19.1 Eigenvalues and Eigenfunctions

A Sturm-Liouville system consists of the information

$$(*)\quad \begin{gathered} \frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + \left[\lambda p(x) - q(x)\right] y = 0 \qquad a<x<b \\ -l_1 y'(a) + h_1 y(a) = 0 \\ l_2 y'(b) + h_2 y(b) = 0 \end{gathered}$$

- The last two equations are linear homogeneous boundary conditions
- $h_1, h_2, l_1, l_2 \in \mathbb{R}$ and independent of the parameter λ.
- The negative signs are simply a matter of convenience.

Think of the system as a family of differential equations dependent on the parameter λ. That is, for each λ, you get some differential equation which may or may not have solutions.

- $y \equiv 0$ is always a solution
- For certain λ, the system has nontrivial solutions. These λ are called eigenvalues, and there are always ∞ many (denoted by λ_n for $n=1,2,3,\ldots$
- A solution of the system corresponding to λ_n is denoted by $y_n(x)$ and is called an eigenfunction

- When $\lambda=0$ is an eigenvalue, we write $\lambda_0 = 0$ with eigenfunction $y_0(x)$.

Theorem: All eigenvalues of Sturm-Liouville system $(*)$ are real, and eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function $p(x)$,

$$\int_a^b p(x)\, y_n(x)\, y_m(x)\, dx = 0$$

This is similar to the language from linear algebra. There, we are given an $n \times n$ matrix A and asked when a vector v satisfies

$$Av = \lambda v$$

(or when does a linear transformation map a vector to a scalar multiple of itself).

$$\Rightarrow Av - \lambda v = 0$$
$$\Rightarrow (A - \lambda I_n)v = 0$$
$$\Rightarrow v \in Ker(A - \lambda I_n)$$

λ is called an eigenvalue and v an eigenvector.

- The eigenvectors corresponding to a fixed λ define an eigen space V_λ.

- Eigenvector from distinct λ_i are linearly independent

- The linear transformation in general can be $\frac{d}{dx}$ applied to the vector space $C^1([a,b])$.

Vectors are functions, and we could have

$$\frac{d}{dx} e^{\lambda x} = \lambda e^{\lambda x}$$

so $e^{\lambda x}$ is an eigen vector (or an eigenfunction) corresponding to λ.

Review of solutions to homogeneous linear ODEs with constant coefficients.

- Since the differential equation is linear, we can write

$$\phi(D)y = F(x)$$

where $\phi(D) = (a_n D^{(n)} + \dots + a_1 D^{(1)} + a_0)$

We call $\phi(m)$ the auxiliary equation.

Theorem: (i) If $\phi(m) = 0$ has a real root m of multiplicity K, then a solution to the differential equation $\phi(D)y = 0$ is

$$(C_1 + C_2 x + \dots + C_k x^{k-1})e^{mx};$$

(ii) If $\phi(m) = 0$ has a pair of complex conjugate roots $a \pm bi$ each of multiplicity k, then a solution is

$$e^{ax}\left[(C_1 + C_2 x + \dots + C_k x^{k-1})\cos bx + (D_1 + D_2 x + \dots + D_k x^{k-1})\sin bx\right].$$

- To find a general solution, superpose all solutions from (i) and (ii)

For example, if $\phi(m) = (m-1)^2(m^2+1)$, then a general solution to the associated ODE is

$$y(x) = (C_1 + C_2 x)e^x + C_3 \cos x + C_4 \sin x$$

where $m = 1$ or $m = \pm i = \overset{a}{0} \pm \overset{b}{1} i$

19.2 Special Sturm-Liouville Systems

For computations needed in chapter 21, we restrict to the special case of $r(x)=1$, $p(x)=1$ and $q(x)=0$.

The differential equation we will be considering is

$$y'' + \lambda y = 0 \qquad 0<x<L$$

with 4 cases of boundary conditions. In each case, we need to consider subcases of $\lambda<0$, $\lambda=0$, $\lambda>0$.

Cases: (1) $y(0)=y(L)=0$, (2) $y'(0)=y'(L)=0$,
(3) $y(0)=y'(L)=0$, (4) $y'(0)=y(L)=0$.

$\lambda<0$: We can write $\lambda=-\alpha^2$ where $\alpha>0$

$y''-\alpha^2 y=0$ has $m^2-\alpha^2=0$ as its auxiliary equation

There are two roots, $m=\pm\alpha$, so the general solution is

$$y(x)=C_1e^{\alpha x}+C_2e^{-\alpha x}$$

The boundary conditions in each case require

(1) $\underbrace{C_1+C_2=0}_{C_1=-C_2}$, $\underbrace{C_1e^{\alpha L}+C_2e^{-\alpha L}=0}_{C_2\underbrace{(e^{-\alpha L}-e^{\alpha L})}_{\neq 0}=0}$ $\Rightarrow C_1=C_2=0$.

