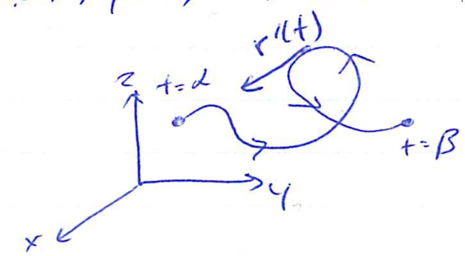
14.2 - Line Integrals

Let C be a space curve defined parametrically by

$$C: \quad x = x(t), \quad y = y(t), \quad z = z(t) \qquad \alpha \le t \le \beta.$$

Recall that C is smooth if $x(t), y(t), z(t) \in C^1([\alpha,\beta])$

- We can also describe C by



$$r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

so that $r'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$ defines a tangent vector to C at the point $r(t)$. (provided $x'(t), y'(t), z'(t)$ don't simultaneously vanish).

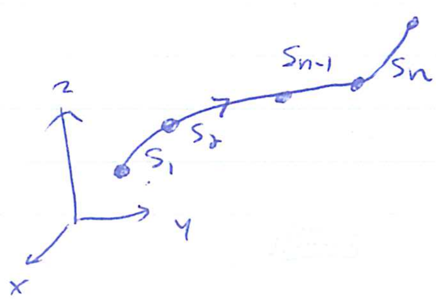
Suppose $f(x,y,z)$ is a continuous function defined over C. How do we compute the integral of f over this curve?

- This is not like the triple integrals over some region with dimension = 3.

- This is not a single variable integral over an interval (although we will change it to like one).

With the usual setup for Riemann sums, subdivide C into finitely many subcurves S_i.

- Pick a test point in each S_i, say (x_i^*, y_i^*, z_i^*)



length of S_i

- Then the sum $\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta S_i$

approximates the area underneath the graph of f. (Note C here is the domain).

Then $\int_C f(x,y,z)\,ds = \lim_{\|\Delta S_i\| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta S_i$

Line integral over C

We say line integral since C has dimension 1 (ie curved line).

- What is ds? The limit uses arclength so we know from previous classes that

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$

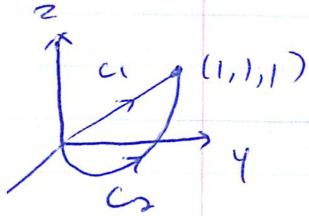
$$\therefore \int_C f(x,y,z)\,ds = \int_\alpha^\beta f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Example 1: Evaluate the line integral of $f(x,y,z) = 8x + 6xy + 30z$ from $A = (0,0,0)$ to $B = (1,1,1)$

(a) Along the straight line joining A to B

$C_1: x = t,\ y = t,\ z = t \qquad 0 \le t \le 1$

(b) Along the straight line using the parameterization



$$C_1: \quad x = 2 - 4t, \quad y = 2 - 4t, \quad z = 2 - 4t \qquad 1/4 \le t \le 1/2$$

(c) Along the curve

$$C_2: \; x = t, \quad y = t^2, \quad z = t^3, \qquad 0 \le t \le 1$$

(a) $\displaystyle\int_{C_1} (8x + 6xy + 30z)\,ds$

$$= \int_0^1 (8t + 6(t)(t) + 30t)\sqrt{(1)^2 + (1)^2 + (1)^2}\,dt$$

$$= \sqrt{3}\int_0^1 (38t + 6t^2)\,dt$$

$$= \sqrt{3}\left(19t^2 + 2t^3\right)\Big|_0^1 \qquad = 21\sqrt{3}$$

(b) $\displaystyle\int_{C_2} (8x + 6xy + 30z)\,ds$

$$= \int_{1/4}^{1/2} 8(2-4t) + 6(2-4t)^2 + 30(2-4t)\sqrt{(-4)^2 + (-4)^2 + (-4)^2}\,dt$$

$$= 4\sqrt{48}\int_{1/4}^{1/2} 19(1-2t) + 6(1-2t)^2\,dt$$

$$= 16\sqrt{3}\left(\frac{19(1-2t)^2}{-4} - (1-2t)^3\right)\Bigg|_{1/4}^{1/2}$$

$$= 16\sqrt{3}\left(0 - \left(-\tfrac{19}{4}(1/4) - 2/16\right)\right) \qquad = 16\sqrt{3}\left(\frac{21}{16}\right) = 21\sqrt{3}$$

