Solutions of Math 3132 Practice Questions Part 7

Part 7 (sections 19.1, 19.2 and 20.1–20.3)

24. Let $f(x) = 3x - 2$ where $0 \le x \le \frac{1}{4}$. Expand $f(x)$ in terms of the eigenfunctions of the Sturm-Liouville system

$$y'' + \lambda y = 0, \quad 0 < x < \frac{1}{4}, \quad y(0) = 0, \quad y'(L) = 0.$$

Solution: Using the Sturm-Liouville table we get

$$y_n(x) = \sin \frac{(2n-1)\pi x}{2(\frac{1}{4})} = \sin 2(2n-1)\pi x, \quad n \ge 1.$$

So we need to find c_n such that

$$f(x) = \sum_{n=1}^{\infty} c_n\, y_n(x) = \sum_{n=1}^{\infty} c_n \sin 2(2n-1)\pi x, \quad 0 < x < \frac{1}{4}.$$

But

$$\begin{aligned} c_n = \frac{2}{L} \int_0^L f(x)\, y_n(x)\, dx &= \frac{2}{\frac{1}{4}} \int_0^{\frac{1}{4}} (3x-2) \sin 2(2n-1)\pi x\, dx \\ &= 8 \int_0^{\frac{1}{4}} (3x-2) \sin 2(2n-1)\pi x\, dx; \end{aligned}$$

using integration by parts, let $u = 3x - 2$ and $dv = \sin 2(2n-1)\pi x\, dx$ then $du = 3dx$ and $v = -\frac{1}{2(2n-1)\pi} \cos 2(2n-1)\pi x$. Hence

$$\begin{aligned} c_n &= 8 \int_0^{\frac{1}{4}} (3x-2) \sin 2(2n-1)\pi x\, dx \\ &= 8\Big[\frac{-(3x-2)}{2(2n-1)\pi} \cos 2(2n-1)\pi x \Big|_0^{\frac{1}{4}} + \frac{3}{2(2n-1)\pi} \int_0^{\frac{1}{4}} \cos 2(2n-1)\pi x\, dx\Big] \\ &= 8\Big[\frac{-2}{2(2n-1)\pi} + \frac{3}{4(2n-1)^2\pi^2} \sin 2(2n-1)\pi x \Big|_0^{\frac{1}{4}}\Big] \\ &= 8\Big[\frac{-2}{2(2n-1)\pi} + \frac{3}{4(2n-1)^2\pi^2} (-1)^{n-1} \Big|_0^{\frac{1}{4}}\Big] \\ &= \frac{2[\,3(-1)^{n-1} - 4(2n-1)\pi\,]}{(2n-1)^2\pi^2}. \end{aligned}$$

Therefore

$$\begin{aligned} f(x) = 3x - 2 &= \sum_{n=1}^{\infty} c_n\, y_n(x) \\ &= \sum_{n=1}^{\infty} \frac{2[\,3(-1)^{n-1} - 4(2n-1)\pi\,]}{(2n-1)^2\pi^2} \sin 2(2n-1)\pi x, \quad 0 < x < \frac{1}{4}. \end{aligned}$$

25. Consider the Sturm-Liouville system

$$\begin{aligned} y'' + 4y' + (1-2\lambda)y &= 0, \qquad 0 < x < L, \\ y(0) &= 0, \\ y(L) &= 0. \end{aligned}$$

(a) Find the standard form of the Sturm-Liouville system.

Solution:

$$e^{\int 4dx} = e^{4x} \Rightarrow e^{4x}y'' + 4e^{4x}y' + (1-2\lambda)e^{4x}y = 0$$

So the standard form of the differential equation is

$$\frac{d}{dx}(e^{4x}y') + (\lambda(-2e^{4x}) - (-e^{4x}))y = 0.$$

(b) If $\lambda < -\frac{3}{2}$, find all eigenvalues and eigenfunctions of the Sturm-Liouville system.

