Solutions of Math 3132 Practice Questions Part 4

13. Evaluate the line integral $\oint_C y^2\,dx + xz^3\,dy + x^3\,dz$ where C is the curve of intersection of sphere $x^2+y^2+z^2 = 8$ and the cone $x^2+y^2 = z^2$ with $z \ge 0$, directed clockwise as viewed from the origin.(Do it with and also without Stokes' s Theorem.)

Solution:

We offer two solutions, one with using Stokes's Theorem and the other without using Stokes's Theorem:

Solution 1:

$$x^2+y^2=z^2 \Rightarrow z^2+z^2=8 \Rightarrow z=\pm 2 \Rightarrow z=2\,.$$

So C is the curve $x^2+y^2=4$, $z=2$. Let S be that part of the plane $z=2$ inside $x^2+y^2=4$. Then $\hat{\mathbf{n}}=\hat{\mathbf{k}}$. But

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & xz^3 & x^3 \end{vmatrix} = -3xz^2\hat{\mathbf{i}} - 3x^2\hat{\mathbf{j}} + (z^3-2y)\hat{\mathbf{k}}\,.$$

Therefore

$$\nabla\times\mathbf{F}\cdot\hat{\mathbf{n}} = (-3xz^2\,,\,-3x^2\,,\,z^3-2y)\cdot(0\,,\,0\,,\,1) = z^3-2y\,;$$

$$dS = \sqrt{1+z_x^2+z_y^2}\,dA = \sqrt{1+0+0}\,dA = dA\,.$$

Now by Stokes's Theorem:

$$\begin{aligned}
\oint_C y^2\,dx + xz^3\,dy + x^3\,dz &= \iint_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\;dS \\
&= \iint_{S_{xy}} (z^3-2y)\,dA \\
&= \iint_{S_{xy}} (8-2y)\,dA \quad \text{(because } z=2 \text{ on } S\text{)} \\
&= \int_0^{2\pi}\int_0^2 r(8-2r\sin\theta)\,drd\theta \\
&= \int_0^{2\pi} (4r^2-\frac{2}{3}r^3\sin\theta)\Big|_0^2\,d\theta \\
&= \int_0^{2\pi} (16-\frac{16}{3}\sin\theta)\,d\theta \\
&= (16\theta+\frac{16}{3}\cos\theta)\Big|_0^{2\pi}\,d\theta \\
&= 32\pi\,.
\end{aligned}$$

Solution 2:

$$x^2+y^2=z^2 \Rightarrow z^2+z^2=8 \Rightarrow z=\pm 2 \Rightarrow z=2\,.$$

So C is the curve $x^2+y^2=4$, $z=2$. Let $x=2\cos\theta$, then

$$C: \quad x=2\cos\theta, \quad y=2\sin\theta, \quad z=2, \quad 0\le\theta\le 2\pi$$

Therefore

$$\oint_C y^2\,dx + xz^3\,dy + x^3\,dz$$
$$= \int_0^{2\pi} 2^2 \sin^2\theta(-2\sin\theta\,d\theta) + (2\cos\theta)(2^3)(2\cos\theta)d\theta + 0$$
$$= \int_0^{2\pi} (32\cos^2\theta - 8\sin^3\theta)\,d\theta$$
$$= \int_0^{2\pi} [16(1+\cos(2\theta)) - 8(1-\cos^2\theta)\sin\theta]\,d\theta$$
$$= \int_0^{2\pi} [16 + 16\cos(2\theta) - 8\sin\theta + 8\sin\theta\cos^2\theta]\,d\theta$$
$$= [16\theta + 8\sin(2\theta) + 8\cos\theta - \frac{8}{3}\cos^3\theta]\Big|_0^{2\pi}\,d\theta$$
$$= [16(2\pi) + 8(0) + 8(1) - \frac{8}{3}(1)] - [16(0) + 8(0) + 8(1) - \frac{8}{3}(1)]$$
$$= 32\pi\,.$$

14. Evaluate $I = \oint_C [\,(xy + 3x^2y^2)\,\hat{\mathbf{i}} + (z + 2x^3y)\,\hat{\mathbf{j}} + (z^2 + x^2z^2)\,\hat{\mathbf{k}}\,] \cdot d\mathbf{r}$, where C is the curve $x^2 + z^2 = 1$, $x^2 + y^2 = 1$, $z = y$, directed counterclockwise as viewed from a point far up the positive z–axis.

