RESEARCH STATEMENT

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In its simplest form, algebraic geometry studies objects which arise as the solutions to polynomial equations. It is a broad subject, having been generalized considerably in the 20th century using the language of schemes. Algebraic geometry appears in theoretical and particle physics, through Calabi-Yau manifolds and amplituhedrons. It has connections to real and complex analysis with correspondences like Serre's GAGA or Artin approximation, and it intersects differential geometry in the study of Kähler manifolds. The use of elliptic curves in number theory and cryptography is also well-known.

Imagine being able to take something as diverse in its applications as an algebraic variety, and reading off its properties from a combinatorial object, like a graph or counting function. Suppose that we could take complex systems, both abstract and applied, and be able to visualize them with fans, polytopes or posets. This is the versatility of algebraic combinatorics, and I am motivated by any question which can be simplified in this way. More specifically, my interests lie in problems which can be reduced to combinatorics or commutative algebra, and often involves varieties that carry a large amount of symmetry (through group actions for example). In the past this has involved Schubert and toric varieties, but more recently has included ideals associated to graphs, K-orbit closures and Hessenberg varieties.

The general goal of my research is to uncover the underlying structure of geometric objects using computational tools and techniques from algebraic combinatorics and commutative algebra. These methods have proven successful in other contexts and offer a powerful way to simplify complicated questions. For instance, algebro-geometric properties of Schubert varieties can often be checked on their defining permutations, and classes in the cohomology or K-theory of flag varieties can be computed using pipe dreams or Knutson-Tao puzzles. I am interested in applying similar techniques to less explored families of varieties, using broad tools that include Frobenius splittings, Gröbner degeneration, and geometric vertex decomposition.

My recent work showcases some of these applications with toric ideals of graphs, Hessenberg varieties, and K-orbit closures. When I study a new problem, I ask: Does this object possess a useful flat degeneration which summarizes key data? Can I decompose this degeneration into simpler ones which respects certain geometric limits? Can I relate this object to a flag variety, and can I resolve its singularities using a Frobenius split smooth scheme? I also enjoy problems which bridge these themes with other areas of mathematics. My previous research has applied some of these ideas to the resolution of singularities, tensor decomposition, algebraic statistics and cryptography. While my work has the specificity to focus on concrete programs like the classification of Hessenberg varieties with a given property, it also has the flexibility to be applied to other areas.

In the coming years, I hope to expand the limits of what is possible with these strategies. While partial results exist for toric ideals of graphs and for Hessenberg varieties, much work remains. Many of these projects provide multiple opportunities for future research, including student research. I have co-supervised several undergraduate projects, especially while at McMaster, all of which are outlined in my CV. Most of the research detailed below is conducive to smaller-scale problems appropriate for student work.

1. Geometric Vertex Decomposition

Geometric vertex decomposition was first introduced by Knutson-Miller-Yong in [13] to study diagonal degenerations of Schubert varieties. Later results on the topic were mostly formulated in the context of Schubert geometry, until very recent work of Klein-Rajchgot in [10] established a connection between liaison theory and geometric vertex decomposition. For homogeneous Cohen-Macaulay ideals, being geometric vertex decomposable is in some sense equivalent to being *glicci* (i.e. is in the Gorenstein liaison class of a complete intersection). The interplay between these two theories can be used to analyze degenerations, construct Gröbner bases, and check unmixedness.

I began working with geometric vertex decomposition as a computational tool over the last year. This includes joint work with Megumi Harada on Hessenberg varieties, and a collaboration between Michael Cummings, Jenna Rajchgot and Adam Van Tuyl on toric ideals of graphs. Each project demonstrates a different use of geometric vertex decomposition, and is detailed in the subsections that follow. Let us first fix notation and introduce important definitions.

