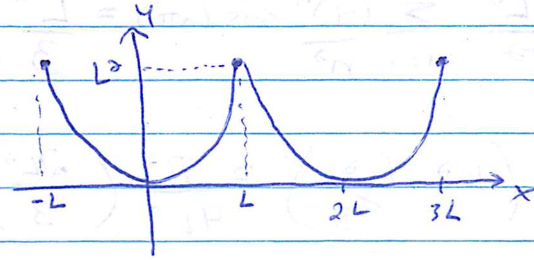
Example 2: Find the Fourier series of the function $f(x)$ that is equal to x^2 for $-L \leq x \leq L$ and is of period $2L$.



- Using $[-L, L]$ instead of $[0, 2L]$ works as before.

$$a_0 = \frac{1}{L}\int_{-L}^{L} x^2\,dx = \frac{1}{L}\left.\frac{x^3}{3}\right|_{-L}^{L} = \frac{1}{L}\left(\frac{2L^3}{3}\right) = \frac{2L^2}{3}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} x^2 \cos\left(\frac{n\pi x}{L}\right)dx$$ using integration by parts

$$= \frac{1}{L}\left. x^2 \sin\left(\frac{n\pi x}{L}\right)\cdot\frac{L}{n\pi}\right|_{-L}^{L} - \frac{1}{L}\int_{-L}^{L} 2x\sin\left(\frac{n\pi x}{L}\right)\cdot\frac{L}{n\pi}\,dx$$

$$= 0 - \frac{2}{n\pi}\left[\left.-x\cos\left(\frac{n\pi x}{L}\right)\frac{L}{n\pi}\right|_{-L}^{L} + \int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\frac{L}{n\pi}\,dx\right]$$

$$= \frac{2L}{n^2\pi^2}\underbrace{\left.\left(x\cos\left(\frac{n\pi x}{L}\right)\right)\right|_{-L}^{L}}_{2L(-1)^n} - \frac{2L}{n^2\pi^2}\underbrace{\left.\left(\sin\left(\frac{n\pi x}{L}\right)\frac{L}{n\pi}\right)\right|_{-L}^{L}}_{0}$$

$$= \frac{4L^2(-1)^n}{n^2\pi^2}$$

(The text uses a reduction formula for $\int x^n \cos ax\,dx$)

Similarly, we can compute to see that $b_n = 0$, $n > 0$. We can also observe that $f(x)$ is an even function, so $b_n = 0$ is required (left as an exercise).

use the expression for a_n as an integral

- We will see that since f is continuous, we can write

$$f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{L}\right)$$

without any issues of convergence.

As a bonus, notice that when $x = L$, $f(L) = L^2$, and

$$L^2 = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi) = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

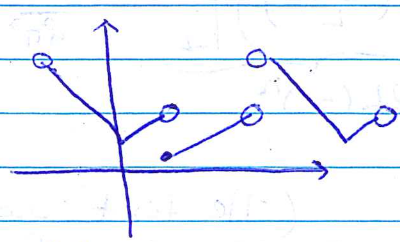
Then $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{4L^2}\left(L^2 - \frac{L^2}{3}\right) = \frac{\pi^2}{4L^2}\left(\frac{2L^2}{3}\right) = \frac{\pi^2}{6}$$

Now $\pi^2 = 6 \sum_{n=1}^{\infty} \frac{1}{n^2}$, which gives another expression for π.

To discuss convergence, we need a few definitions and theorems.

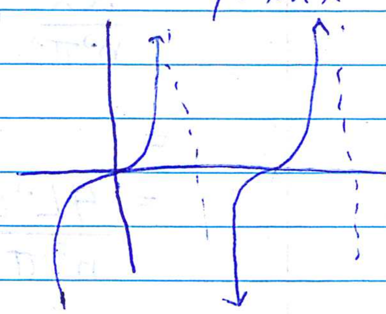
Definition: A function $f(x)$ is piecewise continuous on an interval $[a,b]$ if there exist finitely many points c_i where $a \le c_1 < c_2 < \ldots < c_k \le b$ where f is continuous on each (c_{i-1}, c_i) and $\lim_{x \to c_i^+} f$ and $\lim_{x \to c_i^-} f$ are finite.

