18.2 Fourier Sine and Cosine Series

If f is an even function, and can be represented by a Fourier series, then

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) = f(x)$$

$$= f(-x)$$

$$= \frac{a_0}{2} + \sum_{n=0}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) - b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

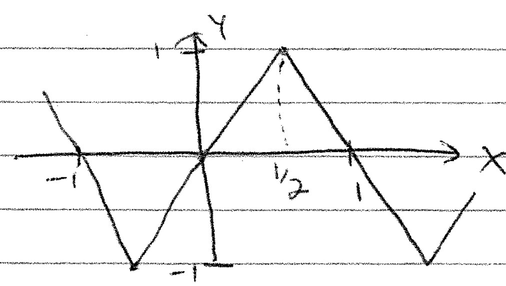
Since cos is even and sin odd.

$$\Rightarrow \sum_{n=1}^{\infty} 2b_n \sin\left(\frac{n\pi x}{L}\right) = 0 \quad \Rightarrow \quad b_n = 0 \quad n \geq 1.$$

Similarly, all of the a_n are zero when $f(x)$ is odd.

- In the case that all $b_n = 0$, we have Fourier cosine series
- When all $a_n = 0$, we have Fourier sine series.

Example 1: Find the Fourier series for the function $f(x)$ whose graph is



By looking at the interval $[-1, 1]$, we see that f is odd with period 2.

Then $2L = 2 \Rightarrow L = 1$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\left(\frac{n\pi x}{L}\right)}_{\text{The product is an even function.}} dx = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$= 2\int_0^{1/2} 2x\sin(n\pi x)\,dx + 2\int_{1/2}^{1} -2(x-1)\sin(n\pi x)\,dx$$

$$= 4\left(-\frac{\cos(n\pi x)}{n\pi}x + \frac{\sin(n\pi x)}{n^2\pi^2}\right)\Bigg|_0^{1/2} - 4\left(\frac{-(x-1)}{n\pi}\cos(n\pi x) + \frac{\sin(n\pi x)}{n^2\pi^2}\right)\Bigg|_{1/2}^{1}$$

using integration by parts

$$= \frac{-2\cos(\frac{n\pi}{2})}{n\pi} + \frac{4\sin\frac{n\pi}{2}}{n^2\pi^2} - 0 - \frac{4}{n^2\pi^2}\overset{0}{\sin(0)} + \frac{4\sin(\frac{n\pi}{2})}{n^2\pi^2} + \frac{2\cos(\frac{n\pi}{2})}{n\pi}$$

$$= \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^k & \text{if } n = 2k+1 \end{cases}$$

- $f(x)$ is continuous $\Rightarrow$ Fourier series converges to $f(x)$ for all $x \in \mathbb{R}$.

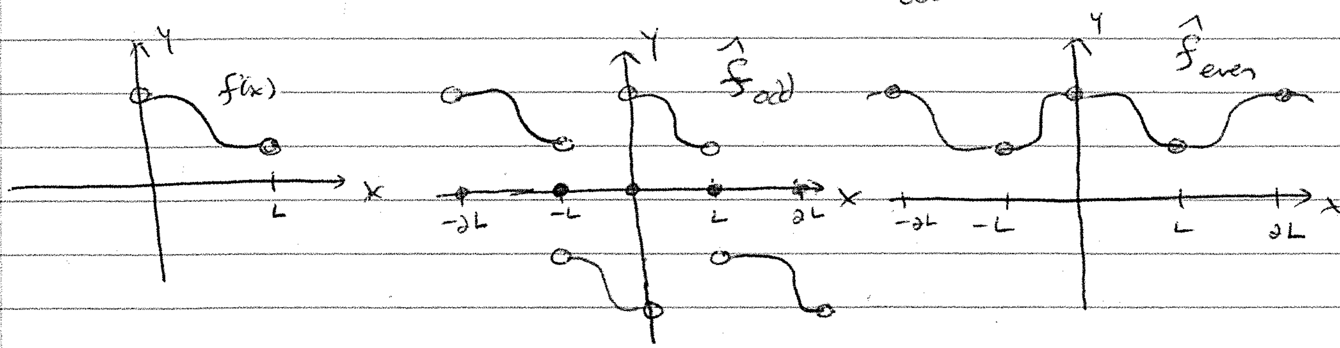
Then $f(x) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)\sin(n\pi x)$ ← Fourier sine series since f is odd

$$= \frac{8}{\pi^2}\sum_{k=0}^{\infty} \frac{(-1)^k \sin((2k+1)\pi x)}{(2k+1)^2}$$

So far we have viewed cosine and sine Fourier series as special cases of the full Fourier series for periodic functions.

Question: Can we represent any piecewise-smooth function $f(x)$ defined on $(0, L)$ using a Fourier series?

