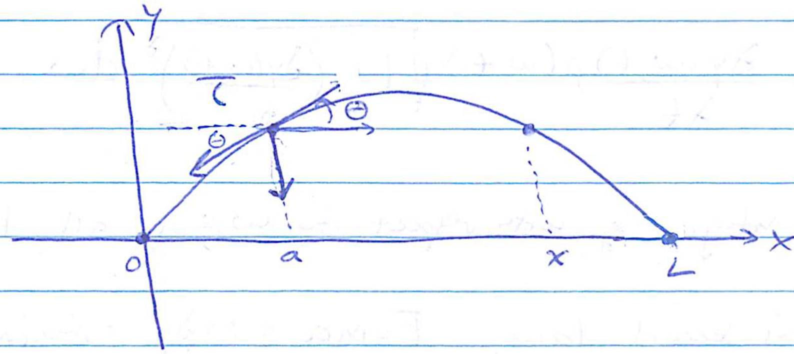
Transverse Vibrations of Strings.

We want to model and solve the following mathematical physics problem: a perfectly flexible string is stretched tightly between $x=0$ and $x=L$. We set the string into motion in the xy-plane by plucking it, and want to study its motion.



- Horizontal displacement is negligible, so we assume only transverse displacements where position $y(x,t)$ is a function of x and time t.

- Looks at the forces involved from a fixed a to a point x. If $\tau(x,t)$ is the magnitude of the tension, then y-component of force due to tension is $\tau \sin\theta$ (Recall that a vector v can be represented by $(|v|\cos\theta, |v|\sin\theta)$)

- If all other forces acting on the segment are grouped into one function $F(x,t)$ (only considering y-components again), then the resultant force is

$$\underbrace{(\tau\sin\theta)\Big|_{x=x} - (\tau\sin\theta)\Big|_{x=a}}_{\text{resultant force from tension}} + \underbrace{\int_a^x F(w,t)\,dw}_{\substack{\text{other forces}\\ \text{(such as gravity)}}}$$

- Our goal now is to relate this quantity to another physical quantity and hope that the resulting relationship provides a useful PDE.

- Momentum at a point can be described as the product of mass and velocity. The latter is just $\frac{\partial y}{\partial t}$, and the former in an infinitesimal length is the product of density $\rho(x,t)$ and $\underbrace{\sqrt{1+\left[\frac{\partial y(\omega,t)}{\partial x}\right]^2}\,d\omega}_{\text{Think of arclength}}$

Then momentum along the segment of string is just

$$\int_a^x \frac{\partial y(\omega,t)}{\partial t}\rho(\omega,t)\sqrt{1+\left(\frac{\partial y(\omega,t)}{\partial x}\right)^2}\,d\omega$$

Note, we are integrating with respect to x and leaving time independent

Now by Newton's second law, $F = ma = m\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}(\text{momentum})$.

Therefore

$$\frac{\partial}{\partial t}\left[\int_a^x \frac{\partial y(\omega,t)}{\partial t}\rho(\omega,t)\sqrt{1+\left(\frac{\partial y(\omega,t)}{\partial x}\right)^2}\,d\omega\right]$$

$$= (\tau\sin\theta)\Big|_{x=x} - (\tau\sin\theta)\Big|_{x=a} + \int_a^x F(\omega,t)\,d\omega.$$

Then differentiate everything with respect to x (using the FTC) and we get

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial y}{\partial t}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\right) = \underbrace{\frac{\partial}{\partial x}(\tau\sin\theta)}_{\text{since the antiderivative can be computed at endpoints by FTC.}} + F(x,t)$$

If we assume that θ is small, then we can take $\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} \approx 1$

and $\frac{\partial y}{\partial x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \approx \sin\theta$. For most applications, density and tension are constant with the relationship $c^2 = \tau/\rho$.

The equation reduces to

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + \frac{F(x,t)}{\rho}$$

which is the one-dimensional wave equation, which models small transverse vibrations of a taut string.

