Example 2: Evaluate $\iint_S z^2\,dS$, where S is the sphere $x^2+y^2+z^2=4$.

- Divide S into two hemispheres

$$S_1: y=\sqrt{4-x^2-z^2} \qquad S_2: y=-\sqrt{4-x^2-z^2}$$

- Then $\left(\frac{\partial y}{\partial x}\right)^2=\left(\frac{\pm 2x}{2\sqrt{4-x^2-z^2}}\right)^2=\frac{x^2}{4-x^2-z^2}$

and $\left(\frac{\partial y}{\partial z}\right)=\left(\frac{\pm 2z}{2\sqrt{4-x^2-z^2}}\right)^2=\frac{z^2}{4-x^2-z^2}$

- The projection to the xz-plane is $\{x^2+z^2\le 4,\ y=0\}=S_{xz}$

$$\iint_S z^2\,dS=\iint_{S_1} z^2\,dS+\iint_{S_2} z^2\,dS$$
$$=2\iint_{S_{xz}} z^2\sqrt{1+\frac{x^2}{4-x^2-z^2}+\frac{z^2}{4-x^2-z^2}}\,dA$$
$$=4\iint_{S_{xz}}\frac{z^2}{\sqrt{4-x^2-z^2}}\,dA$$

Be careful about using symmetry arguments since integrals of other functions over the sphere would be different.

In polar coordinates $\begin{array}{l} x=r\cos\theta \\ z=r\sin\theta \end{array}$

$$=4\int_0^{2\pi}\int_0^2\frac{r^2\sin^2\theta}{\sqrt{4-r^2}}\,r\,dr\,d\theta$$
$$=4\int_0^{2\pi}\underbrace{\sin^2\theta}_{=\frac{1-\cos 2\theta}{2}}\,d\theta\int_0^2\frac{r^3}{\sqrt{4-r^2}}\,dr$$

$$= 4\pi \int_0^2 r^2 \cdot \frac{r}{\sqrt{4-r^2}}\, dr$$

$$= 4\pi \left(-r^2 \cdot (4-r^2)^{1/2} \Big|_0^2 - \int_0^2 -2r\,(4-r^2)^{1/2}\, dr \right)$$

$$= 4\pi \left(-4\,(0)^{1/2} + 0^2 \cdot (4)^{1/2} - \left\{ \frac{(4-r^2)^{3/2}}{3/2} \right\} \Big|_0^2 \right)$$

$$= 4\pi \left(0 + \frac{4^{3/2}}{3/2} \right) = 4\pi \left(\frac{2 \cdot 2^3}{3} \right) = \frac{64\pi}{3}$$

Note: The solution in the text uses the upper and lower hemisphaes, which is easier to solve since z^2 changes to $4-x^2-y^2$ when evaluating the integral.

Parameterizations of Surfaces:

A parameterization of a surface S is a map

$$\gamma : [a,b] \times [c,d] \longrightarrow S$$
$$(s,t) \longmapsto (x(s,t), y(s,t), z(s,t))$$

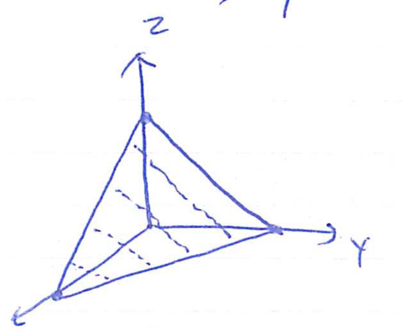
where x, y, z are usually chosen to be smooth.
(or similarly from $\mathbb{R}^2 \to S$).

- When S has the form $z = f(x,y)$, then we can set:

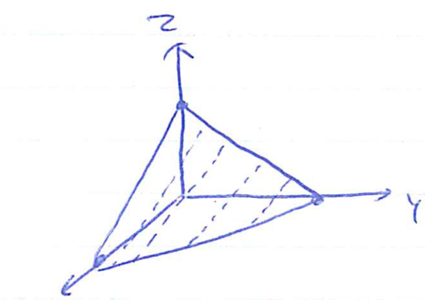
$$x = s \quad , \quad y = t \quad \text{and} \quad z = f(s,t)$$

- For example, the plane $x + y + z = 1$ can be parameterized by

$$x = s \, , \; y = t \, , \; z = 1 - s - t \qquad s, t \in \mathbb{R}$$



Fixed s

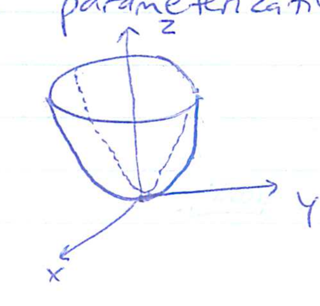


Fixed t

For fixed s, we get a pamaterization for the line
$x = a \quad , \quad y + z = 1 - a$.

- For $z = x^2 + y^2$, the paraboloid, the parameterization

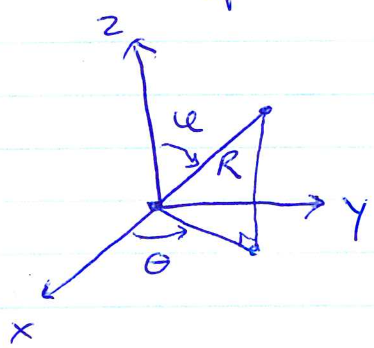
$$x = s \, , \; y = t \, , \; z = s^2 + t^2 \qquad s, t \in \mathbb{R}$$



also parameterizes each cross section
$\{ z = a^2 + y^2, x = a \}, \{ z = x^2 + b^2, y = b \}$

- A less trivial example is the sphere $x^2+y^2+z^2=a^2$. Using $x=s$ and $y=t$ would require a separate parameterization for the upper and lower hemispheres.