Solution: Let S be that part of the plane $z = y$ inside C. Then

$$\hat{\mathbf{n}} = \frac{\nabla(z-y)}{|\nabla(z-y)|} = \frac{< 0, -1, 1 >}{\sqrt{2}}, \text{ and}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy + 3x^2y^2 & z + 2x^3y & z^2 + x^2z^2 \end{vmatrix} = -\hat{\mathbf{i}} - 2xz^2\hat{\mathbf{j}} - x\hat{\mathbf{k}}\,.$$

Therefore

$$\nabla \times \mathbf{F} \cdot \hat{\mathbf{n}} = (-1\,,\, -2xz^2\,,\, -x) \cdot \frac{1}{\sqrt{2}}(0\,,\, -1\,,\, 1) = \frac{1}{\sqrt{2}}(2xz^2 - x)\,, \text{ and}$$

$$dS = \sqrt{1 + z_x^2 + z_y^2}\,dA = \sqrt{1+0+1}\,dA = \sqrt{2}\,dA\,.$$

Now by Stokes's Theorem:

$$
\begin{aligned}
&\oint_C [\,(xy + 3x^2y^2)\,\hat{\mathbf{i}} \;+\; (z + 2x^3y)\,\hat{\mathbf{j}} \;+\; (z^2 + x^2z^2)\,\hat{\mathbf{k}}\,] \cdot d\mathbf{r} \\
&= \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS \\
&= \iint_{S_{xy}} \frac{1}{\sqrt{2}}(2xz^2 - x)\;\sqrt{2}\,dA \\
&= \iint_{S_{xy}} (2xy^2 - x)\,dA \quad \text{(because } z = y \text{ on } S\text{)} \\
&= \int_0^{2\pi}\int_0^1 [2(r\cos\theta)(r^2\sin^2\theta) - r\cos\theta]\,(rdr)d\theta \\
&= \int_0^{2\pi}\int_0^1 [2r^4\cos\theta\sin^2\theta \;-\; r^2\cos\theta]\,drd\theta \\
&= \int_0^{2\pi} [\frac{2}{5}r^5\cos\theta\sin^2\theta \;-\; \frac{1}{3}r^3\cos\theta\,]\,\Big|_0^1\,d\theta \\
&= \int_0^{2\pi} [\frac{2}{5}\cos\theta\sin^2\theta \;-\; \frac{1}{3}\cos\theta\,]\,d\theta \\
&= [\frac{2}{15}\sin^3\theta \;-\; \frac{1}{3}\sin\theta\,]\,\Big|_0^{2\pi} \\
&= 0\,.
\end{aligned}
$$

15. Assuming that $y \;=\; \sum_{n=0}^{\infty} a_n\,(x-1)^n$ is a solution of the differential equation

$$(x-1)^2\,y'' \;-\; (x-1)\,y' \;-\; (x^2 - 2x)\,y \;=\; 0\,,$$

find a recurrence relation for a_n and simplify it as much as possible. (Do not continue after finding the recurrence relation).

Solution: $y' \;=\; \sum_{n=1}^{\infty} na_n\,(x-1)^{n-1}$ and $y'' \;=\; \sum_{n=2}^{\infty} n(n-1)a_n\,(x-1)^{n--2}$. Also note that $x^2 - 2x = (x-1)^2 - 1$. Now putting in the differential equation we get

