2 changes to $4-x^2-y^2$ when evaluating the integral.