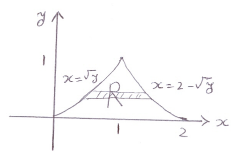
Solutions of Math 3132 Practice Questions Part 2

7. Let $\mathbf{F}(x,y) = (\frac{2}{3}\,xy\,\sqrt{y}\,)\,\hat{\mathbf{i}} + (\frac{3}{2}\,x^2\,\sqrt{y}\,)\,\hat{\mathbf{j}}$. Find the work done by $\mathbf{F}$ on a particle that moves along C where C traverses once counter-clockwise around the region in the $xy-$plane bounded by the parabolas $y = x^2$, $y = (x-2)^2$ and the line $y = 0$.

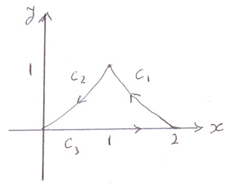
Solution: We offer two solutions:

Solution 1: Using Green's Theorem



$$
\begin{aligned}
W = \oint_C \mathbf{F}\cdot d\mathbf{r} &= \int\int_R \Big[\frac{\partial}{\partial x}(\frac{3}{2}x^2\sqrt{y}) - \frac{\partial}{\partial y}(\frac{2}{3}xy\sqrt{y})\Big]\,dA \\
&= \int\int_R (3x\sqrt{y} - x\sqrt{y})\,dA \\
&= \int\int_R 2x\sqrt{y}\,dA \\
&= \int_0^1\int_{\sqrt{y}}^{2-\sqrt{y}} 2x\sqrt{y}\,dx\,dy \\
&= \int_0^1 x^2\sqrt{y}\,\Big|_{\sqrt{y}}^{2-\sqrt{y}}\,dy \\
&= \int_0^1 \sqrt{y}\,[(2-\sqrt{y})^2 - (\sqrt{y})^2]\,dy \\
&= \int_0^1 (4\sqrt{y} - 4y)\,dy \\
&= (\frac{8}{3}y^{\frac{3}{2}} - 2y^2)\,\Big|_0^1 \\
&= \frac{2}{3}.
\end{aligned}
$$

Solution 2: We divide C into three parts C_1, C_2 and C_3:



On C_1: $\quad x = 2 - t, \quad y = t^2, \quad 0 \le t \le 1$

$$\int_{C_1} (\frac{2}{3} xy \sqrt{y})\, dx + (\frac{3}{2} x^2 \sqrt{y}) dy = \int_0^1 \frac{2}{3}(2-t)(t^2)(t)(-dt) + \frac{3}{2}(2-t)^2(t)(2tdt)$$

$$= \int_0^1 \left[\frac{11}{3}t^4 - \frac{40}{3}t^3 + 12t^2\right] dt)$$

$$= \left[\frac{11}{15}t^5 - \frac{10}{3}t^4 + 4t^3\right]\Big|_0^1$$

$$= \frac{21}{15}.$$

On C_2: $\quad x = 1 - t, \quad y = (1-t)^2, \quad 0 \le t \le 1$

$$\int_{C_2} (\frac{2}{3} xy \sqrt{y})\, dx + (\frac{3}{2} x^2 \sqrt{y}) dy$$

$$= \int_0^1 \frac{2}{3}(1-t)(1-t)^2(1-t)(-dt) + \frac{3}{2}(1-t)^2(1-t)(-2(1-t)dt)$$

$$= \int_0^1 \left[-\frac{2}{3}(1-t)^4 - 3(1-t)^4\right] dt$$

$$= \int_0^1 -\frac{11}{3}(1-t)^4\, dt$$

$$= \frac{11}{15}(1-t)^5 \Big|_0^1$$

$$= -\frac{11}{15}.$$

On C_3: $\quad x = t, \quad y = 0, \quad 0 \le t \le 2$

$$\int_{C_3} (\frac{2}{3} xy \sqrt{y})\, dx + (\frac{3}{2} x^2 \sqrt{y}) dy = \int_0^2 \frac{2}{3}(t)(0)(0)(dt) + \frac{3}{2}(t^2)(0)(0) = 0$$

Therefore since $C = C_1 \cup C_2 \cup C_3$ we get

$$W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (\frac{2}{3} xy \sqrt{y})\, dx + (\frac{3}{2} x^2 \sqrt{y}) dy = \frac{21}{15} + (-\frac{11}{15}) + 0 = \frac{2}{3}$$

8. Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution: Let $x = a\cos t$, $y = b\sin t$ where $0 \le t \le 2\pi$. Then

$$\text{Area of ellipse} = \frac{1}{2}\oint_C xdy - ydx = \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)dt \; - \; (b\sin t)(-a\sin t)dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\, dt$$

$$= ab\pi$$

Therefore the area of that part of the ellipse not in the third quadrant is $\frac{3}{4}(ab\pi)$.