Let $R = k[x_1, \dots, x_n]$ and fix a variable $y = x_i$. For any $f \in R$, we can write $f = \sum_i \alpha_i y^i$ with $\alpha_i \neq 0$, and define $\mathrm{init}_y(f)$ to be the coefficient of the highest power of y^i appearing in the sum. For any ideal I of R, we set $\mathrm{init}_y(I)$ to be the ideal generated by all the $\mathrm{init}_y(f)$ for $f \in I$. Let $<$ be a monomial ordering on R such that $\mathrm{init}_<(f) = \mathrm{init}_<(\mathrm{init}_y(f))$ for all $f \in R$, in which case we say that $<$ is y-compatible. Suppose that $\mathcal{G}(I) = \{g_1, \dots, g_m\}$ is a the Gröbner basis of I with respect to $<$. Write each g_i as $g_i = y^{d_i} q_i + r_i$, where y does not divide any term of q_i. Then define two ideals:

$$C_{y,I} = \langle q_1, \dots, q_m \rangle \text{ and } N_{y,I} = \langle q_i \mid d_i = 0 \rangle.$$

An ideal I of $R = k[x_1, \dots, x_n]$ is *geometrically vertex decomposable* (or GVD for short) if I is unmixed and

1. (1) $I = \langle 1 \rangle$, or I is generated by a (possibly empty) subset of variables of R, or
2. (2) there is a variable $y = x_i$ in R and a y-compatible monomial ordering $<$ such that

$$\mathrm{init}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle),$$

and the contractions of $C_{y,I}$ and $N_{y,I}$ to the ring $k[x_1, \dots, \hat{x}_i, \dots, x_n]$ are geometrically vertex decomposable.

There are two principal uses of a GVD that I would like to highlight:

- If a Gröbner basis is known for I, then we can select a variable y and compute $N_{y,I}$ and $C_{y,I}$ directly. If these ideals have a similar structure to I, then we can continue the process to inductively find a GVD. If I is homogeneous, then the existence of a GVD implies that R/I is Cohen-Macaulay, and I is glicci.
- If a Gröbner basis is *not* known for a homogeneous I, then liaison theory can be used to construct a GVD and Gröbner basis for I. Theorem 6.1 from [10] constructs a GVD for I from C and N using a certain R/N-module isomorphism $\varphi : C/N \to I/N$. Furthermore, Gröbner bases for C and N can be extended to a Gröbner basis for I.

1.1. **Toric ideals of graphs.** The basic definitions and properties of toric ideals of graphs can be found in [14]. Given a finite simple graph G, we can define a universal Gröbner basis for I_G using generators which correspond to primitive closed even walks in G. My joint work in the subject included results on the interaction between geometric vertex decomposition and other standard operations from commutative algebra. For example, an ideal being GVD behaves well with respect to ideal extensions and tensor products, allowing us to assume that G is connected.

Lemma 1.1. *Suppose that $I \subseteq R = k[x_1, \dots, x_n]$ is geometrically vertex decomposable. Then the extension of I in $R' = k[x_1, \dots, x_n, x_{n+1}]$, i.e., IR', is also geometrically vertex decomposable.*

Lemma 1.2. *Let $I \subseteq R$ and $J \subseteq S$ be proper ideals. Then I and J are geometrically vertex decomposable if and only if $(I + J)$ is geometrically vertex decomposable in $R \otimes S$.*

There is a large family of toric ideals of graphs which we showed are GVD. These geometric vertex decompositions are also compatible with a vertex decomposition for the Stanley-Reisner complex associated to $\mathrm{init}_<(I_G)$.

Theorem 1.3. *Let I_G be a toric ideal of a finite simple graph G. If there exists a lexicographic ordering $<$ on R such that $\mathrm{init}_<(I_G)$ is squarefree, then I_G is geometric vertex decomposable.*

By [1, Theorem 3.9], all gap-free graphs satisfy the conditions of the theorem. As another corollary, any I_G whose universal Gröbner basis is quadratic will automatically be GVD.