So there are no nontrivial solutions in this case, so λ cannot be an eigenvalue when $\lambda<0$.

(2) $\alpha C_1-\alpha C_2=0$, $C_1\alpha e^{\alpha L}-C_2\alpha e^{-\alpha L}$ $\Rightarrow C_1=C_2=0$
$\Rightarrow C_1=C_2$, $C_1\alpha(e^{\alpha L}-e^{-\alpha L})$

Similarly for (3) and (4) $\therefore$ $\lambda<0$ cannot be an eigenvalue.

$\lambda = 0$: The auxiliary equation for $y'' = 0$ is $m^2 = 0$, so there is a double root.

Then $y(x) = (C_1 + C_2 x)e^{0x} = C_1 + C_2 x$ ← we could have just used the Fundamental Theorem of Calculus to find this.

The boundary conditions imply:

(1) $C_1 + 0 = 0$, $C_1 + C_2 L = 0$ $\Rightarrow$ $C_2 = 0 = C_1$

(2) $C_2 = 0$, $C_2 = y'(L) = 0$ $\Rightarrow$ $C_2 = 0$

C_1 is free, so $y(x) = C_1$ is a nontrivial solution for $C_1 \neq 0$, and $\lambda = 0$ is an eigenvalue.

(3) $C_1 = 0$, $C_2 = y'(L) = 0$ $\Rightarrow$ $C_1 = C_2 = 0$

(4) $C_1 + C_2 L = 0$, $C_2 = 0$ $\Rightarrow$ $C_1 = C_2 = 0$.

$\lambda > 0$: The auxiliary equation for $y'' + \alpha^2 y = 0$ has $m^2 + \alpha^2 = 0$ where $\lambda = \alpha^2$, $\alpha > 0$.

This has two complex roots, $\pm \alpha i$, and a general solution is:

$$y(x) = \underbrace{e^{0x}}_{1}\left[C_1 \cos \alpha x + C_2 \sin \alpha x\right]$$

The boundary conditions imply

(1) $0 = y(0) = C_1$, $0 = y(L) = C_1 \cos \alpha L + C_2 \sin \alpha L$

$\Rightarrow$ $C_2 \sin \alpha L = 0$.

If $C_2 = 0$, then we couldn't have a nontrivial solution. Instead, observe that $\sin \alpha L = 0$ if

$$\alpha L = n\pi \quad \text{for} \quad n \in \mathbb{Z}$$

$$\Rightarrow \alpha = \frac{n\pi}{L} \Rightarrow \lambda = \alpha^2 = \frac{n^2\pi^2}{L^2}$$

Then $\lambda_n = \frac{n^2\pi^2}{L^2}$ for $n > 0$ are eigen values

corresponding to eigenfunctions $y_n(x) = C_2 \sin\left(\frac{n\pi x}{L}\right)$

- We call $[\lambda_n, y_n(x)]$ an eigenpair, and it is clear that $[\lambda_n, Cy_n(x)]$ is also an eigenpair for $C \neq 0$ (The equations in the system are homogeneous.

- It is customary to omit the constant C_2 from $y_n(x)$ since it is understood that $y_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ is not unique, and any nonzero scaler multiple is also a solution.

(2) $y'(x) = -\alpha C_1 \sin\alpha x + \alpha C_2 \cos\alpha x$

Then $0 = y'(0) = \alpha C_2 \Rightarrow C_2 = 0$ since $\alpha > 0$

$0 = y'(L) = -\alpha C_1 \sin\alpha L + \alpha C_2 \cos\alpha L$

$\Rightarrow \alpha C_1 \sin\alpha L$

To get nontrivial solutions, we need $\alpha L = n\pi$ $n \in \mathbb{Z}$

Then $\lambda_n = \alpha^2 = \frac{n^2\pi^2}{L}$ $n > 0$, $y_n(x) = \cos\left(\frac{n\pi x}{L}\right)$

are eigenpairs for this system

(3) $0 = y(0) = C_1$

$0 = y'(L) = -\alpha C_1 \sin(\alpha L) + \alpha C_2 \cos\alpha L$

$\Rightarrow \alpha C_2 \cos\alpha L = 0$

For there to be nontrivial solutions, we need

$\alpha L = \pi/2 - n\pi = \frac{\pi(1-2n)}{2}$ $n \in \mathbb{Z}$

use − so that index can start at $n = 1$.

$\Rightarrow \alpha = \frac{\pi(1-2n)}{2L}$

$\Rightarrow \lambda = \alpha^2 = \frac{(2n-1)^2\pi^2}{4L^2}$

Then $\lambda_n = \dfrac{(2n-1)^2\pi^2}{4L^2}$, $n \geq 1$, $y_n(x) = \sin\left(\dfrac{(2n-1)\pi x}{2L}\right)$

are eigenpairs for each n for this system.