$F(x,t)$ could be given by

$-\rho g$	$-\beta \frac{\partial y}{\partial t}$, $\beta > 0$	$-ky$, $k>0$
gravity	damping force proportional to velocity	restoring force proportional to displacement.

Initial conditions describe displacement and velocity at some initial time (usually $t=0$):

$$y(x,0) = f(x) \quad , \quad \frac{\partial y}{\partial t}(x,0) = y_t(x,0) = g(x) \qquad x \in I$$

where I is the interval where the string is stretched (usually $I = [0,L]$)

Boundary conditions can be specified by <u>Dirichlet</u> boundary conditions and <u>Neumann</u> boundary conditions. The first specifies the values that a solution must take along the boundary of the domain for the problem. The second specifies the values that the derivative of a solution must take along the boundary.

Dirichet conditions: $y(0,t) = f_1(t)$, $y(L,t) = f_2(t)$, $t > 0$

Neumann conditions: $\frac{\partial y}{\partial x}(0,t) = g_1(t)$, $\frac{\partial y}{\partial x}(L,t) = g_2(t)$, $t > 0$

See the text for an example of a possible g_1, g_2.

Example 1: Formulate the initial boundary value problem for transverse vibrations of a string stretched tightly along the x-axis between $x=0$ and $x=L$, two fixed points. The string is initially at rest along the x-axis, and is then allowed to drop under its own weight.

- Here $F(x,t) = -\rho g$ so the PDE is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - g, \qquad 0 < x < L, \qquad t > 0,$$

- At $t=0$, the string starts at the x-axis and is not moving so $y(x,0) = 0 = y_t(x,0) \quad 0 < x < L$.

- The physical endpoints do not change for the problem, so

$$y(0,t) = 0 = y(L,t), \qquad t > 0.$$

Let us now solve the initial boundary value problem

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \qquad 0<x<L, \quad t>0$$

$$\begin{aligned} y(0,t) &= 0 = y(L,t) && t>0 \\ y(x,0) &= f(x), && 0<x<L \\ y_t(x,0) &= 0, && 0<x<L \end{aligned}$$

Assume there is a solution of the form $y(x,t) = X(x)T(t)$.

Then $$\frac{\partial^2 T}{\partial t^2} X = c^2 \frac{\partial^2 X}{\partial x^2} T$$

$$\Rightarrow \frac{\partial^2 T}{\partial t^2}\left(\frac{1}{c^2 T}\right) = \frac{\partial^2 X}{\partial x^2}\frac{1}{X} \Rightarrow \begin{array}{l} X'' + \lambda X = 0 \\ T'' + \lambda c^2 T = 0 \end{array}$$

using the separation principle with $-\lambda$.

• Since $y(0,t)=0 \Rightarrow X(0)T(t)=0 \quad t>0$

We cannot have $T(t)\equiv 0$ (otherwise the solution $y(x,t)=0$ and the string would not be moving), so $X(0)=0$.
Similarly, the other initial/boundary conditions imply $X(L)=0$ and $T'(0)=0$. We then have

$$X''+\lambda X=0, \quad 0<x<L \qquad \text{and} \qquad T''+\lambda c^2 T=0, \quad t>0$$
$$X(0)=0 \qquad\qquad T'(0)=0$$
$$X(L)=0$$

Note: Nonhomogeneous conditions are not considered when separating variables since $X(x)T(0)=f(x)$ provides no information about $X(x)$ or $T(t)$ separately.

Using Section 19.2, we can apply our knowledge of Sturm-Liouville systems to solve the above.

