14.1 - Vector Fields

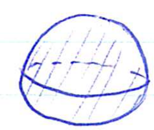
Let $S \subseteq \mathbb{R}^n$. Define the open n-ball of radius r centered at the point $a \in \mathbb{R}^n$ by

$$B_r(a) = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid (x_1-a_1)^2 + \dots + (x_n-a_n)^2 < r\}.$$

For $n=1$ $n=2$ $n=3$

open interval

area bounded by circle



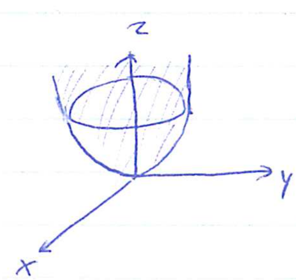
area bounded by sphere.

A point $P \in \mathbb{R}^n$ is said to be:

- An interior point of S, if there is a ball $B_r(P) \subseteq S$
- An exterior point of S, if there is a $B_r(P) \subseteq \mathbb{R}^n \setminus S$

  points of $\mathbb{R}^n$ not in S.
- A boundary point of S if every $B_r(P)$ contains points in S and $\mathbb{R}^n \setminus S$.

Example 1: Let $S = \{z > x^2 + y^2\} \subseteq \mathbb{R}^3$



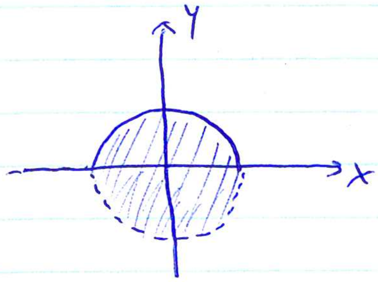
Then $\text{int}(S) = S$

$\text{ext}(S) = \{z < x^2 + y^2\}$

$\text{bdry}(S) = \{z = x^2 + y^2\}$

Note: We can also use ∂S for the boundary, although we will use this notation for a different concept at the end of the chapter.

Example 2: Let $S = \{y \leq \sqrt{1-x^2}\} \cup \{y > -\sqrt{1-x^2}\} \subseteq \mathbb{R}^2$.



$\text{int}(S) = \{x^2+y^2 < 1\}$

$\text{ext}(S) = \{x^2+y^2 > 1\}$

$\text{bdry}(S) = \{x^2+y^2 = 1\}$

Definition: Let $S \subseteq \mathbb{R}^n$. S is said to be an open set if all points of S are interior points ($\text{int}(S) = S$). S is said to be a closed set if it contains all of its boundary points ($\text{bdry}(S) \subseteq S$).

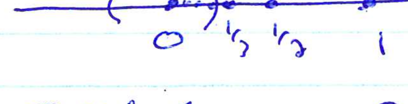
Example 1 was an example of an open set, while example 2 was neither open or closed.

Example 3: Is the set $S = \{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\} \subseteq \mathbb{R}$ closed?

We see that $\text{int}(S) = \emptyset$ since S contains no open interval.

It is easy to see that $S \subseteq \text{bdry}(S)$ since any open interval centered at $1/n$ will contain irrational number

However, since $\lim_{n\to\infty} \frac{1}{n} = 0$, the point $0 \in \text{bdry}(S)$ since any open interval centered at 0 contains $1/n$ for some n.



Therefore $\text{bdry}(S) = S \cup \{0\}$ so S is not closed.

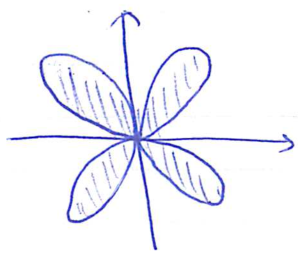
Note: One way to characterize the boundary of a set is by the limits of converging sequences of points in S and $\mathbb{R}^n \setminus S$.

Definition: $S \subseteq \mathbb{R}^n$ is a connected set if every pair of points in S can be joined by a piecewise-smooth curve lying entirely in S.

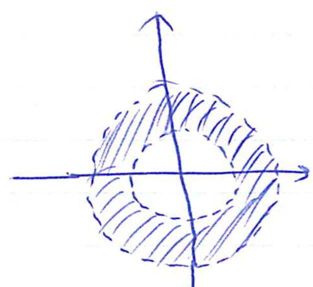
Note: This is really the notion of path connected which is different from connected, but we will not worry about such distinctions in this class.

A domain is an open, connected set (different from a function domain).

