Solutions to PDEs using separation of Variables

A partial differential equation is an equation involving a function and its partial derivatives. Taking a partial derivative is a linear operation since

$$\frac{\partial}{\partial x_i}(f+g) = \frac{\partial f}{\partial x_i} + \frac{\partial g}{\partial x_i} \quad \text{and} \quad \frac{\partial}{\partial x_i}(cf) = c\frac{\partial f}{\partial x_i}$$

A linear PDE is one which is linear in the unknown function and its derivatives. The most general form of a second-order linear PDE for a function $u(x,y)$ with x,y independent is

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u = F(x,y)$$

As usual, the PDE is homogeneous if $F(x,y) \equiv 0$. Just like the homogeneous ODEs that we have studied, we can use the concept of superposition to combine solutions.

Theorem: If $u_1, \ldots, u_n$ are solutions to the same linear, homogeneous PDE, then so is

$$u = \sum_{i=1}^{n} c_i u_i \quad , \quad c_i \in \mathbb{R}.$$

A similar statement holds for linear, homogeneous boundary conditions.

- The idea behind the separation of variables is to look for solutions of the form $u(x,y) = f(x)g(y)$. Of course we can't expect that all solutions to a PDE have this form, but it provides a starting point to build up a general solution.

Example 1: Use the separation of variables to find a solution to $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial y}{\partial t}$.

Assume that a solution $y(x,t)$ can be written as

$$y(x,t) = X(x)\,T(t).$$

Substituting into the PDE, we get

$$\frac{\partial^2 X}{\partial x^2} T(t) = X(x) \frac{\partial T}{\partial t}$$

$$\Rightarrow \quad \frac{1}{X(x)} \frac{\partial^2 X}{\partial x^2} = \frac{1}{T(t)} \frac{\partial T}{\partial t}$$

Now the LHS is a function of x alone, and the RHS is a function of t alone, so the variables have been separated.

Now if we differentiate both sides with respect to x, we get $\dfrac{\partial}{\partial x}\left(\dfrac{1}{X(x)} \dfrac{\partial^2 X}{\partial x^2} \right) = 0 \Rightarrow \dfrac{1}{X(x)} \dfrac{\partial^2 X}{\partial x^2} = C_1$

and similarly if we differentiate with respect to t, then

$$\frac{\partial}{\partial t}\left(\frac{1}{T(t)} \frac{\partial T}{\partial t} \right) = 0 \Rightarrow \frac{1}{T(t)} \frac{\partial T}{\partial t} = C_2.$$

Then the constants are equal $C_1 = C_2 = \lambda$ and we write

$$\frac{1}{X(x)} \frac{\partial^2 X}{\partial x^2} = \lambda = \frac{1}{T(t)} \frac{\partial T}{\partial t}$$

called the separation principle.

$$\Rightarrow \quad \frac{\partial^2 X}{\partial x^2} - \lambda X = 0 \quad \text{and} \quad \frac{\partial T}{\partial t} - \lambda T = 0.$$

We are left with two ODEs which we can solve using previous techniques

Example 2: Find a solution using separation of variables to the PDE:

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 2u^2$$

Assume $u(x,y) = f(x)g(y)$. Then

$$\left(\frac{\partial f}{\partial x} g(y)\right)^2 + \left(\frac{\partial g}{\partial y} f(x)\right)^2 = 2(f(x)g(y))^2$$

$$\Rightarrow \quad \frac{1}{f^2 g^2}\left[\left(\frac{\partial f}{\partial x}\right)^2 g^2 + \left(\frac{\partial g}{\partial y}\right)^2 f^2\right] = 2$$

$$\Rightarrow \quad \frac{1}{f^2}\left(\frac{\partial f}{\partial x}\right)^2 = 2 - \left(\frac{\partial g}{\partial y}\right)^2 \frac{1}{g^2}$$

$$\Rightarrow \quad \left(\frac{\partial f}{\partial x}\right)^2 - \lambda f^2 = 0 \quad \text{and} \quad \left(\frac{\partial g}{\partial y}\right)^2 - (2-\lambda) g^2 = 0$$

using the separation principle.

The first ODE can be solved using the FTC.

$$\left(\frac{\partial f}{\partial x} - \sqrt{\lambda} f\right)\left(\frac{\partial f}{\partial x} + \sqrt{\lambda} f\right) = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = \pm\sqrt{\lambda} f \quad \Rightarrow \quad f(x) = C_1 e^{\pm\sqrt{\lambda}x}$$

and similarly $\quad g(y) = C_2 e^{\pm\sqrt{2-\lambda}\,y}$

Then we can combine our solutions to define

$$u_1(x,y) = Ce^{\sqrt{\lambda}x + \sqrt{2-\lambda}y} \qquad u_2(x,y) = Ce^{\sqrt{\lambda}x - \sqrt{2-\lambda}y}$$

$$u_3(x,y) = Ce^{-\sqrt{\lambda}x + \sqrt{2-\lambda}y} \qquad u_4(x,y) = Ce^{-\sqrt{\lambda}x - \sqrt{2-\lambda}y}$$

where $C, \lambda \in \mathbb{R}$, and for most λ, provides 4 linearly independent solutions to the PDE.

Taking $\lambda = 1$ for example, we get that

$e^{x+y}, e^{x-y}, e^{y-x}, e^{-x-y}$ are all solutions to the PDE.

However $e^{x+y} + e^{x-y}$ is not a solution. Indeed, the PDE is not linear, so while the separation of variables might still work, we cannot expect to use superposition to add solutions.

As practice, try using the separation principle on some of the equations in 21.1 #1–10. There is no need to solve the ODEs, just determine whether the principle can be used, and what system of ODEs you need to solve. (You can, for example, apply separation of variables to #1).