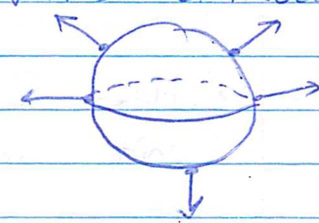
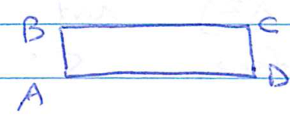
14.8 - Surface Integrals Involving Vector Fields

An orientable surface S is a surface for which a normal vector can be chosen consistently at each point, that is, a normal vector that varies continuously over S.

- The sphere is orientable since we can choose the outward unit normal vector that varies continuously as we go around the sphere

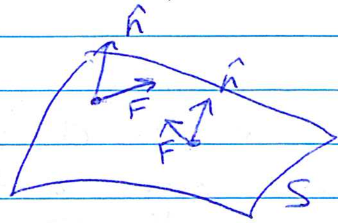


- The Möbius strip is not orientable. To form this surface, take  and glue the side AB to CD (so that A is glued to C and B to D). If you choose a unit normal vector at a point, follow it around a curve going around S back to the point, then it will be in the opposite direction when you return to the point. (there are videos of this online as it is difficult to draw on paper).

- The precise definition of orientable requires terminology beyond the course, so this visual description will have to suffice for now.

In 14.3 we considered line integrals of the tangential component of a vector field. A normal direction along a curve is not unique. For surfaces, the tangential direction is not unique, but the normal direction is.

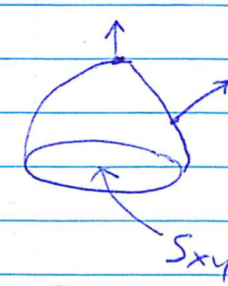
- We will restrict to $\iint_S f(x,y,z)\,dS$ where $f(x,y,z)$ is the normal component of a vector field F defined on S.



- $f(x,y,z)$ = component of $F(x,y,z)$ in direction of $\hat{n}$.
  $= F \cdot \hat{n}$

- Here S is an orientable surface and $\hat{n}$ is the unit normal on one side of S.

Example 1: Evaluate $\iint_S F \cdot \hat{n}\, dS$ where $F = x^2 y\,\hat{i} + xz\,\hat{j}$ and $\hat{n}$ is the upper normal to the surface $S: z = 4 - x^2 - y^2,\ z \geq 0$.



- An outward normal vector to S is

$$n = \nabla(z + x^2 + y^2 - 4) = (2x, 2y, 1)$$

(at $(0,0,4)$ for example, it is $(0,0,1)$).

Then $\hat{n} = \dfrac{(2x, 2y, 1)}{\sqrt{4x^2 + 4y^2 + 1}}$ and $F \cdot \hat{n} = \dfrac{2x^3 y + 2xyz}{\sqrt{4x^2 + 4y^2 + 1}}$

Then $\displaystyle\iint_S F \cdot \hat{n}\, dS = \iint_{S_{xy}} \frac{2x^3 y + 2xyz}{\sqrt{4x^2 + 4y^2 + 1}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$

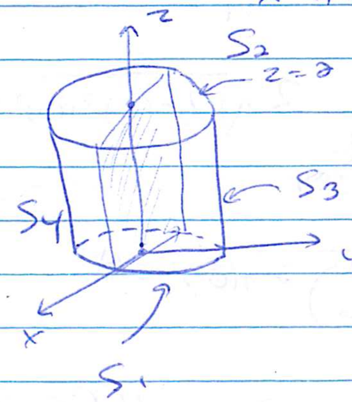
but $\left(\dfrac{\partial z}{\partial x}\right)^2 = (-2x)^2$ and $\left(\dfrac{\partial z}{\partial y}\right)^2 = (-2y)^2$ and $z = 4 - x^2 - y^2$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2x^3 y + 2xy(4 - x^2 - y^2)\, dy\, dx$$

$$\ldots = 0$$

Example 2: Evaluate $\iint_S F\cdot\hat{n}\,dS$, where $F = x\hat{i} + y\hat{j} + z\hat{k}$ and $\hat{n}$ is the unit outward pointing normal to the surface bounded by $x^2+y^2=4$, $z=0$, $z=2$.

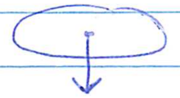
Method 1:

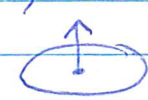


In order to evaluate the surface integral, we need to break up S into pieces defined by a function.

S_1: $z=0$, $x^2+y^2 \le 4$
S_2: $z=2$, $x^2+y^2 \le 4$
S_3: $y=\sqrt{4-x^2}$, $0 \le z \le 2$
S_4: $y=-\sqrt{4-x^2}$, $0 \le z \le 2$