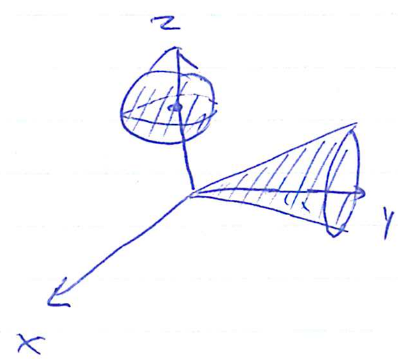
Example 4:



connected
(not a domain)



domain

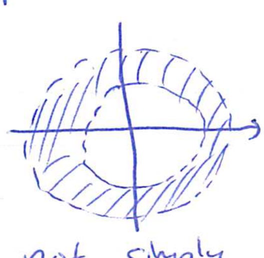


not connected

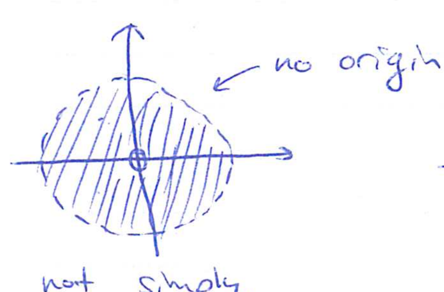
Definition: A domain S is said to be simply connected if every closed curve in S contains in its interior only points of S.

This essentially says that S has no holes.

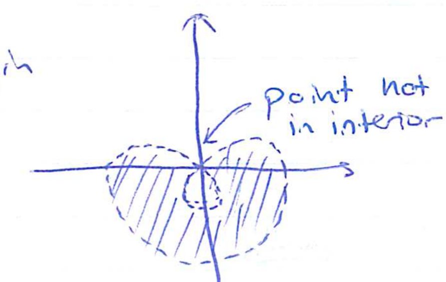
Example 5:



not simply connected



not simply connected



simply connected

A vector field F over a region $D \subseteq \mathbb{R}^n$ assigns a vector in $\mathbb{R}^n$ to each point in D.

That is, F is a vector-valued function

$$F : D \longmapsto \mathbb{R}^n.$$

- In $\mathbb{R}^3$ we can write

$$F = F(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$$

We say that F is continuous, differentiable etc if P, Q, R are.

- A common vector field seen in everyday life is wind speed and direction at any given point on a weather map.

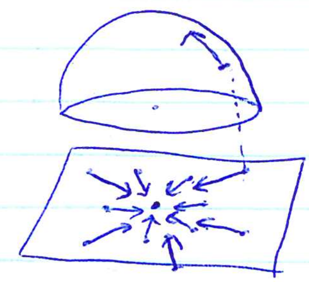
Example 6: Recall that given $f(x,y,z)$,

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

is a vector field. We know that ∇f points in the direction where f increases most rapidly. Indeed $|\nabla f|$ is the rate of change in that direction.

In the sphere $z = \sqrt{1-x^2-y^2}$ for example,

$$\nabla z = \left(\frac{-x}{\sqrt{1-x^2-y^2}}, \frac{-y}{\sqrt{1-x^2-y^2}} \right)$$



- We can also write

$$\nabla f = \underbrace{\left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right)}_{= \nabla} f$$

$= \nabla$ the vector differential operator called the del operator

It takes a differentiable function and produces a vector field (i.e. $\nabla : C^1(\mathbb{R}^n) \longrightarrow \{ \text{vector fields} \}$)

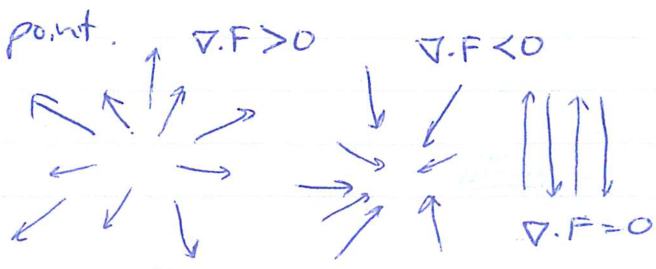
- In the other direction we have the notion of divergence. Given $F = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$, over a region D, the divergence of F is a scalar field in D defined by

$$\operatorname{div} F = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Note: $\operatorname{div} F$ takes a vector field and produces a function. It is not the same as the gradient of a function. (which is a vector).

We will discuss the physical interpretation in more detail later, but essentially divergence measures whether there are more field vectors exiting than entering in an infinitesimal region. Put another way, it measures the degree to which the vector field flux behaves like a "source" or "sink" at a point.