$$(x-1)^2 \sum_{n=2}^{\infty} n(n-1)a_n\,(x-1)^{n-2} - (x-1)\sum_{n=1}^{\infty} na_n\,(x-1)^{n-1} - [(x-1)^2 - 1]\sum_{n=0}^{\infty} a_n\,(x-1)^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n\,(x-1)^n - \sum_{n=1}^{\infty} na_n\,(x-1)^n - \sum_{n=0}^{\infty} a_n\,(x-1)^{n+2} + \sum_{n=0}^{\infty} a_n\,(x-1)^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n\,(x-1)^n - \sum_{n=1}^{\infty} na_n\,(x-1)^n - \sum_{n=2}^{\infty} a_{n-2}\,(x-1)^n + \sum_{n=0}^{\infty} a_n\,(x-1)^n = 0$$

$$\Rightarrow a_0 + a_1(x-1) - a_1(x-1) \;+\; \sum_{n=2}^{\infty}\Big[[n(n-1) - n + 1]a_n \;-\; a_{n-2}\Big](x-1)^n \;= 0$$

$$\Rightarrow a_0 \;+\; \sum_{n=2}^{\infty}\Big[(n-1)^2\,a_n \;-\; a_{n-2}\Big](x-1)^n \;= 0$$

$$\Rightarrow a_0 \;=\; 0\,, \quad (n-1)^2\,a_n \;-\; a_{n-2} \;= 0\,, \quad n \geq 2$$

Therefore $a_n \;=\; \dfrac{1}{(n-1)^2}\,a_{n-2}$ where $n = 2, 3, 4, \cdots$.

16. For the differential equation $xy'' + 3y' - xy = 0$, when you use the power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ to solve the differential equation, you get

$$3a_1 + \sum_{n=2}^{\infty} [\, n(n+2)\, a_n + a_{n-2} \,]\, x^{n-1} = 0 .$$

You do not need to prove this relation. Use it to find the solution of the differential equation. Write your solution using sigma notation and **simplify as much as possible**.

Solution:

$$3a_1 = 0 \ , \quad n(n+2)a_n + a_{n-2} = 0 \ , \quad n \geq 2$$

So $a_1 = 0$ and $a_n = -\dfrac{1}{n(n+2)}\, a_{n-2}$, $n \geq 2$. Now

$$n = 2 \quad \Rightarrow \quad a_2 = -\frac{1}{2(4)}\, a_0$$

$$n = 3 \quad \Rightarrow \quad a_3 = -\frac{1}{3(5)}\, a_1 = 0$$

$$n = 4 \quad \Rightarrow \quad a_4 = -\frac{1}{4(6)}\, a_2 = \frac{1}{2(4^2)6}\, a_0$$

$$n = 5 \quad \Rightarrow \quad a_5 = -\frac{1}{3(5)}\, a_3 = 0$$

$$n = 6 \quad \Rightarrow \quad a_6 = -\frac{1}{6(8)}\, a_4 = -\frac{1}{2(4^2)(6^2)8}\, a_0 .$$

Therefore $a_{2k-1} = 0$ where $k = 1, 2, 3, \cdots$ and

$$a_{2k} = \frac{(-1)^k}{2(4^2)(6^2)(8^2)\cdots(2k)^2(2k+2)}\, a_0 \ , \quad k = 2, 3, 4, \cdots$$

Hence

$$\begin{aligned}
y(x) = \sum_{n=0}^{\infty} a_n x^n &= a_0 + (a_1 x + a_3 x^3 + a_5 x^5 + \cdots) + (a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots) \\
&= a_0 + (0 + 0 + 0 + \cdots) + \Big(-\frac{1}{8} a_0 x^2 + a_0 \sum_{n=2}^{\infty} \frac{(-1)^n}{2(4^2)(6^2)(8^2)\cdots(2n)^2(2n+2)}\, x^{2n} \Big) \\
&= a_0 + (0 + 0 + 0 + \cdots) + \Big(-\frac{1}{8} a_0 x^2 + a_0 \sum_{n=2}^{\infty} \frac{(-1)^n}{[2^2(4^2)(6^2)(8^2)\cdots(2n)^2](n+1)}\, x^{2n} \Big) \\
&= a_0 - \frac{1}{8} a_0 x^2 + a_0 \sum_{n=2}^{\infty} \frac{(-1)^n}{[2(4)(6)\cdots(2n)]^2\,(n+1)}\, x^{2n} \\
&= a_0 - \frac{1}{8} a_0 x^2 + a_0 \sum_{n=2}^{\infty} \frac{(-1)^n}{[2^n\, n!]^2\,(n+1)}\, x^{2n} \\
&= a_0 - \frac{1}{8} a_0 x^2 + a_0 \sum_{n=2}^{\infty} \frac{(-1)^n}{2^{2n}\,(n!)^2\,(n+1)}\, x^{2n} \\
&= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}\,(n!)^2\,(n+1)}\, x^{2n} \\
&= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}\, n!\,(n+1)!}\, x^{2n}
\end{aligned}$$

