The University of Manitoba

MATH 3132: Engineering Mathematical Analysis 3
(Winter Term 2020)

Term Test 2
March 12, 2020

Time: 90 minutes

Total Marks: 35

Last Name (IN CAPITAL LETTERS): ______________________________

First Name (IN CAPITAL LETTERS): ______________________________

Student Number: ______________________________

Signature: ______________________________

(I acknowledge that cheating is a serious offense.)

Instructions:
Please ensure that your paper has a total of 7 pages (including this page). Read the questions thoroughly and carefully before answering them. You must **show your work** clearly in order to get any marks for your answers.

You are **not allowed** to use any of the following: calculators, notes, books, dictionaries or electronic communication devices (e.g., cellphones or pagers).

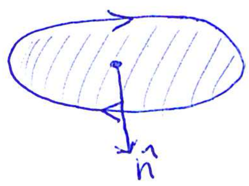
	Obtained	Maximum
Q1		7
Q2		8
Q3		6
Q4		7
Q5		7
Bonus		5
Total		**35**

Q1. [7 pts] Let C be the curve of intersection between $x^2 + y^2 + z^2 = 18$ and $z^2 = x^2 + y^2$ with $z \geq 0$, directed counterclockwise as viewed from the origin. Using Stokes' theorem, evaluate the line integral

$$\oint_C y^2\,dx + xz^3\,dy + x^3\,dz.$$

- The curve of intersection between the sphere and cone is the circle defined by $x^2+y^2=9$, $z=3$.
  (since $x^2+y^2+(x^2+y^2)=18 \Rightarrow x^2+y^2=9 \Rightarrow z^2=x^2+y^2=9$)

- Let S be the surface in $\mathbb{R}^3$ defined by $x^2+y^2 \leq 9$, $z=3$.



Since C is directed as counterclockwise as viewed from the origin, $\hat{n}$ must be $-\hat{k}$.

- By Stokes' theorem, the line integral is equal to

$$\iint_S (\nabla \times F)\cdot \hat{n}\,dS$$

- $$\nabla \times F = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & xz^3 & x^3 \end{pmatrix} = (-3xz^2, -3x^2, z^3-2y)$$

and $(\nabla \times F)\cdot \hat{n} = (-3xz^2, -3x^2, z^3-2y)\cdot(0,0,-1)$

$= 2y - z^3.$

Then $$\iint_S (\nabla\times F)\cdot\hat{n}\,dS = \iint_S 2y - z^3\,dS$$

$$= \iint_{S_{xy}} 2y - 27\,dA$$ since $z=3$, $dA=dS$. $S_{xy} = \{x^2+y^2 \leq 9, z=0\}$

$$= \int_0^{2\pi}\int_0^3 (2r\sin\theta - 27)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left\{\frac{2r^3}{3}\sin\theta - \frac{27r^2}{2}\right\}\Big|_0^3 d\theta$$

$$= \int_0^{2\pi} 18\sin\theta - \frac{243}{2}\,d\theta$$

$$= -\frac{243}{2}\cdot 2\pi$$

$$= -243\pi$$

Q3. Consider the differential equation

$$xy'' + (\sin x)y' + (\tan x)y = 0.$$

(a) [2 pts] Determine all singular points of this differential equation.

$$\frac{Q(x)}{P(x)} = \frac{\sin x}{x} \quad \text{and} \quad \frac{R(x)}{P(x)} = \frac{\tan x}{x}$$

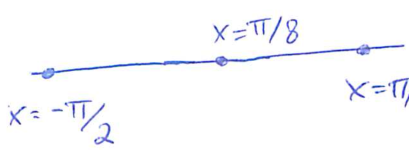
- If we fill-in the removable discontinuity for $\frac{\sin x}{x}$, then the resulting function has a Taylor series expansion at $x=0$.
- Since $\lim_{x\to 0} \frac{\tan x}{x} \overset{LH}{=} \frac{\sec^2 x}{1} = 1$, then $\frac{\tan x}{x}$ has a removable discontinuity. and so we can define a Maclaurin series there.
- Let $x_n = \pi/2 + n\pi$. Since $\lim_{x\to x_n^+} \frac{\tan x}{x} = \infty$, we do not have a Taylor series expansion for $\frac{\tan x}{x}$ at $x = x_n$.