(c) $$\int_{C_2} (8x+6xy+30z)\,ds = \int_0^1 (8t+6t^3+30t^3)\cdot \left(\sqrt{(1)^2+(2t)^2+(3t^2)^2}\right) dt$$

$$= \int_0^1 (8t+36t^3)\sqrt{1+4t^2+9t^4}\,dt$$

Let $u = 1+4t^2+9t^4$
$du = 8t+36t^3$

$$= \int_1^{14} u^{1/2}\,du$$

$$= \frac{2u^{3/2}}{3}\Big|_1^{14}$$

$$= \tfrac{2}{3}(14\sqrt{14}-1)$$

We notice that
- Line integrals seem to be independent of parameterization
- They do depend on the path between two points in general.

Example 2: Show that line integrals over C are independent of parameterization.

Given a parameterization $r(t)$, there is always (locally) a $t=\varphi(u)$ so that $g(u)=r(\varphi(u))$ is a parameterization by arclength.

Let C_r and C_g be the parameterized curves.

Then $$\int_{C_r} f(x,y,z)\,ds = \int_a^b f(r(t))\cdot|r'(t)|\,dt$$

Let $t=\varphi(u)$
$dt = \varphi'(u)\,du$

By substitution

$$\stackrel{\downarrow}{=} \int_c^d f(r(\varphi(u))) \cdot |r'(\varphi(u))| \varphi'(u)\, du$$

$\leftarrow$ assuming increasing

$$= \int_c^d f(g(u)) \cdot |r'(\varphi(u)) \varphi'(u)|\, du$$

but $\quad g'(u) = r'(\varphi(u)) \cdot \varphi'(u)$

$$= \int_c^d f(g(u))\, |g'(u)|\, du \quad = \int_{C_g} f\, ds$$

Example 3: Evaluate the integral of $f(x,y) = x^2+y^2$ once clockwise around the circle $x^2+y^2=4$.

By example 2, we can use any parameterization (preserving direction), so we use

$$C: \; x = -2\sin(3t), \quad y = 2\cos(3t), \quad z=0$$

$$\frac{2\pi}{3} \le t \le \frac{4\pi}{3}$$

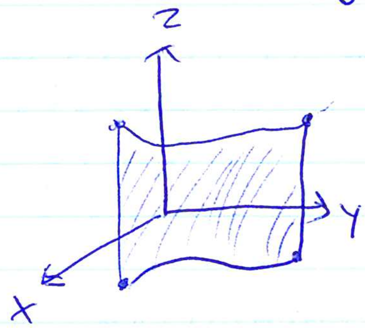
Then $\displaystyle\int_C (x^2+y^2)\, ds = \int_{\frac{2\pi}{3}}^{4\pi/3} (4) \sqrt{(-6\cos(3t))^2 + (-6\sin(3t))^2}\, dt$

$$= 24 \int_{2\pi/3}^{4\pi/3} dt$$

$$= \frac{2\pi}{3}(24) = 16\pi$$

Note: The amount of times the curve winds around matters

- When C is contained in a plane, then we can visualize $\int_C ds$ as the area of a "wall":

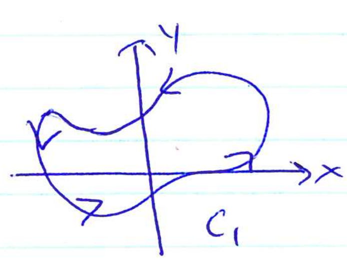


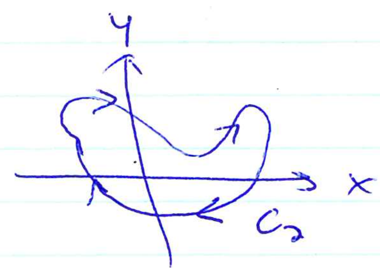
- Length of curve $C = \int_C ds$.

- When C is a closed curve, we use $\oint_C f(x,y,z)\,ds$

For example, $x^2+y^2=z$ intersected with $x+z=1$ is a closed curve.

- When C is on the xy-plane, we can use $\circlearrowleft$ or $\circlearrowright$ to indicate direction.





$\oint_{C_1} f(x,y)\,ds$ $\qquad$ $\oint_{C_2} f(x,y)\,ds$