18.1 Fourier Series

We have previously discussed when a function $f(x)$ can be described as a sum

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

What if we wanted infinite sums using other functions? For example, a Taylor series for cyclic function can converge very slowly.

- $e^x = 1 + x + \frac{x^2}{2!} + \dots$ so $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$

  To get 2 decimal places, we simply need 6 terms ($2.71\overline{6}$)

- $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4} - \frac{x^6}{6!} + \dots$, we know $\cos(0) = \cos(2\pi) = 1$.

  but substituting 2π above doesn't yield a quickly convergent sequence

To work more generally, let V be a vector space over $\mathbb{R}$. An inner product on V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ (also called a scalar product), where for $x, y, z \in V$, $a \in \mathbb{R}$:

- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle ax, y \rangle = a \langle x, y \rangle$ and $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle x, x \rangle > 0$, $x \in V \setminus \{0\}$

This gives a notion of length of a vector.

- For example, if $V = \mathbb{R}^n$, we can take $\langle u, v \rangle$ to be the usual dot product between vectors.

- In $\mathbb{R}^3$ we can write $V = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$

  and $V_x = V \cdot \hat{i}$, $V_y = V \cdot \hat{j}$, $V_z = V \cdot \hat{k}$.

- Here $\hat{i}, \hat{j}, \hat{k}$ form an orthonormal basis for $\mathbb{R}^3$. That is, they are mutually orthogonal (so $\hat{i}\cdot\hat{j} = 0$ etc). and they have length 1 ($\hat{i}\cdot\hat{i} = 1$).

- If we used a different basis v_1, v_2, v_3 (such as $\hat{i}+\hat{j}$, $\hat{i}-\hat{j}$, $3\hat{k}$ in the text), then we can write

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3$$

Then $$v \cdot v_1 = a_1 v_1 \cdot v_1 + a_2 v_2 \cdot v_1 + a_3 v_3 \cdot v_1$$

We want to isolate a_1, so we hope that $v_2 \cdot v_1 = v_3 \cdot v_1 = 0$ (otherwise we get a system of equations). Assuming this, we get

$$v \cdot v_1 = a_1 v_1 \cdot v_1$$
$$\Rightarrow \quad a_1 = \frac{v \cdot v_1}{|v_1|^2}$$

If v_1 has length 1, then $|v_1| = 1$ and we have $a_1 = v \cdot v_1$.

Definition: Given an inner product $\langle \cdot, \cdot \rangle$ on V, $u, v \in V$ are orthogonal if $\langle u, v \rangle = 0$ and orthonormal if u, v are orthogonal and $\langle u, u \rangle = \langle v, v \rangle = 1$.

Example 1: Let V be the vector space of continuous functions on $[a,b]$ (that is, $V = C^0([a,b])$).

check the inner product conditions → Define $$\langle f, g \rangle = \int_a^b w(x) f(x) g(x)\, dx$$

where $w(x) \geq 0$ on $[a,b]$. This is a sort of generalization of the dot product where we "sum" over infinitely many components

- f, g are orthogonal on $[a,b]$ with respect to $w(x)$ if $\int_a^b w(x) f(x) g(x)\, dx = 0$.

- A sequence of nonzero functions $\{f_n(x)\}$ is orthogonal on $[a,b]$ w.r.t $w(x)$ if for every pair

$$\langle f_n, f_m \rangle = \int_a^b w(x) f_n(x) f_m(x)\, dx = 0 \quad \text{when } n \neq m.$$

- If $f_n(x) = \sin(nx)$, $w(x) = 1$ and $0 \leq x \leq 2\pi$, then

$$\int_0^{2\pi} \sin(nx)\sin(mx)\, dx = \int_0^{2\pi} \tfrac{1}{2}\left[\cos(n-m)x - \cos(n+m)x\right] dx$$

(using the identity $\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$)

$$= \frac{1}{2}\left(\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m} \right)\Bigg|_0^{2\pi}$$

$$= 0$$

- The same is true on $[0,\pi]$.

Question: When is a function $f(x)$ equal to the infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) \; ?$$

- The notation here is a matter of convention.
- These functions have period $\dfrac{2\pi}{n\pi/L} = \dfrac{2L}{n}$, so $f(x)$ better be a periodic function with period $2L$.

Theorem: $\{1, \cos(n\pi x/L), \sin(n\pi x/L)\}$ for $n = 1, 2, 3, \ldots$ are a set of orthogonal functions on $0 \leq x \leq 2L$ with respect to $w(x) = 1$
(or, they are orthogonal w.r.t the inner product

$$\langle f, g \rangle = \int_0^{2L} f(x) g(x)\, dx$$

How do we find the coefficients?

Forgetting about convergence issues for now, notice that

$$\langle f(x), h(x)\rangle = \left\langle \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right), h(x)\right\rangle$$

$$= \frac{a_0}{2}\langle 1, h(x)\rangle + \sum_{n=1}^{\infty} a_n\left\langle \cos\left(\frac{n\pi x}{L}\right), h(x)\right\rangle + b_n\left\langle \sin\left(\frac{n\pi x}{L}\right), h(x)\right\rangle$$

by linearity. Now choose $h(x)$ as one of the vectors from our orthogonal set.

- $h(x)=1$, then $\left\langle \cos\left(\frac{n\pi x}{L}\right), 1\right\rangle = \left\langle \sin\left(\frac{n\pi x}{L}\right), 1\right\rangle = 0$ and

$$\int_0^{2L} f(x)\,dx = \int_0^{2L} \frac{a_0}{2}\,dx = \frac{a_0(2L)}{2}$$

$$\Rightarrow \quad a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx \quad \text{or the average value of } f(x) \text{ on } [0, 2L]$$

- $h(x) = \cos(k\pi x/L)$ means all terms are 0 except

$$\langle f, h\rangle = a_k\left\langle \cos\left(\frac{k\pi x}{L}\right), \cos\left(\frac{k\pi x}{L}\right)\right\rangle$$

It is not hard to see that $\int_0^{2L} \cos^2\left(\frac{k\pi x}{L}\right)dx = L$

$$\Rightarrow \quad a_k = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{k\pi x}{L}\right)dx \qquad k > 0$$

- If $h(x) = \sin(k\pi x/L)$ we get

$$b_k = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

This expression for $f(x)$ is called the Fourier series for f, and the coefficients a_k, b_k, a_0 are called the Fourier coefficients.