Solutions of Math 3132 Practice Questions Part 5

18. Consider the differential equation

$$(\sin x)\, y'' \;+\; \frac{\sin x}{x^2+16}\, y' \;+ (x\cos x)\, y \;=\; 0\,.$$

(a) Is $x = 0$ a singular point for the differential equation? Why?

Solution: $P(x) = \sin x$, $Q(x) = \dfrac{\sin x}{x^2+16}$, and $R(x) = x\cos x$.Now

$\dfrac{Q(x)}{P(x)} = \dfrac{1}{x^2+16}$,

$\dfrac{R(x)}{P(x)} = \dfrac{x\cos x}{\sin x} = x\cot x$,

$\lim\limits_{x\to 0} x\cot x = \lim\limits_{x\to 0} \dfrac{x}{\tan x} = 1$.

So $x = 0$ is not a singular point.

(b) Find all real or complex singular points of the differential equation.

Solution: If $x^2 + 16 = 0$, then $x = \pm 4i$ are singular points. Also $x\cot x$ is not defined if $x = k\pi$, $k = \pm 1, \pm 2, \cdots$. Therefore $x = \pm 4i$ and $x = k\pi$, $k = \pm 1, \pm 2, \cdots$, are all singular points.

(c) What can be said about the radius of convergence of a power series solution about $x = 3$ for the differential equation?
(You are **not** asked to solve the differential equation.)

Solution: $x = 3$ is a ordinary point and

$d_1 = \sqrt{(3-0)^2 + (0-(\pm 4))^2} = 5$,

$d_2 = \pi - 3$,

So $R \geq \text{Min}\{5, \pi - 3\} = \pi - 3$.

Hence the radius of convergence of a power series solution about $x = 3$ is *at least* $\pi - 3$.

19. Consider the differential equation

$$(x^2 - 2x + 2)\, y'' \;+\; x^2 y' \;-\; (\sin^2 x)\, y \;=\; 0\,.$$

(a) Find all real or complex singular points of the differential equation.

Solution: $P(x) = x^2 - 2x + 2$, $Q(x) = x^2$, and $R(x) = -\sin^2 x$.Now

$\dfrac{Q(x)}{P(x)} = \dfrac{x^2}{x^2-2x+2}$,

$\dfrac{R(x)}{P(x)} = \dfrac{-\sin^2 x}{x^2-2x+2}$,

So if $x^2 - 2x + 2 = 0$ then $x = 1 + i$ and $x = 1 - i$ are singular points.

(b) If $y(x) = \sum\limits_{n=0}^{\infty} a_n (x+1)^n$ is used to solve this differential equation, will the result be a general solution ? What can be said about the radius of convergence of this power series solution ? Justify your conclusions.(You do not need to solve the differential equation.)

Solution:
$x = -1$ is a ordinary point and

$$d_1 = \sqrt{(-1-1)^2 + (0-1)^2} = \sqrt{5},$$

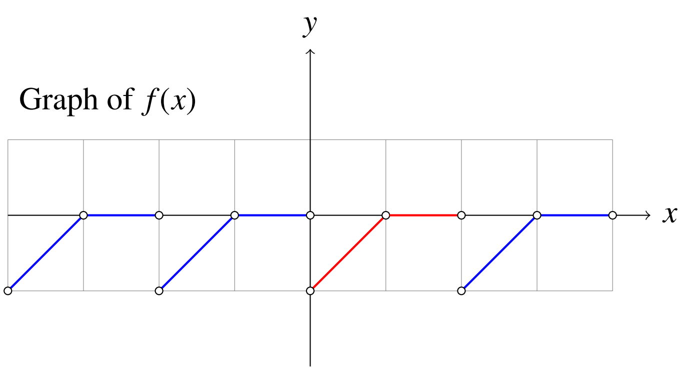
$$d_2 = \sqrt{(-1-1)^2 + (0+1)^2} = \sqrt{5},$$