For example



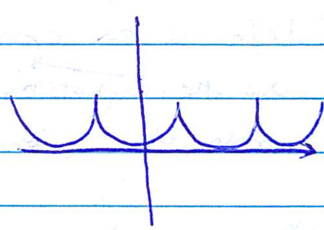
piecewise continuous

$y = \tan x$



Not piecewise continuous

• Piecewise smooth on $[a,b]$ = $f(x)$ and $f'(x)$ both piecewise continuous



is not piecewise smooth since $f'(x)$ will go off to infinity near the cusps.

Theorem: The Fourier series of a periodic, piecewise continuous function $f(x)$ converges to $[f(x+) + f(x-)]/2$ at any point at which $f(x)$ has both a left and right derivative.

Here $\lim_{\varepsilon \to 0^+} f(x+\varepsilon) = f(x+)$ (right-hand limit)

and similarly for $f(x-)$

Corollary: When $f(x)$ is a periodic, piecewise smooth function, its Fourier series converges to $\frac{f(x+) + f(x-)}{2}$

We can therefore write

$$\frac{f(x+) + f(x-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

When $f(x)$ is continuous at $x=a$, $\lim_{x\to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$

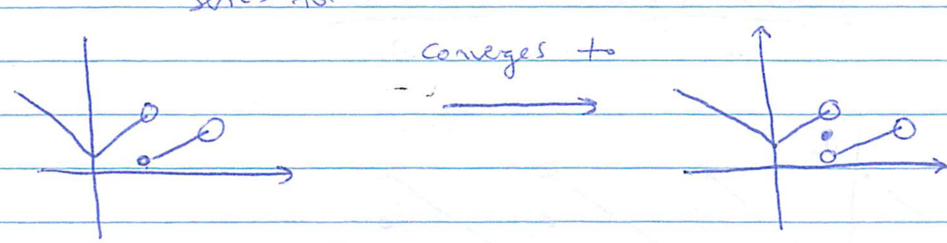
$$\text{so} \quad \frac{f(a+) + f(a-)}{2} = \frac{2f(a)}{2} = f(a)$$

- There is no need to use interval $[0, 2L]$. We can use any interval of length $2L$.

- Use the convention that $f(x)$ shall be defined by the average of its right- and left-hand limits. Then

$f(x) = \frac{f(x+) + f(x-)}{2}$ at all points and the Fourier series converges to $f(x)$ at all points.

Fourier series for



Example 3: Find the Fourier series of the function $f(x)$ that is equal to x for $0 < x < 2L$ and is $2L$-periodic (ie, $f(x+2L) = f(x)$). What does the series converge to?

Fourier coefficients:

$$a_0 = \frac{1}{L}\int_0^{2L} x\,dx = \frac{1}{L}\left\{\frac{x^2}{2}\right\}\Big|_0^{2L} = \frac{4L^2}{2L} = 2L$$

$$a_n = \frac{1}{L}\int_0^{2L} x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left(x\sin\left(\frac{n\pi x}{L}\right)\frac{L}{n\pi}\Big|_0^{2L} - \int_0^{2L}\sin\left(\frac{n\pi x}{L}\right)\frac{L}{n\pi}\,dx\right)$$

$$= 0 + \frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\frac{L}{n\pi}\Big|_0^{2L} = 0 \qquad n > 0$$

$$b_n = \frac{1}{L}\int_0^{2L} x\sin\left(\frac{n\pi x}{L}\right)dx = \frac{-1}{L}x\cos\left(\frac{n\pi x}{L}\right)\frac{L}{n\pi}\Big|_0^{2L} - 0$$

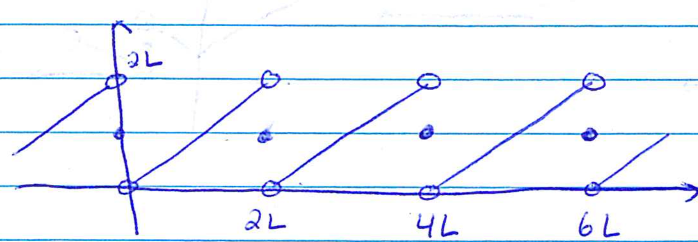
$$= \frac{-2L}{n\pi}\cos(2n\pi) = \frac{-2L}{n\pi}$$

$$\therefore f(x) = L + \sum_{n=1}^{\infty} -\frac{2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right) = L\left(1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right)\right)$$

as long as $f(x)$ is defined as $\frac{f(x+)-f(x-)}{2}$ at points of discontinuity.