9. Evaluate the surface integral $\iint_S xy\,dS$ where S is that part of the paraboloid $z = \frac{1}{2}x^2 + \frac{1}{2}y^2$ inside the sphere $x^2 + y^2 + z^2 = 3$ in the first octant.

Solution: To find S_{xy} we find the intersection of the two surfaces:

$$2z + z^2 = 3 \Rightarrow (z-1)(z+3) = 0 \Rightarrow z = 1,\ \ z = -3 \text{ NA} \Rightarrow x^2 + y^2 = 2,$$

that is

$$S_{xy} = \{(x,y) \,|\, x^2 + y^2 \le 2,\ \ x \ge 0,\ \ y \ge 0\} = \{(r,\theta) \,|\, 0 \le r \le \sqrt{2},\ \ 0 \le \theta \le \frac{\pi}{2}\}$$

$$\begin{aligned}
\iint_S xy\,dS &= \iint_{S_{xy}} xy\sqrt{1 + z_x^2 + z_y^2}\,dA \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}} (r\cos\theta)(r\sin\theta)\sqrt{1+r^2}\,(r\,dr\,d\theta) \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}} (r^3\sqrt{1+r^2})(\sin 2\theta)\,dr\,d\theta \quad (\text{let } u = 1 + r^2) \\
&= \frac{1}{4}\int_0^{\frac{\pi}{2}}\int_1^{3} (u-1)\sqrt{u}\,(\sin 2\theta)\,du\,d\theta \\
&= \frac{1}{4}\int_0^{\frac{\pi}{2}}\int_1^{3} (u^{\frac{3}{2}} - u^{\frac{1}{2}})(\sin 2\theta)\,du\,d\theta \\
&= \frac{1}{4}\int_0^{\frac{\pi}{2}} (\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}})\Big|_1^3(\sin 2\theta)\,d\theta \\
&= \frac{6\sqrt{3}+1}{15}\int_0^{\frac{\pi}{2}} \sin 2\theta\,d\theta \\
&= -\frac{6\sqrt{3}+1}{30}\cos 2\theta\Big|_0^{\frac{\pi}{2}} \\
&= \frac{6\sqrt{3}+1}{15}.
\end{aligned}$$

10. Evaluate the surface integral of $f(x,y,z) = 2y^2z$ over the surface S, where S is that part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$.

Solution: The sphere and the cone intersect at $x^2 + y^2 + (\sqrt{x^2 + y^2})^2 = 4$ that is at the circle $x^2 + y^2 = 2$, $z = \sqrt{2}$. So then
$S_{xy} = \{(x,y) \,|\, x^2 + y^2 \le 2\} = \{(r,\theta) \,|\, 0 \le r \le \sqrt{2},\ 0 \le \theta \le 2\pi\}$.
Also from $x^2 + y^2 + z^2 = 4$ we get $z = \sqrt{4 - x^2 - y^2}$, so

$$
\begin{aligned}
dS &= \sqrt{1 + (z_x)^2 + (z_y)^2}\, dA \\
&= \sqrt{1 + (\frac{-x}{\sqrt{4 - x^2 - y^2}})^2 + (\frac{-y}{\sqrt{4 - x^2 - y^2}})^2}\, dA \\
&= \sqrt{\frac{4 - x^2 - y^2 + x^2 + y^2}{4 - x^2 - y^2}}\, dA \\
&= \frac{2}{\sqrt{4 - x^2 - y^2}}\, dA \\
&= \frac{2}{z}\, dA\,.
\end{aligned}
$$

Now

$$
\begin{aligned}
\iint_S f(x, y, z)\, dS &= \iint_{S_{xy}} 2y^2 z\, dS \\
&= \iint_{S_{xy}} 2y^2 z\, (\frac{2}{z}\, dA) \\
&= \iint_{S_{xy}} 4y^2\, dA \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} 4r^3 \sin^2\theta\, dr d\theta \\
&= \int_0^{2\pi} (r^4 \Big|_0^{\sqrt{2}}) \sin^2\theta\, d\theta \\
&= \int_0^{2\pi} 4 \sin^2\theta\, d\theta \\
&= \int_0^{2\pi} 2(1 - \cos 2\theta)\, d\theta \\
&= (2\theta - \sin 2\theta) \Big|_0^{2\pi} \\
&= 4\pi\,.
\end{aligned}
$$