$\therefore$ $\pi/2 + n\pi$ for $n \in \mathbb{Z}$ are the only singular points.

(b) [2 pts] Will there be a general power series solution centered at the point $x = \frac{\pi}{8}$? What can be said about its radius of convergence?

Since $x = \pi/8$ is an ordinary point for the differential equation, we know from a theorem in the text that a general power series solution exists at $\pi/8$.

We can bound the radius of convergence from below by its distance to the closest singular point.



Radius of convergence $\geq |\pi/2 - \pi/8| = 3\pi/8$

(c) [2 pts] Is each singular point regular or irregular? Can we guarantee that there exists a nonzero Frobenius solution at each singular point? Explain.

We can check that $(x-x_n)^2 \frac{R(x)}{P(x)}$ does not have a Taylor series at $x = x_n$.

In fact $\lim_{x\to x_n} (x-x_n)^2 \tan x = \lim_{x\to x_n} \frac{(x-x_n)^2}{1/\tan x} \overset{LH}{=} \lim_{x\to x_n} \frac{2(x-x_n)}{\frac{-\sec^2 x}{\tan^2 x}} = \lim_{x\to x_n} -2(x-x_n)\sin^2 x = 0$

egular singular points
$\Rightarrow$ Frobenius solution exists by a theorem from the text.

So $(x-x_n)^2 \frac{R(x)}{P(x)}$ has a removable discontinuity $\left(\text{Since } \lim_{x\to x_n} \frac{(x-x_n)^2 \tan x}{x} = \frac{\lim_{x\to x_n} (x-x_n)^2 \tan x}{\lim_{x\to x_n} x} = 0\right)$

- By centering the Taylor series for $\tan x$ at $x = x_n$ and multiplying by $(x-x_n)^2$, we can see that $(x-x_n)^2 R/P$ has a Taylor series at $x = x_n$.

In other words, each singular point is regular. This is more work than I originally intended so for full marks, the correct definition and conclusion is needed.

Q2. [8 pts] Use $y(x) = \sum_{n=0}^{\infty} a_n x^n$ to find a general solution to the differential equation

$$y'' + xy' - y = 0.$$

Simplify as much as possible. What is the radius of convergence of the series?

Since the differential equation doesn't have any singular points, then there is a Taylor series solution centered at every point with an infinite radius of convergence.

Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is a solution. Then

$$0 = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

Since the first two terms in the first series are 0, all indices start at $n=0$.

Then $(n+2)(n+1)a_{n+2} + n a_n - a_n = 0$

$$\Rightarrow \quad a_{n+2} = \frac{-(n-1)a_n}{(n+2)(n+1)} \qquad n \geq 0$$

If n is even, $a_2 = \frac{a_0}{2}$, $a_4 = \frac{-a_2}{4 \cdot 3} = \frac{-a_0}{4 \cdot 3 \cdot 2}$, $a_6 = \frac{-3a_4}{6 \cdot 5} = \frac{3a_0}{6!}$

If n is odd, $a_3 = 0$, $a_5 = 0$, ...

To simplify, notice that if n is even, $n = 2k$ and if $k > 1$

$$a_n = \frac{(-1)^{k+1} 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-3)a_0}{n!} = \frac{(-1)^{k+1}(n-3)!\, a_0}{2 \cdot 4 \cdot \ldots \cdot (n-4) \cdot n!}$$

$$= \frac{(-1)^{k+1}(n-3)!\, a_0}{2^{k-2}(k-2)!\, n!}$$

since $(n-4)/2 = (2k-4)/2 = k-2$

Then $$y(x) = a_1 x + a_0\left(1 + \frac{x^2}{2} + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}(2k-3)!}{2^{k-2}(k-2)!(2k)!} x^{2k}\right)$$

Q4. Consider the differential equation

$$x^2y'' + x(x+1)y' - 4y = 0.$$

(a) [1 pt] Is the point $x = 0$ a regular or irregular singular point? Explain.