On S_1, $\hat{n} = -\hat{k}$ and on S_2, $\hat{n} = \hat{k}$





On S_3 and S_4, $\hat{n} = \dfrac{\nabla(x^2+y^2-4)}{|\nabla(x^2+y^2-4)|} = \dfrac{(2x, 2y, 0)}{\sqrt{4x^2+4y^2}}$

but a point on the cylinder $x^2+y^2=4$
so $\sqrt{4x^2+4y^2} = \sqrt{4(x^2+y^2)} = \sqrt{16} = 4$

so $\hat{n} = \dfrac{(x, y, 0)}{2}$

- S_1 and S_2 project down to $S_{xy} = \{z=0, x^2+y^2 \le 4\}$
- S_3 and S_4 project to S_{xz} defined using the $y=0$ cross-section $x^2=4$, $0 \le z \le 2$ so

$$S_{xz} = \{-2 \le x \le 2,\ 0 \le z \le 2\}$$

$$-(x\hat{i}+y\hat{j}+z\hat{k})\cdot\hat{k}$$

$$\iint_S F\cdot\hat{n}\,dS = \iint_{S_1} -z\,dS + \iint_{S_2} z\,dS$$

$$+ \iint_{S_3} \frac{x^2+y^2}{2}\,dS + \iint_{S_4} \frac{x^2+y^2}{2}\,dS$$

$$= \iint_{S_{xy}} -(0)\sqrt{1+(0)^2+(0)^2}\,dA + \iint_{S_{xy}} 2\sqrt{1+(0)^2+(0)^2}\,dA$$

← since $y^2 = 4-x^2$

$$+ 2\iint_{S_{xz}} \frac{x^2+4-x^2}{2}\sqrt{1+\left(\frac{\pm x}{\sqrt{4-x^2}}\right)^2+(0)^2}\,dA$$

The surface integral over S_3 and S_4 are the same

$$= 0 + 2(\text{area of } S_{xy}) + 4\iint_{S_{xz}} \frac{\sqrt{4-x^2+x^2}}{\sqrt{4-x^2}}\,dA$$

$$= 2\pi(2)^2 + 8\int_{-2}^{2}\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dz\,dx$$

$$= 8\pi + 16\int_{-2}^{2} \frac{1}{\sqrt{4-x^2}}\,dx$$

Let $x = 2\sin\theta$ $\quad dx = 2\cos\theta\,d\theta$

$$\int_{-2}^{2} \frac{1}{\sqrt{4-x^2}}\,dx = \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\,\frac{1}{|\cos\theta|}\,2\cos\theta\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} d\theta \quad \text{since } \cos\theta \geq 0 \text{ on } [-\pi/2, \pi/2]$$

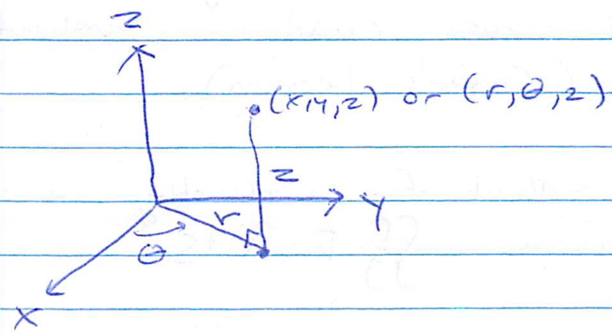
$$= \pi$$

So $\iint F\cdot\hat{n}\,dS = 8\pi + 16\pi = 24\pi$

Method 2:

- What if we wanted to compute the surface integral over the cylinder using cylindrical coordinates?

- Recall that we can change to cylindrical coordinates using



$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$r > 0 \qquad 0 \le \theta < 2\pi$

- We can parameterize the cylinder $S: x^2 + y^2 = a^2,\ c \le z \le d$ by

$\sigma: \quad x = a\cos\theta \quad , \quad y = a\sin\theta \quad , \quad z = z$

$$0 \le \theta < 2\pi \quad , \quad c \le z \le d.$$

- Then $\sigma_\theta = (-a\sin\theta,\ a\cos\theta,\ 0)$
  $\sigma_z = (0,\ 0,\ 1)$

- $\sigma_\theta \times \sigma_z = (a\cos\theta,\ a\sin\theta,\ 0)$

- $|\sigma_\theta \times \sigma_z| = \sqrt{a^2\cos^2\theta + a^2\sin^2\theta} = a$

Then in cylindrical coordinates, the surface integral over a cylinder is

$$\iint_S f(x,y,z)\,dS = \int_0^{2\pi}\int_c^d f(a\cos\theta, a\sin\theta, z)\,a\,dz\,d\theta$$

(here $a=2$)

Applying this to the example, we get

$$\iint_{S_3 \cup S_4} F\cdot\hat{n}\,dS = \int_0^{2\pi}\int_0^2 \frac{a^2\cos^2\theta + a^2\sin^2\theta}{2}\,a\,dz\,d\theta$$

$$= 4\int_0^{2\pi}\int_0^2 dz\, d\theta = 8\int_0^{2\pi} d\theta = 16\pi$$

In 14.6, we showed that line integrals of vector fields over closed curves could be evaluated using a double integral (Green's theorem).

We seek a similar method for evaluating surface integrals of the form $\oiint_S F\cdot\hat{n}\,dS$.

Method 3:

By the divergence theorem, if V is the region enclosed by S, then

$$\oiint_S F\cdot\hat{n}\,dS = \iiint_V \nabla\cdot F\,dV$$

Here $\nabla\cdot F = \operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 1+1+1$

$$= \iiint_V 3\,dV$$

$$= 3\,\text{volume}(V)$$

$$= 3\pi(2)^2(2) = 24\pi$$

Using the divergence theorem converts the surface integral into a simple triple integral because the vector field $F = x\hat{i} + y\hat{j} + z\hat{k}$ had a particularly easy divergence.