Q1. [6 pts] (a) Let C be the curve $z = x^2 + y^2$, $x + y = 1$ from the point $(1, 0, 1)$ to the point $(2, -1, 5)$. Evaluate the line integral

$$\int_C 16(x - y)\, ds.$$

We can parameterize C by

$$x = t, \quad y = 1 - t, \quad z = t^2 + (1-t)^2 = 2t^2 - 2t + 1 \qquad 1 \le t \le 2$$

Then $$\int_C 16(x-y)\,ds = \int_1^2 16(t-(1-t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= 16\int_1^2 (2t-1)\sqrt{(1)^2 + (-1)^2 + (4t-2)^2}\,dt$$

Let $u = 2 + (4t-2)^2$, $du = 2(4t-2)\cdot 4\,dt = 16(2t-1)\,dt$

So $$\int_C 16(x-y)\,ds = \int_6^{38} u^{1/2}\,du$$

$$= \frac{2u^{3/2}}{3}\Big|_6^{38}$$

$$= \frac{2}{3}\left(38^{3/2} - 6^{3/2}\right)$$

(b) [1 pt] Let $-C$ be the the curve $z = x^2 + y^2$, $x + y = 1$ from the point $(2, -1, 5)$ to the point $(1, 0, 1)$. Using your answer in part (a), evaluate

$$\int_{-C} 16(x - y)\, ds.$$

Since line integrals of scalar functions is independent of parameterization

$$\int_{-C} 16(x-y)\,ds = \int_C 16(x-y)\,ds = \frac{2}{3}\left(38^{3/2} - 6^{3/2}\right)$$

Q2. Consider the vector field defined by

$$\mathbf{F}(x,y,z) = \left(\frac{y}{x} - \ln y\right)\hat{\mathbf{i}} + \left(\ln x - \frac{x}{y}\right)\hat{\mathbf{j}} + \ln(z)\hat{\mathbf{k}}.$$

Let $D \subset \mathbb{R}^3$ be the domain where $\mathbf{F}$ is defined.

(a) [3 pts] Provide the definition of a simply connected domain. Is D simply connected? Why?

- A domain $\subseteq \mathbb{R}^2$ is simply connected if every closed curve in the domain contains in its interior only points of the domain
- $\ln(x)$, $\ln(y)$ and $\ln(z)$ are defined only if $x,y,z>0$. We also need $x \neq y \neq 0$. Therefore $D = \{(x,y,z) \in \mathbb{R}^3 \mid x,y,z>0\}$
- While we have not discussed the general definition of simply connected besides in $\mathbb{R}^2$, it is clear that D does not have any points missing that would obstruct any curves contained in a plane from having interiors also contained in D. In particular, the region D does not have any holes, and so is simply connected.

(b) [3pts] Show that $\int \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D.

- Since D is simply connected, $\int F \cdot dr$ is independent of path iff $\nabla \times F = \vec{0}$

We check that $\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{y}{x} - \ln y & \ln x - \frac{x}{y} & \ln z \end{vmatrix}$

$= \left(0, 0, \frac{1}{x} - \frac{1}{y} - \left(\frac{1}{x} - \frac{1}{y}\right)\right)$

$= (0,0,0)$

- Alternatively, $\int F \cdot dr$ is independent of path iff there is a $f(x,y,z)$ where $\nabla f = F$. (we don't need that D is simply connected)

Observe that $\nabla\big(y\ln x - x\ln y + z(\ln z - 1)\big) = F$

(c) [2 pts] Let C be any closed curve in D. Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

In general, $\oint_C F \cdot dr$ is independent of path iff $\oint_C F \cdot dr = 0$ for any closed curve C

By (b), $\oint_C F \cdot dr = 0$

- Alternatively, by (b) we know that there exists an $f(x,y,z)$ with $\nabla f = F$ on D (even if we didn't find f in (b)).

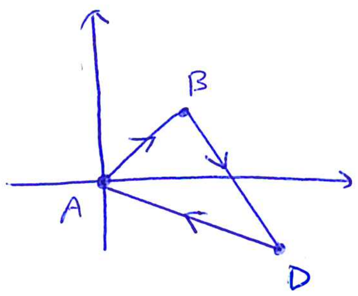
Then $\oint_C F \cdot dr = f(P_{final}) - f(P_{initial}) = 0$

since the initial and final points are the same for a closed curve

Q3. [6 pts] Evaluate the line integral

$$\oint_C \left(xy^4 + e^{x+y}\right) dx + \left(x + \sin(y^2) + 2x^2y^3 + e^{x+y}\right) dy,$$

where C is the boundary of the triangle joining the points $(1,1)$, $(0,0)$ and $(2,-1)$.