Since $x\frac{Q(x)}{P(x)} = x+1$ and $x^2\frac{R(x)}{P(x)} = -4$

have Maclaurin series, $x=0$ is a regular singular point.

(b) [3 pts] Find the roots of the indicial equation for a Frobenius solution $\sum_{n=0}^{\infty} a_n x^{n+r}$.

Assume that $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ is a solution. Then

$$0 = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - 4\sum_{n=0}^{\infty} a_n x^{n+r}$$

$$\left(\text{since } \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} = \sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r}\right)$$

$$= [r(r-1)a_0 + ra_0 - 4a_0]x^r + \sum_{n=1}^{\infty} C_n x^{n+r}$$

The indicial equation is $r^2 - r + r - 4 = r^2 - 4 = (r+2)(r-2)$,

So $r = \pm 2$ are the two indicial roots.

(c) [3 pts] Using the Frobenius solution in part (*b*), find a recurrence relation for the coefficients a_n of a nonzero solution which is analytic at $x = 0$. Do NOT actually solve for the a_n.

If we choose $r=2$, then we guarantee that the exponents in $\sum_{n=0}^{\infty} a_n x^{n+r}$ are not negative, providing an analytic solution at $x=0$. Let $r=2$.

Then C_n in part (b) can be written as

$$(n+2)(n+2-1)a_n + (n+2-1)a_{n-1} + (n+2)a_n - 4a_n$$

$$= (n^2+3n+2)a_n + (n+1)a_{n-1} + (n+2)a_n - 4a_n$$

$$= (n^2+4n)a_n + (n+1)a_{n-1}$$

We require that $C_n = 0$ in (b), so we get the recurrence relation

$$a_n = \frac{-(n+1)a_{n-1}}{n(n+4)}, \quad n \geq 1$$

Q5. Consider the periodic function

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 1 & 1 \le x < 2 \end{cases}, \qquad f(x+2) = f(x).$$

$2L = 2$
$\Rightarrow L = 1$

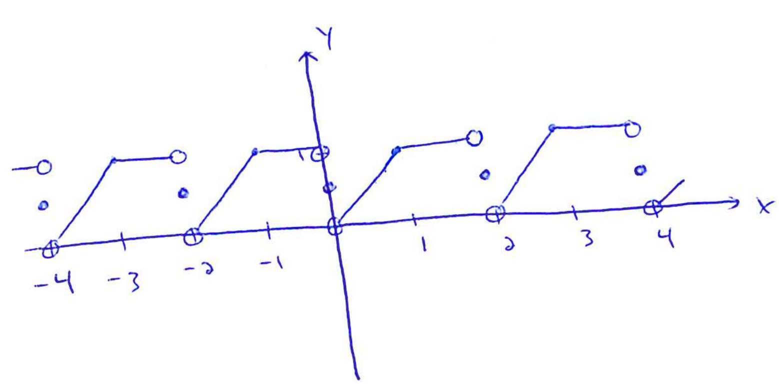
(a) [5 pts] Find the a_n coefficients in the Fourier series for $f(x)$. Do NOT calculate the coefficients b_n for the $\sin(n\pi x/L)$ terms.