For the first system, eigenvalues are $\lambda_n = \frac{n^2\pi^2}{L}$ with $y_n(x)=\sin\left(\frac{n\pi x}{L}\right)$

For the second, $m^2+\frac{n^2\pi^2c^2}{L^2}=0 \Rightarrow m=\pm\frac{n\pi c i}{L}$

using $\lambda=\lambda_n$ from the first ODE. A general solution is then
$$T(t)=F\cos\left(\frac{n\pi ct}{L}\right)+G\sin\left(\frac{n\pi ct}{L}\right)$$
(using our knowledge of solutions to homogeneous linear ODEs).

and $T'(0)=0 \Rightarrow G=0$

Then $y(x,t)=X(x)T(t)=b\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$

where $b\in\mathbb{R}$, $n\in\mathbb{Z}$. This solves the original system except for the condition $y(x,0)=f(x)$.

$\Rightarrow \quad X(x)T(0) = f(x)$

$\Rightarrow \quad b\sin\left(\frac{n\pi x}{L}\right) = f(x) \qquad$ since $\cos\left(\frac{n\pi c(0)}{L}\right) = 1$

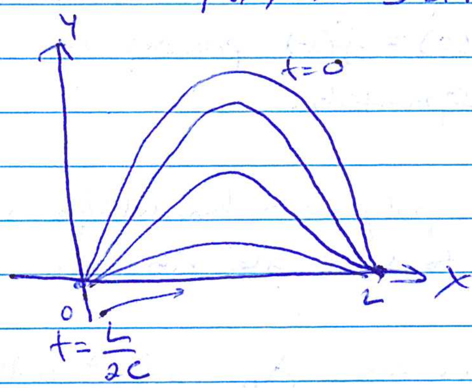
So b and n must satisfy this equation.

For example, if $f(x) = 3\sin\left(\frac{\pi x}{L}\right)$, then

$$3\sin\left(\frac{\pi x}{L}\right) = b\sin\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad 0<x<L$$

so we can take $b=3$ and $n=1$. The solution is then

$$y(x,t) = 3\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi c t}{L}\right)$$



- If instead $f(x) = 3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{2\pi x}{L}\right)$, then it is clear that there is no n and b where

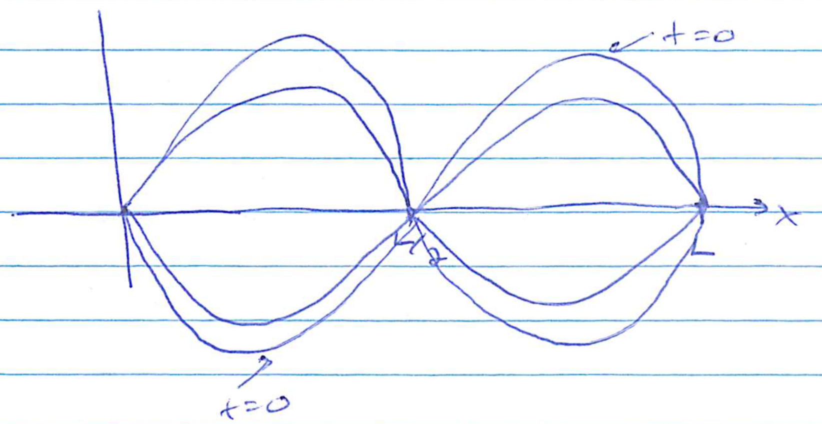
$$3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{2\pi x}{L}\right) = b\sin\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad 0<x<L$$

This means that the solution is not separable. However, we can use superposition of separable solutions, even if the sum is not separable. For example,

$$y(x,t) = b\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right) + d\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{m\pi c t}{L}\right)$$

is a solution to the homogeneous system but is not separable.

Equating with $f(x)$ we get $b=3$, $d=-1$, $n=1$ and $m=2$ which gives the solution to initial boundary value problem



This is a solution with more than one mode.

Finally, look at the example $f(x) = x(L-x)$ in your text, to see an example where an infinite sum is required to find a solution to the initial boundary value problem.

Try some of the exercises from 20.2 and 21.2 Part B to gain comfort with setting up the initial boundary value problem and solving it.