Let $A=(0,0)$, $B=(1,1)$ and $D=(2,-1)$,

and $P(x,y) = xy^4 + e^{x+y}$

$Q(x,y) = x + \sin y^2 + 2x^2y^3 + e^{x+y}$

Let R be the triangle ΔABD with boundary C.

Then $\oint_C P\,dx + Q\,dy = -\oint_C P\,dx + Q\,dy$

$= -\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ by Green's theorem

$= -\iint_R (1 + 4xy^3 + e^{x+y}) - (4xy^3 + e^{x+y})\, dA$

$= -\iint_R dA$

$= -\text{area}(R)$

Since $\vec{AB} = (1,1,0)$ and $\vec{AD} = (2,-1,0)$ viewed in the xy-plane in $\mathbb{R}^3$,

then $\vec{AB} \times \vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 2 & -1 & 0 \end{vmatrix} = (0,0,-3)$

and $\text{area}(R) = \dfrac{|\vec{AB} \times \vec{AD}|}{2} = \dfrac{\sqrt{0^2+0^2+(-3)^2}}{2} = \dfrac{3}{2}$

$\therefore \oint_C P\,dx + Q\,dy = -\frac{3}{2}$

Q4. [8 pts] Let S be the surface defined by the portion of the sphere $x^2 + y^2 + z^2 = 9$ where $1 \le z \le 2$. Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

where $\mathbf{F}(x, y, z) = 2xz\,\hat{\mathbf{i}} + 2yz\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$, and $\hat{\mathbf{n}}$ is the unit outward-pointing normal to the sphere.

- The unit outward-pointing normal to the sphere is

$$\hat{n} = \frac{\nabla(x^2+y^2+z^2-9)}{|\nabla(x^2+y^2+z^2-9)|} = \frac{(2x, 2y, 2z)}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$$

$$= \tfrac{1}{3}(x,y,z) \quad \text{since } x^2+y^2+z^2=9 \text{ on } S.$$

- $F\cdot\hat{n} = (2xz, 2yz, z^2)\cdot\dfrac{(x,y,z)}{3} = \dfrac{2x^2z + 2y^2z + z^3}{3}$

Since S is defined by $z = \sqrt{9-x^2-y^2}$ with $1 \le z \le 2$,

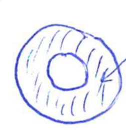
$$\iint_S F\cdot\hat{n}\,dS = \iint_{S_{xy}} F\cdot\hat{n}\sqrt{1+\left(\tfrac{\partial z}{\partial x}\right)^2+\left(\tfrac{\partial z}{\partial y}\right)^2}\,dA$$

- Here $\dfrac{\partial z}{\partial x} = \dfrac{-x}{\sqrt{9-x^2-y^2}} = \dfrac{-x}{z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-y}{z}$

so $\sqrt{1+\left(\tfrac{\partial z}{\partial x}\right)^2+\left(\tfrac{\partial z}{\partial y}\right)^2} = \dfrac{\sqrt{x^2+y^2+z^2}}{z} = \dfrac{3}{z}$

- Equations for the $z=1$ and $z=2$ cross-sections of the sphere are $\{z=1,\ x^2+y^2=8\}$ and $\{z=2,\ x^2+y^2=5\}$, so

$$S_{xy} = \{z=0,\ 5 \le x^2+y^2 \le 8\}$$



S_{xy}

Then

$$\iint_{S_{xy}} F\cdot\hat{n}\sqrt{1+\left(\tfrac{\partial z}{\partial x}\right)^2+\left(\tfrac{\partial z}{\partial y}\right)^2}\,dA = \iint_{S_{xy}} 2x^2+2y^2+9-x^2-y^2\,dA$$

$$= \int_0^{2\pi}\int_{\sqrt{5}}^{\sqrt{8}} (r^2+9)\,r\,dr\,d\theta \quad \text{in polar coordinates}$$

$$= 2\pi\left(\frac{r^4}{4}+\frac{9r^2}{2}\right)\Big|_{\sqrt{5}}^{\sqrt{8}}$$

$$= \frac{93\pi}{2}$$

Q5. [6 pts] Let $\mathbf{F}$ be the vector field defined by

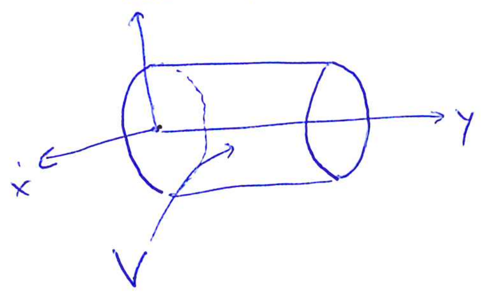
$$\mathbf{F}(x,y,z) = x\cos(y)\,\hat{\mathbf{i}} + (e^y + \tan(z) + 3y^2x^2)\,\hat{\mathbf{j}} + 2yz^3\,\hat{\mathbf{k}}.$$