Solution: Since $\lambda < -\frac{3}{2}$ so $3 + 2\lambda < 0$ and

$$m^2 + 4m + (1-2\lambda) = 0 \Rightarrow m = -2 \pm \sqrt{3+2\lambda} = -2 \pm \sqrt{-(3+2\lambda)}\,i.$$

Therefore $y(x) = e^{-2x}\left(c_1\cos(\sqrt{-(3+2\lambda)}\,x) + c_2\sin(\sqrt{-(3+2\lambda)}\,x)\right)$. Now

$$0 = y(0) = 1(c_1(1) + c_2(0)) \Rightarrow c_1 = 0.$$

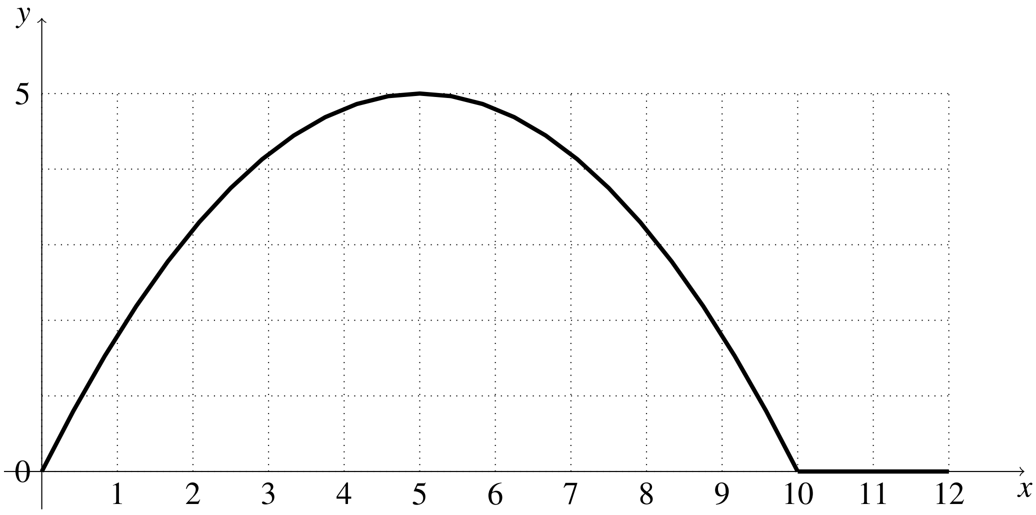
Also

$$0 = y(L) = e^{-2L}(0 + c_2\sin(\sqrt{-(3+2\lambda)}\,L)) \Rightarrow \sin(\sqrt{-(3+2\lambda)}\,L) = 0.$$

Hence $\sqrt{-(3+2\lambda)}\,L = n\pi$ (where $n = 1, 2, \cdots$) which means

$\lambda_n = -\frac{3}{2} - \frac{n^2\pi^2}{L^2}$ and therefore $y_n(x) = c_2\,e^{-2x}\sin\frac{n\pi x}{L}$ where $n = 1, 2, \cdots$.

26. A string with constant linear density ρ is stretched tightly between the points $x = 0$ and $x = 12$ on the x-axis. The tension in the string is a constant τ. The displacement of the string at time $t = 0$ is shown in the figure below, and from this position, it is released. The right end of the string is fixed on the x-axis, but the left end is looped around a vertical rod, and can move vertically without friction. A restoring force proportional to displacement and also gravity are taken into account. What is the initial-value problem for displacement $y(x, t)$ of the string? Include the partial differential equation, and all boundary and initial conditions, and include intervals on which they must be satisfied.



Solution:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - g - \frac{ky}{\rho}, \quad 0 < x < 12\,, \quad t > 0, \quad c^2 = \frac{\tau}{\rho}, \quad k > 0$$

$$\begin{aligned} y(12,t) &= 0\,, \quad t > 0 \\ y_x(0,t) &= 0\,, \quad t > 0 \\ y_t(x,0) &= 0\,, \quad 0 < x < 12 \end{aligned}$$

$$y(x,0) = \begin{cases} 2x - \frac{1}{5}x^2 & \text{if } 0 \le x \le 10 \\ 0 & \text{if } 10 \le x \le 12 \end{cases}.$$

[In fact $y = ax^2 + bx + c$ and it goes through $(0,0)$, $(5,5)$, $(10,0)$ so then $c = 0$ and $10a + b = 0$, $5a + b = 1$ and therefore $a = -\frac{1}{5}$, $b = 2$; so $y = -\frac{1}{5}x^2 + 2x$.]