17. Use $y = \sum_{n=0}^{\infty} a_n x^n$ to solve the differential equation

$$x^2 y'' + xy' + (x^2 - 1) y = 0 .$$

Simplify as much as possible. Is this solution a general solution? What is the interval of convergence?

Solution: $y' = \sum_{n=1}^{\infty} na_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ .Now putting in the differential equation we get

$$x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x \sum_{n=1}^{\infty} na_n x^{n-1} + (x^2 - 1) \sum_{n=0}^{\infty} a_n x^n = 0 ,$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} - \sum_{n=0}^{\infty} a_n x^n = 0 ,$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0 ,$$

$$\Rightarrow -a_0 - a_1 x + a_1 x + \sum_{n=2}^{\infty} \Big[[n(n-1) + n - 1]a_n + a_{n-2} \Big] x^n = 0 ,$$

$$\Rightarrow -a_0 + \sum_{n=2}^{\infty} \Big[(n^2 - 1)a_n + a_{n-2} \Big] x^n = 0 .$$

Therefore $a_0 = 0$ and $a_n = \dfrac{-1}{n^2 - 1} a_{n-2} = -\dfrac{1}{(n-1)(n+1)} a_{n-2}$ where $n \geq 2$.

$$n = 2 \quad \Rightarrow \quad a_2 = -\frac{1}{1(3)} a_0 = 0$$

$$n = 4 \quad \Rightarrow \quad a_4 = -\frac{1}{3(5)} a_2 = 0$$

$$n = 6 \quad \Rightarrow \quad a_6 = -\frac{1}{5(7)} a_4 = 0 .$$

Therefore $a_{2k} = 0$ for $k = 0, 1, 2, 3, \cdots$. Also

$$n = 3 \quad \Rightarrow \quad a_3 = -\frac{1}{2(4)} a_1$$

$$n = 5 \quad \Rightarrow \quad a_5 = -\frac{1}{3(5)} a_3 = + \frac{1}{2(4)(4)(6)} a_1 = + \frac{1}{4^2 (2!)(3!)} a_1$$

$$n = 7 \quad \Rightarrow \quad a_7 = -\frac{1}{6(8)} a_5 = - \frac{1}{2(4)(4)(6)(6)(8)} a_1 = - \frac{1}{4^3 (3!)(4!)} a_1$$

Therefore $a_{2k-1} = \dfrac{(-1)^{k-1}}{4^{k-1} (k-1)!\, k!} a_1$ for $k = 1, 2, 3, \cdots$. Hence

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ &= 0 + a_1 x + a_3 x^3 + a_5 x^5 + \cdots \\ &= a_1 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4^{n-1} (n-1)!\, n!} x^{2n-1} . \end{aligned}$$

It is not a general solution because it has only one parameter while a general solution needs two parameters. To find the radius of convergence R :

$$R = \lim_{n \to \infty} \Big| \frac{a_n}{a_{n+1}} \Big| = \lim_{n \to \infty} \Big| \frac{(-1)^{n-1}}{4^{n-1} (n-1)!\, n!} \cdot \frac{4^n\, n!\, (n+1)!}{(-1)^n} \Big| = \lim_{n \to \infty} 4n(n+1) = \infty .$$

That is the interval of convergence is $(-\infty , \infty)$.