Evaluate the surface integral

$$\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$$

where S is the surface enclosing the volume bounded by $x^2+z^2=4$, $y=0$ and $y=\pi$, and $\hat{\mathbf{n}}$ is the unit outward-pointing normal to S.

Let V be the volume bounded by $x^2+z^2=4$, $y=0$ and $y=\pi$.



Then $\oiint_S F\cdot\hat{n}\,dS$

$= \iiint_V \nabla\cdot F\,dV$ by the divergence theorem

$= \iiint_V \cos y + e^y + 6yx^2 + 6yz^2\,dV$

Using the cylindrical coordinates

$x = r\cos\theta$ $\quad r>0$

$y = y$ $\quad 0\le\theta<2\pi$

$z = r\sin\theta$ $\quad 0\le y\le\pi$

$= \int_0^{2\pi}\int_0^2\int_0^\pi (\cos y + e^y + 6yr^2)\,r\,dy\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^2 (\sin y + e^y + 3y^2r^2)\Big|_0^\pi r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^2 (e^\pi - 1 + 3\pi^2r^2)\,r\,dr\,d\theta$

$= \int_0^{2\pi} \left(\frac{(e^\pi-1)r^2}{2} + \frac{3\pi^2r^4}{4}\right)\Big|_0^2 d\theta$

$= 2\pi\left(2(e^\pi-1) + 12\pi^2\right)$

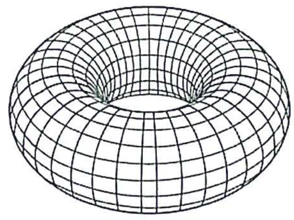
$= 4\pi\left(e^\pi + 6\pi^2 - 1\right)$

Bonus Question: [5 pts] Consider the surface S (shown below) defined by the parametric equations

$$\sigma: \qquad x = (R + r\cos\theta)\cos\varphi, \quad y = (R + r\cos\theta)\sin\varphi, \quad z = r\sin\theta,$$

where r and R are constants, and $0 \leq \theta, \varphi \leq 2\pi$. Find the surface area of S using a surface integral that can be computed using this parameterization.

Let $K = R + r\cos\theta$.



- Holding θ constant, we see that $\sigma_\varphi = (-K\sin\varphi, K\cos\varphi, 0)$

  Similarly, $\sigma_\theta = (-r\sin\theta\cos\varphi, -r\sin\theta\sin\varphi, r\cos\theta)$

- $$\sigma_\varphi \times \sigma_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -K\sin\varphi & K\cos\varphi & 0 \\ -r\sin\theta\cos\varphi & -r\sin\theta\sin\varphi & r\cos\theta \end{vmatrix}$$

$$= \left(rK\cos\varphi\cos\theta, rK\sin\varphi\cos\theta, \overbrace{rK\sin\theta\sin^2\varphi + rK\sin\theta\cos^2\varphi}^{= rK\sin\theta \text{ since } \sin^2\varphi + \cos^2\varphi}\right)$$

$$= rK(\cos\varphi\cos\theta, \sin\varphi\cos\theta, \sin\theta)$$

- $$|\sigma_\varphi \times \sigma_\theta|^2 = r^2K^2(\cos^2\varphi\cos^2\theta + \sin^2\varphi\cos^2\theta + \sin^2\theta)$$

$$= r^2K^2(\cos^2\theta(\cos^2\varphi + \sin^2\varphi) + \sin^2\theta)$$

$$= r^2K^2(\cos^2\theta + \sin^2\theta)$$

$$= r^2K^2$$

$$\therefore \quad dS = r(R + r\cos\theta)\,d\theta\,d\varphi$$

$$\text{area of } S = \iint_S dS = \int_0^{2\pi}\int_0^{2\pi} r(R + r\cos\theta)\,d\theta\,d\varphi$$

$$= \int_0^{2\pi} \left(rR\theta + r^2\sin\theta\right)\Big|_0^{2\pi} d\varphi$$

$$= \int_0^{2\pi} rR(2\pi)\,d\varphi$$

$$= 4\pi^2 rR$$