Think of airflow for example.



- For example, if $F = 2xy\hat{i} + z\hat{j} + x^2\cos(yz)\hat{k}$,

then
$$\nabla \cdot F = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}\left(x^2\cos(yz)\right)$$
$$= 2y + 0 + -x^2\sin(yz)\,y$$

- We can also define the curl of $F = P\hat{i} + Q\hat{j} + R\hat{k}$ by

$$\text{curl}\, F = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

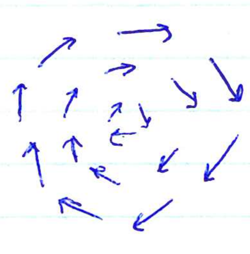
$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

Note: Unlike ∇ and div, the curl doesn't generalize so easily to higher dimensions.

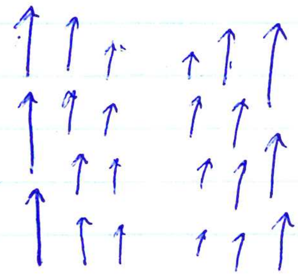
Using the above example, we can see that

$$\text{curl}\, F = \left[-x^2 z\sin(yz) - 1\right]\hat{i} + \left[-2x\cos(yz)\right]\hat{j} + (-2x)\hat{k}$$

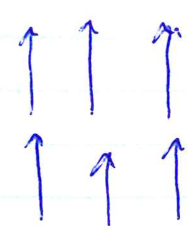
- Curl measures tendency for flow in a vector field to be circular rather than in a straight line.



$\nabla \times F \neq 0$



$\nabla \times F \neq 0$



$\nabla \times F = 0$

Definition: A vector field F is said to be irrotational in a region D if in D

$$\nabla \times F = 0.$$

- There are a number of properties on pg 987 between divergence, curl and vector field operations.

For example $\operatorname{div}(\operatorname{curl} F) = 0$. Why?

$$\operatorname{curl} F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

$\overrightarrow{\text{vector field}}$

So
$$\operatorname{div}(\operatorname{curl} F) = \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y}$$

$$= 0 \quad \text{assuming } P, Q, R \in C^2(D)$$

continuous second-order partial derivatives on D

When is a vector field a gradient?

In Math 2130 you likely saw a question similar to the following:

Find all functions $F(x,y,z)$, if there are any, such that

$$\nabla F = (2xy^3 + yze^{xyz})\hat{i} + (3x^2y^2 + xze^{xyz})\hat{j} + (xye^{xyz} + y)\hat{k}$$

Assume such a function exists. Then

$$\frac{\partial F}{\partial x} = 2xy^3 + yze^{xyz}$$

$$\Rightarrow F(x,y,z) = \frac{2x^2y^3}{2} + e^{xyz} + \underbrace{g(y,z)}_{\text{some function differentiable}}$$

- But $\frac{\partial F}{\partial y} = 3x^2y^2 + xze^{xyz} + \frac{\partial g}{\partial y}$

which by assumption should equal $3x^2y^2 + xze^{xyz}$

$$\Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z).$$

- Finally, $\frac{\partial F}{\partial z} = xye^{xyz} + \frac{\partial h}{\partial z}$

but also we need $\frac{\partial F}{\partial z} = xye^{xyz} + y$

$$\Rightarrow \frac{\partial h}{\partial z} = y$$ which contradicts the fact that h was not a function of y.

$\therefore$ While the gradient of any differentiable scalar function produces a vector field, not every vector field is the gradient of some function.

In other words, there is a map

continuously differentiable functions over D $\rightarrow$ vector fields on D

$$\Omega^0(D) \longrightarrow \Omega^1(D)$$

$$f \longmapsto \nabla f$$

which is not an onto map in general. Comparing images and kernels of such maps leads to notions like de Rham cohomology.

When is a vector field in the image of this map?

Theorem: Let $F = P\hat{i} + Q\hat{j} + R\hat{k}$ with $P, Q, R \in C^1(D)$ domain
If there exists a function $f(x,y,z)$ defined on D such that $\nabla f = F$, then $\nabla \times F = 0$.
Conversely, if D is simply connected and $\nabla \times F = 0$ in D, then there exists a function $f(x,y,z)$ such that $\nabla f = F$ in D.

- If $F = \nabla f$, then $\nabla \times \nabla f = 0$ by property 14.14
- Converse requires Stokes's theorem

Note: The converse requires that D be simply connected. The nature of the region is important for the result to be true.