To compute Fourier series, we need a periodic function. Given $f(x)$, we can define $\hat{f}_{even}$ and $\hat{f}_{odd}$ as



where f is extended to be an odd function with $\hat{f}_{odd}$ and an even one with $\hat{f}_{even}$.

- Notice that f piecewise smooth $\Rightarrow$ we should define $\hat{f}_{even}(0) = \frac{f(0+) + f(0-)}{2} = f(0+)$

and similarly for $\hat{f}_{even}(L)$ so that $\hat{f}_{even}$ is continuous

- Notice that for $\hat{f}_{odd}$, we should define $\hat{f}_{odd}(0) = \dfrac{f(0+)+f(0-)}{2} = 0$

and similarly for kL where $k \in \mathbb{Z}$

Both $\hat{f}_{odd}$ and $\hat{f}_{even}$ are piecewise smooth and periodic, and the Fourier series for each converges to $\hat{f}_{odd}$ and $\hat{f}_{even}$ respectively.

Then $\hat{f}_{odd}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ and $\hat{f}_{even}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$

which are each a Fourier series expansion for $f(x)$ on $0 < x < L$.

Note: we could have simply defined $\hat{f}_{per}$ as the function with period L defined as $f(x)$ on $(0,L)$ and $\hat{f}_{per}(x+L) = \hat{f}_{per}(x)$ for all other x. We could find a Fourier series for this. However, $\hat{f}_{per}$ is neither odd nor even in general, so we won't always have all $a_n = 0$ or $b_n = 0$. The above technique simplifies computations

Example 2: Find a Fourier series representation for $1+2x$ on the interval $(0,3)$.

Using sine series: Extend $f(x) = 1+2x$ via $\hat{f}_{odd}$, with period $2L = 6$ (with $L = 3$)

Then $\hat{f}_{odd}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right)$ where

$$b_n = \frac{1}{3}\int_{-3}^{3} \underbrace{\hat{f}_{odd}(x)\sin\frac{n\pi x}{3}}_{\text{even function}}\,dx$$

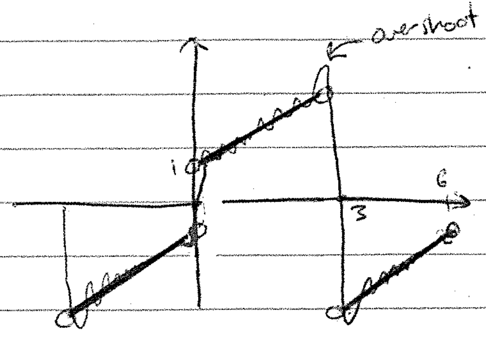
$$= \frac{2}{3}\int_0^3 (1+2x)\sin\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[-(1+2x)\cos(n\pi x/3)\,\frac{3}{n\pi} + 2\sin\left(\frac{n\pi x}{3}\right)\frac{3}{n\pi}\right]\Bigg|_0^3$$

$$= \frac{2}{n\pi}\left[-7(-1)^{n+1} + (1+2(0))\cos(0)\right] = \frac{2}{n\pi}\left[1+7(-1)^{n+1}\right]$$

So $1+2x = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1+7(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{3}\right)$ for $0<x<3$

- By previous note, we know that the series converges to 0 at $x=0$ and $x=3$ (not to $1+2x$).



- Approximation of $\hat{f}_{odd}$ using first 20 terms
- Every partial sum crosses $x=0$ and $x=3$
- The "overshoot" near the discontinuities is always there, no matter how many terms in the partial sum. Known as Gibbs phenomenon.

- For large n, the partial sum overshoots the curve at a discontinuity by about 9% of the size of the jump.

Using cosine series: Extend $f(x)$ to an even function $\hat{f}_{even}$.

Then $\hat{f}_{even}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right)$ where

$$a_0 = \frac{1}{3}\int_{-3}^{3} \hat{f}_{even}\, dx$$

$$= \tfrac{2}{3}\int_0^3 (1+2x)\,dx = \tfrac{2}{3}(x+x^2)\Big|_0^3 = 8$$

$$a_n = \tfrac{2}{3}\int_0^3 (1+2x)\cos\left(\frac{n\pi x}{3}\right)dx$$

$$= \tfrac{2}{3}\cos\left(\frac{n\pi x}{3}\right)\left(\frac{3}{n\pi}\right)^2 \cdot 2\Big|_0^3$$ using integration by parts

$$= \frac{12}{n^2\pi^2}\left[(-1)^n - 1\right]$$

Actually we can use $0 \le x \le 3$ by construction of $\hat{f}_{even}$.

Then $1+2x = 4 + \frac{12}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}\cos\left(\frac{n\pi x}{3}\right)$, $0<x<3$

For n even, $(-1)^n - 1 = 0$, so if $n = 2k+1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}\cos\left(\frac{n\pi x}{3}\right) = \sum_{k=0}^{\infty} \frac{-2}{(2k+1)^2}\cos\left(\frac{(2k+1)\pi x}{3}\right)$$