Recall the spherical coordinates



$$x = R\sin\varphi\cos\theta$$
$$y = R\sin\varphi\sin\theta$$
$$z = R\cos\varphi$$

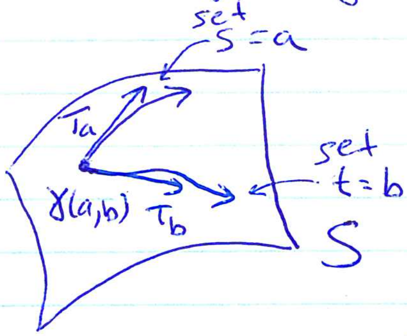
with $R>0$, $0\le\theta<2\pi$, $0\le\varphi<\pi$.

In spherical coordinates, the sphere is just $R=a$. On the other hand, a point (x,y,z) on the sphere can be described by

$$x = a\sin\varphi\cos\theta, \quad y=\sin\varphi\sin\theta, \quad z=\cos\varphi$$
with $0\le\theta<2\pi$ and $0\le\varphi<\pi$.

Surface Integrals over Parameterized Surfaces:

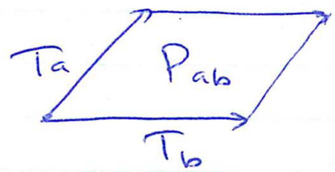
Assume that S can be parameterized by $\gamma: x=x(s,t),\ y=y(s,t),\ z=z(s,t)$. We can approximate the surface area of S using tangent vectors.



- At a point in S given by $\gamma(a,b)$, we can consider two curves, $\gamma(s,b)$ and $\gamma(a,t)$.

- The tangent space $T_{\gamma(a,b)}S$ is generated as a vector space by $\left.\frac{\partial\gamma(s,b)}{\partial s}\right|_{s=a}=T_a$ and $\left.\frac{\partial\gamma(a,t)}{\partial t}\right|_{t=b}=T_b$.

- The surface area S_{ab} can be approximated by the area of the parallelogram



Recall that the area of P_{ab} is given by $|T_a \times T_b|$.

- Therefore, $dS = |\gamma_s \times \gamma_t| \, dA$

For the sphere, the parameters are φ and θ.

Let $\gamma(\varphi, \theta) := (a \sin\varphi \cos\theta)\hat{i} + (a \sin\varphi \sin\theta)\hat{j} + (a\cos\varphi)\hat{k}$,

then $\gamma_\varphi = (a\cos\varphi\cos\theta,\ a\cos\varphi\sin\theta,\ -a\sin\varphi)$

$\gamma_\theta = (-a\sin\varphi\sin\theta,\ a\sin\varphi\cos\theta,\ 0)$

and
$$\gamma_\varphi \times \gamma_\theta = (a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\cos^2\theta\cos\varphi\sin\varphi + a^2\sin^2\theta\cos\varphi\sin\varphi)$$
$$= (a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\cos\varphi\sin\varphi)$$

and
$$|\gamma_\varphi \times \gamma_\theta|^2 = a^4\sin^4\varphi\cos^2\theta + a^4\sin^4\varphi\sin^2\theta + a^4\cos^2\varphi\sin^2\varphi$$
$$= a^4\sin^4\varphi + a^4\cos^2\varphi\sin^2\varphi$$
$$= a^4\sin^2\varphi(\sin^2\varphi + \cos^2\varphi)$$
$$= a^4\sin^2\varphi$$
$$\Rightarrow |\gamma_\varphi \times \gamma_\theta| = a^2\sin\varphi$$

- Many of the surfaces that we have used in Math 2130 can be parameterized using Cartesian, cylindrical or spherical coordinates.

Example 3: Evaluate $\iint_S z^2 \, dS$, using spherical coordinates, where S is $x^2+y^2+z^2=4$.

By previous remarks, since $z = R\cos\varphi$

$$\iint_S z^2 \, dS = \int_0^{2\pi}\int_0^{\pi} (2\cos\varphi)^2 \cdot (2)^2 \sin\varphi \, d\varphi \, d\theta$$

$$= 16 \int_0^{2\pi}\int_0^{\pi} \cos^2\varphi \sin\varphi \, d\varphi \, d\theta$$

$$= 16(2\pi)\left[-\frac{\cos^3\varphi}{3}\right]\Bigg|_0^{\pi}$$

$$= 32\pi\left[-\left(-\tfrac{1}{3}\right) + \left(\tfrac{1}{3}\right)\right] = \frac{64\pi}{3}$$

Example 4: Let S be defined by $z = g(x,y)$. Write an expression for $\iint_S f(x,y,z)\, dS$ in Cartesian coordinates.

domain of g is same as projection

- Using the parameterization $\gamma: x = x$, $y = y$, $z = g(x,y)$, $(x,y) \in S_{xy}$

we get $\gamma_x = (1, 0, g_x)$, $\gamma_y = (0, 1, g_y)$

and $$\gamma_x \times \gamma_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = (-g_x, -g_y, 1).$$

Then $|\gamma_x \times \gamma_y| = \sqrt{1 + g_x^2 + g_y^2}$

and $$\iint_S f(x,y,z)\, dS = \iint_{S_{xy}} f(x,y,g(x,y)) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, d$$

as we had before.