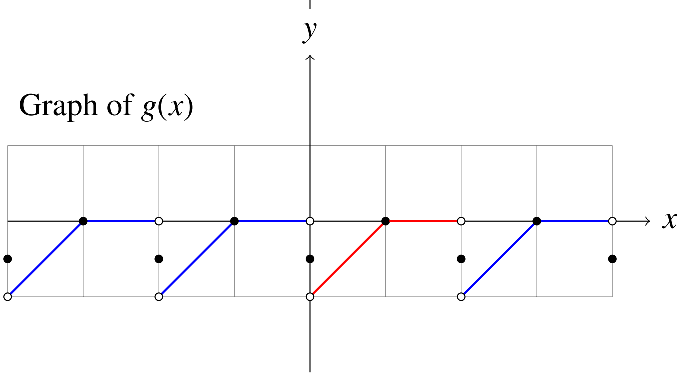
So $R \geq \sqrt{5}$, that is the radius of convergence of a power series solution about $x = -1$ is *at least* $\sqrt{5}$.

20. Let $f(x) = \begin{cases} x-1 & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 < x < 2 \end{cases}$ with $f(x+2) = f(x)$.

(a) On the interval $-4 \leq x \leq 4$, draw the graph of $f(x)$; also draw the graph of the function $g(x)$ to which the Fourier series of $f(x)$ converges. Describe $g(x)$.

Solution:





In fact $g(x) = \begin{cases} f(x) & \text{if } x \neq 0, \pm 1, \pm 2, \cdots \\ 0 & \text{if } x = 2k-1, \quad k = 0, \pm 1, \pm 2, \cdots \\ -\dfrac{1}{2} & \text{if } x = 2k, \quad k = 0, \pm 1, \pm 2, \cdots \end{cases}.$

(b) Find the Fourier series for the periodic function $f(x)$. Simplify your answer as much as possible.

Solution: $2L = 2$ so $L = 1$. Now

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx = \frac{1}{1}\Big[\int_0^1 (x-1)\,dx + \int_1^2 0\,dx\Big] = \Big(\frac{1}{2}x^2 - x\Big)\Big|_0^1 = -\frac{1}{2};$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{1}\Big[\int_0^1 (x-1)\cos n\pi x\,dx + \int_1^2 0\,dx\Big]$$
$$= \int_0^1 (x-1)\cos n\pi x\,dx;$$

using integration by parts let $u = x-1$ and $dv = \cos n\pi x\,dx$ then $du = dx$ and $v = \frac{1}{n\pi}\sin n\pi x$. Hence

$$\begin{aligned} a_n &= \int_0^1 (x-1)\cos n\pi x\,dx \\ &= \frac{x-1}{n\pi}\sin n\pi x\Big|_0^1 - \frac{1}{n\pi}\int_0^1 \sin n\pi x\,dx \\ &= 0 + \frac{1}{n^2\pi^2}\cos n\pi x\Big|_0^1 \\ &= \frac{1}{n^2\pi^2}[(-1)^n - 1]. \end{aligned}$$

$$\begin{aligned} b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,dx &= \frac{1}{1}[\int_0^1 (x-1)\sin n\pi x\,dx + \int_1^2 0\,dx] \\ &= \int_0^1 (x-1)\sin n\pi x\,dx; \end{aligned}$$

using integration by parts let $u = x-1$ and $dv = \sin n\pi x\,dx$ then $du = dx$ and $v = -\frac{1}{n\pi}\cos n\pi x$. Hence

$$\begin{aligned} b_n &= \int_0^1 (x-1)\sin n\pi x\,dx \\ &= -\frac{x-1}{n\pi}\cos n\pi x\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi x\,dx \\ &= 0 + \frac{-1}{n\pi} + \frac{1}{n^2\pi^2}\sin n\pi x\Big|_0^1 \\ &= -\frac{1}{n\pi}. \end{aligned}$$

Therefore the Fourier series of $f(x)$ is

$$\begin{aligned} g(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}] \\ &= \frac{-\frac{1}{2}}{2} + \sum_{n=1}^{\infty}[\frac{1}{n^2\pi^2}[(-1)^n - 1]\cos n\pi x - \frac{1}{n\pi}\sin n\pi x] \\ &= -\frac{1}{4} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)\pi x - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x. \end{aligned}$$

(c) Use part (b) to find the sum $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$.