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx = \int_0^1 x\,dx + \int_1^2 dx = \frac{x^2}{2}\Big|_0^1 + x\Big|_1^2$$
$$= 3/2$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$
$$= \int_0^1 x\cos(n\pi x)\,dx + \int_1^2 \cos(n\pi x)\,dx$$
$$= x\frac{\sin(n\pi x)}{n\pi}\Big|_0^1 - \int_0^1 \frac{\sin(n\pi x)}{n\pi}\,dx + \frac{\sin(n\pi x)}{n\pi}\Big|_1^2$$
$$= 0 + \frac{\cos(n\pi x)}{n^2\pi^2}\Big|_0^1 + 0$$
$$= \frac{\cos(n\pi) - 1}{n^2\pi^2}$$
$$= \frac{(-1)^n - 1}{n^2\pi^2}$$
$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-2}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

(b) [2 pts] Explain why the Fourier series in part (a) converges for every $x \in \mathbb{R}$. Sketch the function that it converges to.

f is discontinuous at $x_k = 2k$ for $k \in \mathbb{Z}$, and $\lim_{x\to x_k^+} f(x)$ and $\lim_{x \to x_k^+} f(x)$ are both finite. It also easily follows that f is piecewise smooth. Then the Fourier series converges to $\dfrac{f(x+) + f(x-)}{2}$ for every $x \in \mathbb{R}$.



Bonus Question: Let V be the vector space of continuous functions on the interval $[-1, 1]$. Given $f, g \in V$, we can define an inner product on V by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$$

Define a sequence of polynomials $P_n(x)$, where $P_0(x) = 1$, $P_1(x) = x$, and $P_k(x)$ is orthogonal (with respect to the inner product) to $P_0(x), P_1(x), \ldots, P_{k-1}(x)$ for $k > 1$. Each $P_n(x)$ has degree n and satisfies $P_n(1) = 1$.

(a) [2 pts] Compute $P_2(x)$.

Since $P_2(x)$ has degree 2, we can write $P_2(x) = ax^2+bx+c$. Then

$$0 = \langle P_0(x), P_2(x)\rangle = \int_{-1}^{1} ax^2+bx+c\,dx = 2\left(\frac{ax^3}{3} + cx\right)\Big|_0^1 = \tfrac{2}{3}a + 2c$$

$$0 = \langle P_1(x), P_2(x)\rangle = \int_{-1}^{1} x(ax^2+bx+c)dx = 2\left(\frac{bx^2}{2}\right)\Big|_0^1 = b$$

and $P_2(1) = 1$ so $a+b+c=1$. Since $b=0$ we are left with

$a+3c=0$ and $a+c=1$

$\Rightarrow 1-c+3c=0$

$\Rightarrow 2c=-1$

$\Rightarrow c=-1/2$ and $a = 3/2$

$$\therefore P_2(x) = \frac{3x^2-1}{2}$$

(b) [1 pt] Show that the sequence $\{P_n(x)\}_{n=0}^{\infty}$ is not orthonormal.

$$\langle P_1(x), P_1(x)\rangle = \int_{-1}^{1} x^2 dx = \frac{2x^3}{3}\Big|_0^1 = 2/3 \neq 1,$$

So the sequence of P_n are orthogonal, but not orthonormal with respect to the inner product.

(c) [2 pts] We can uniquely write the polynomial $f(x) = x^3 + x + 1$ as the sum

$$f(x) = a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + a_3P_3(x),$$

where $a_i \in \mathbb{R}$. Use the inner product to compute a_1.

Applying the inner product with $P_1(x)$ results in

$$\langle f, P_1\rangle = a_0\langle P_0, P_1\rangle + a_1\langle P_1, P_1\rangle + a_2\langle P_2, P_1\rangle + a_3\langle P_3, P_1\rangle$$

$\Rightarrow \langle f, P_1\rangle = a_1\langle P_1, P_1\rangle$ since $\langle P_k, P_1\rangle = 0$ if $k \neq 1$.

$$\Rightarrow a_1 = \frac{\langle f, P_1\rangle}{\langle P_1, P_1\rangle} = \tfrac{3}{2}\int_{-1}^{1}(x^3+x+1)x\,dx$$

$$= \frac{3}{2}\cdot 2\left(\frac{x^5}{5} + \frac{x^3}{3}\right)\Big|_0^1$$

$$= 3\left(1/5 + 1/3\right)$$

$$= 3\left(\frac{8}{15}\right) = 8/5$$

These polynomials $P_i(x)$ are well-studied and called Legendre polynomials