1.2. **Building Gröbner bases for Hessenberg varieties.** Let n be a positive integer. We call a function $h : [n] := \{1, 2, \dots, n\} \to [n] := \{1, 2, \dots, n\}$ an *indecomposable Hessenberg function* if it satisfies the conditions $h(i) \geq i + 1$ for all i and $h(i + 1) \geq h(i)$ for $1 \leq i \leq n - 1$. If $A : \mathbb{C}^n \to \mathbb{C}^n$ is a linear operator, then the *Hessenberg variety associated to A and h* is the subvariety of $\mathrm{Flags}(\mathbb{C}^n)$ defined by

$$\mathrm{Hess}(A, h) := \{V_\bullet \in \mathrm{Flags}(\mathbb{C}^n) | AV_i \subseteq V_{h(i)}, \forall i\}.$$

Reworking certain results about Schubert varieties for Hessenberg varieties first requires a well-chosen Gröbner basis. Geometric vertex decomposition proved to be the best tool to build a Gröbner basis for this purpose. In my joint work with Megumi Harada, the Lie type A case for regular nilpotent operators was studied. We adapted many of the results from [10] to homogeneous ideals with a non-standard grading:

Lemma 1.4. *Suppose we are in the situation of* [10, Theorem 6.1]. *Suppose in addition that the ideals N, C and I are homogeneous with respect to some integer grading on R. Assume that the sets $\{q_1, \dots, q_k, h_1, \dots, h_\ell\}$ and $\{h_1, \dots, h_\ell\}$ define Gröbner bases for C and N respectively, with respect to the y-compatible monomial order $<$. Suppose r_i for $1 \leq i \leq k$ are polynomials in R which do not contain any y's such that $yq_i + r_i \in I$. Then $\{yq_1 + r_1, \dots, yq_k + r_k, h_1, \dots, h_\ell\}$ is a Gröbner basis for I with respect to $<$.*

We denote the ideal defining $\mathrm{Hess}(A, h)$ in the w-chart $\mathcal{N}_w$ by $I_{w,h}$. Certain generators $f^w_{i,j}$ for $I_{w,h}$ were defined in [2]. Two important properties hold when we restrict to the w_0-chart.

Theorem 1.5. *The ideal $I_{w_0,h}$ is geometric vertex decomposable. Furthermore, the generators $f^{w_0}_{i,j}$ are a Gröbner basis for $I_{w_0,h}$.*

Together with Michael Cummings and Jenna Rajchgot, we are continuing this analysis in the other charts. Preliminary data suggests that the only chart for which all $I_{w,h}$ are GVD for any indecomposable Hessenberg function h is the w_0-chart.

2. FROBENIUS SPLITTINGS

The notion of a Frobenius splitting has been used in various ways, from studying F-purity in singularity theory to checking reducedness in algebraic combinatorics. For example, the B-canonical Frobenius splitting of the flag variety G/B is used in [5] to prove that Schubert varieties are normal and Cohen-Macaulay. Allen Knutson used the Frobenius splitting as a tool for analyzing degenerations in [11] and [12].

My first use of Frobenius splittings was in my PhD thesis [6] on the Gorensteinization for Schubert varieties. I showed that the blow-up of a Schubert variety X^w along its boundary divisor ∂X^w is Gorenstein. This is a result of a corresponding blow-up of the subword complex along its boundary in a local degeneration of X^w. The Frobenius splitting of X^w is used in a key way to define a "boundary divisor" for the blow-up.

Frobenius splitting made subsequent appearances in my postdoctoral work, including joint work with Jenna Rajchgot on K-orbit closures and determinantal facet ideals. Some of these projects are summarized in the subsections below. As before, let us start by fixing notation and introducing definitions.

Let R be a commutative ring. One way to check whether R is reduced is if the map $x \to x^n$ only sends 0 to 0 for $n > 0$. If n is prime and R contains the field $\mathbb{F}_p$, then the map is linear and we can write this condition as $\ker(x \to x^n) = 0$. Then R being reduced implies that there exists a one-sided inverse to the Frobenius morphism. This motivates the next definition (from [5]).