Solution: For $x = 0$ since $g(0) = -\frac{1}{2}$ so by substitution in part (b) we get

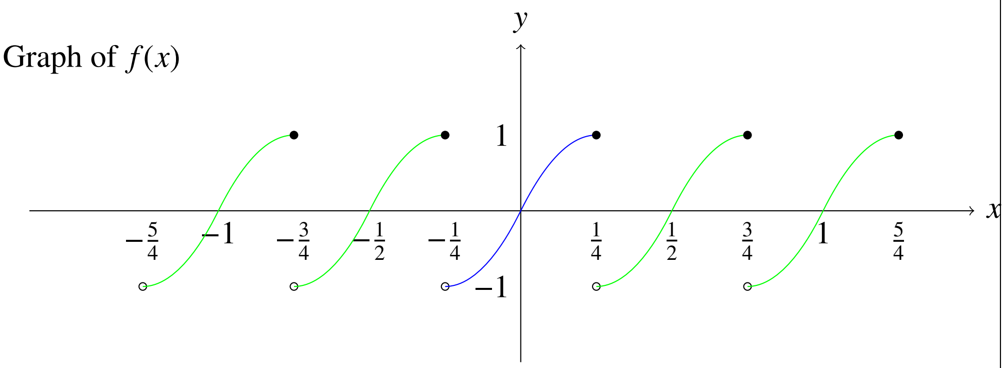
$$-\frac{1}{2} = -\frac{1}{4} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}(1) - \frac{1}{\pi}(0) \Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

21. Let $f(x) = \begin{cases} 8x + 16x^2 & \text{if } -\frac{1}{4} < x \le 0 \\ 8x - 16x^2 & \text{if } 0 < x \le \frac{1}{4} \end{cases}$ with $f(x + \frac{1}{2}) = f(x)$.

(a) Draw the graph of $f(x)$ in the interval $-\frac{5}{4} \le x \le \frac{5}{4}$.

Solution:

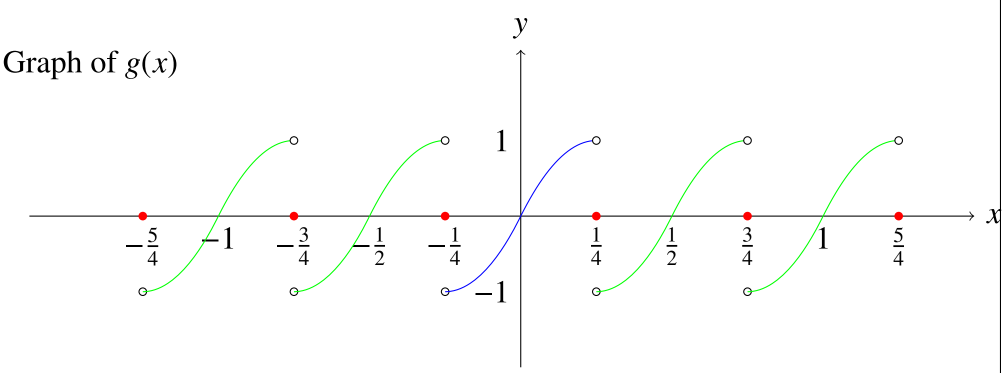
Graph of $f(x)$



(b) Draw the graph of the function $g(x)$ to which the Fourier series of $f(x)$ converges to, in the interval $-\frac{5}{4} \le x \le \frac{5}{4}$. Describe $g(x)$.

Solution:

Graph of $g(x)$



In fact

$$g(x) = \begin{cases} f(x) & \text{if } x \neq \pm\dfrac{2n+1}{4}, \quad \text{where } n = 0, 1, 2, 3, \cdots \\ 0 & \text{if } x = \pm\dfrac{2n+1}{4}, \quad \text{where } n = 0, 1, 2, 3, \cdots \end{cases}$$