Since $\lim_{x\to 0^-} f(x) = 2L$ and $\lim_{x\to 0^+} f(x) = 0$

then $\frac{f(x+)+f(x-)}{2} = \frac{2L+0}{2} = L$



• Note, for example 2, since $f(x)$ was piecewise-smooth with no discontinuities, the Fourier series for $f(x)$ converges to $f(x)$ everywhere (no need to redefine $f(x)$ at certain points)

Example 4: Find the Fourier series for the 2π-periodic function defined as $f(x) = \begin{cases} \sin x & x \in [0,\pi] \\ 0 & x \in (\pi, 2\pi] \end{cases}$

Since the period $2L$ is 2π, $L = \pi$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}\left\{-\cos x\right\}\Big|_0^{\pi} = \frac{2}{\pi}$$

$\frac{\pi}{L} = 1$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$$

$$= \frac{1}{\pi}\begin{cases} \frac{1}{2}\sin^2 x\Big|_0^{\pi} & \text{if } n=1 \\ \left(\frac{\cos(n-1)x}{2(n-1)} - \frac{\cos(n+1)x}{2(n+1)}\right)\Big|_0^{\pi}, & n>1 \end{cases}$$

Look at note on next page.

$$= \begin{cases} 0 & \text{if } n=1 \\ -\frac{1+(-1)^n}{\pi(n^2-1)} & n>1 \end{cases}$$

$$= \frac{\cos(n-1)\pi}{2(n-1)} - \frac{\cos(n+1)\pi}{2(n+1)} - \frac{\cos 0}{2(n-1)} + \frac{\cos(0)}{2(n+1)}$$

$$= \frac{(-1)^{n+1}}{2(n-1)} - \frac{(-1)^{n+1}}{2(n+1)} - \frac{1}{2(n-1)} + \frac{1}{2(n+1)}$$

$$= \frac{1}{2}\left[\frac{(n+1)\left[(-1)^{n+1}-1\right] + \overbrace{(n-1)\left[1-(-1)^{n+1}\right]}^{= -(n-1)\left[(-1)^{n+1}-1\right]}}{n^2-1}\right]$$

$$= \frac{(-1)^{n+1}-1}{n^2-1} = -\frac{\left(1+(-1)^n\right)}{n^2-1}$$

Similarly, $b_n = \begin{cases} 1/2, & n=1 \\ 0, & n>1. \end{cases}$

Since $f(x)$ is continuous everywhere (check this).

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{n=2}^{\infty} -\frac{1+(-1)^n}{\pi(n^2-1)}\cos nx$$

When n is odd, terms vanish, so let $n = 2k$

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kx)}{4k^2-1}$$

Note: To get the trig identities like the one used in the example, use complex identities.

Recall that $e^{ix} = \cos x + i\sin x$ (check this using Taylor series)

so $e^{-ix} = \cos x - i\sin x$

and $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$ $\qquad \sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$

Then
$$\sin x\cos nx = \frac{(e^{ix}-e^{-ix})(e^{inx}+e^{-inx})}{4i}$$
$$= \frac{1}{4i}\left[e^{(n+1)ix} + e^{(1-n)ix} - e^{(n-1)ix} - e^{-(n+1)ix}\right]$$
$$= \frac{e^{(n+1)ix}-e^{-(n+1)x}}{4i} - \left(\frac{e^{(n-1)ix}-e^{-(n-1)ix}}{4i}\right)$$
$$= \frac{1}{2}\left[\sin(n+1)x - \sin(n-1)x\right]$$