(4) $0 = y'(0) = \alpha c_2 \Rightarrow c_2 = 0$

$0 = y(L) = c_1 \cos\alpha L + c_2 \sin\alpha L$

$\Rightarrow \cos\alpha L = 0$

$\Rightarrow \alpha L = \dfrac{\pi}{2} - n\pi \qquad n \in \mathbb{Z}$

Then $\lambda_n = \dfrac{(2n-1)^2\pi^2}{4L^2}$, $n \geq 1$ $y_n(x) = \cos\left(\dfrac{(2n-1)\pi x}{2L}\right)$

In summary:

Boundary Conditions	Eigenvalues	Eigenfunctions
$y(0) = y(L) = 0$	$\dfrac{n^2\pi^2}{L^2}$, $n \geq 1$	$y_n(x) = \sin\dfrac{n\pi x}{L}$
$y'(0) = y'(L) = 0$	$\dfrac{n^2\pi^2}{L^2}$, $\underbrace{n \geq 0}_{*}$	$y_n(x) = \cos\dfrac{n\pi x}{L}$
$y(0) = y'(L) = 0$	$\dfrac{(2n-1)^2\pi^2}{4L^2}$, $n \geq 1$	$y_n(x) = \sin\dfrac{(2n-1)\pi x}{2L}$
$y'(0) = y(L) = 0$	$\dfrac{(2n-1)^2\pi^2}{4L^2}$, $n \geq 1$	$y_n(x) = \cos\dfrac{(2n-1)\pi x}{2L}$

* This includes the $\lambda_0 = 0$ eigenvalue case where $y_0(x) = 1$

- Since each $y_n(x)$ is a function coming from a Fourier series, by 18.2, we can expand any piecewise smooth function on $0 < x < L$ in terms of eigenfunctions of a Sturm-Liouville system.

- Is this true of any Sturm-Liouville

Theorem: • $p, q, r, r', (pr)'' \in C^0([a,b])$ with $p, r > 0$ on $[a,b]$
• $l_1, l_2, h_1, h_2 \in \mathbb{R}$ independent of λ.

Then (1) The Sturm-Liouville system ∞ many eigenvalues $\lambda_1 < \lambda_2 < \dots$, all real, with only finitely many which are negative, and $\lim_{n\to\infty} \lambda_n = \infty$.

(2) If f is piecewise smooth on $[a,b]$, then on (a,b)

$$\frac{f(x+) + f(x-)}{2} = \sum_{n=1}^{\infty} c_n y_n(x)$$

where $c_n = \frac{1}{F_n} \int_a^b p(x) f(x) y_n(x)\, dx$

and $F_n = \int_a^b p(x) [f(x)]^2 dx$.

• In otherwords, we can expand any such f with a basis of functions for Sturm-Liouville systems, called eigenfunction expansions.

• We can also write $c_n = \frac{\langle f, y_n \rangle}{|f|^2}$ where $\langle g(x), h(x) \rangle = \int_a^b p(x) g(x) h(x)\, dx$

Example: Expand $f(x) = 2x - 1$, $0 \le x \le 4$ in terms of eigenfunctions of the Sturm-Liouville system

$$y'' + \lambda y = 0, \quad 0 < x < 4$$
$$y'(0) = 0 = y(4)$$

$L = 4$, and by the table $y_n(x) = \frac{\cos(2n-1)\pi x}{8}$ $n \ge 1$

Here

$$c_n = \frac{2}{\underset{L}{4}}\int_0^4 (2x-1)\,y_n(x)\,dx$$

$$= \frac{1}{2}\left\{ (2x-1)\sin\left(\frac{(2n-1)\pi x}{8}\right)\frac{8}{(2n-1)\pi}\Big|_0^4 - \int_0^4 2\sin\left(\frac{(2n-1)\pi x}{8}\right)\frac{8}{(2n-1)\pi}\right]$$

$$= \frac{4}{(2n-1)\pi}\Big[7\underbrace{\sin\left(\frac{(2n-1)\pi}{2}\right)}_{(-1)^{n+1}}\Big] - \frac{8}{(2n-1)\pi}\underbrace{\left(-\cos\left(\frac{(2n-1)\pi x}{8}\right)\frac{8}{(2n-1)\pi}\right)\Big|_0^4}_{\frac{8}{(2n-1)\pi}}$$

$$= \frac{4}{\pi^2(2n-1)^2}\left[7(-1)^{n+1}(2n-1)\pi - 16\right]$$

$$\Rightarrow \quad 2x-1 = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[7(2n-1)\pi(-1)^{n+1} - 16\right]}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi x}{8}\right)$$

on $0<x<4$