A *Frobenius splitting* of an $\mathbb{F}_p$-algebra R is a map $\varphi : R \to R$ which satisfies:

(1) $\varphi(a+b) = \varphi(a) + \varphi(b)$
(2) $\varphi(a^p b) = a\varphi(b)$
(3) $\varphi(1) = 1$

We say that an ideal $I \subset R$ is *compatibly split* if $\varphi(I) \subset I$. Such ideals have useful properties, detailed more fully in [11]. For example, $I + J$ and $I \cap J$ are compatibly split if I and J are.

This definition can be generalized to schemes. Let X be a separated scheme of finite type over an algebraically closed field of characteristic $p > 0$. The absolute Frobenius morphism $F_X : X \to X$ is the identity map on X and the p-th power map on $\mathcal{O}_X$. We say that X is Frobenius split if the $\mathcal{O}_X$-linear map $F^\# : \mathcal{O}_X \to F_*\mathcal{O}_X$ splits. Frobenius split schemes are reduced and all smooth affine varieties can be split (see [5, Theorems 1.1.6 and 1.2.1]).

2.1. **Frobenius splittings of K-orbit closures.**

Let G be a simple complex Lie group, and fix a maximal torus T as well as a Borel subgroup B containing T. Let K be a symmetric subgroup of G defined as the fixed points of some holomorphic involution of G. We can study the K-orbit closures in G/B (or the B-orbit closures in G/K in a similar process), and compare the results to the known results for Schubert varieties (which are the B-orbit closures in G/B). There are some noticeable differences with the two cases. While Schubert varieties are always Cohen-Macaulay and normal, the same is not true for K-orbit closures in general. Additionally, in most cases there are multiple closed K-orbits, and these closed orbits need not have dimension 0. See [8] for more details.

There is a natural generalization of a Bott-Samelson variety that can be used to resolve singularities for K-orbit closures. A Barbasch-Evens-Magyar variety $\mathcal{BEM}^{Y_0,Q}$ can be described using a sequence of flags, just as in the Bott-Samelson resolution, except that the base flag is not the standard flag. Here Q is a list of simple reflections and Y_0 a closed K-orbit. We can show a number of results:

Theorem 2.1. *Any $\mathcal{BEM}^{Y_0,Q}$ is Frobenius split.*

Corollary 2.2. *Any normal K-orbit closure is compatibly split by some Frobenius splitting of G/B induced from $\mathcal{BEM}^{Y_0,Q} \to G/B$.*

There generally isn't a Frobenius splitting which simultaneously compatibly splits all K-orbit closures. In the case $G = GL_{a+b}$ and $K = GL_a \times GL_b$, we also have an explicit description of the components of the anticanonical divisor defined by the splitting.

2.2. **Resolution of singularities in positive characteristic.** Can Frobenius splittings be used to resolve singularities? A polynomial which defines a Frobenius splitting of $\mathbb{A}^n$ has a particular local normal form. Its order of vanishing is less than p, so a center of blowup can be defined using a local desingularization invariant $\mathrm{inv}_X(a)$ (as in [4]). Under certain conditions, the Frobenius splitting extends to the strict transform, producing ideal conditions for adapting characteristic 0 techniques to the positive characteristic setting.

The centers used must have a certain bounded codimension. This bound would allow for smoothness to be achieved for most curves and surfaces, and could provide an algorithm to simplify F-pure singularities in higher dimensions. A desingularization of the compatibly split varieties is also possible.

Theorem 2.3. *Suppose $f \in k[x_1, \ldots, x_n]$ defines a Frobenius splitting of $\mathbb{A}^n_k$ by $\varphi = \mathrm{Tr}(f^{p-1}\bullet)$ and let $a \in X = V(f)$. Then $\mathrm{inv}_X(a)$ is well-defined for the following cases:*

- $n = 2, p > 0$
- $n = 3, p \neq 3, 5$
- $n > 3, p = 2$

Going forward, it would be interesting to explore the following question: Does there exist a Frobenius split representative for any birational equivalence class such that desingularization in positive characteristic can be reduced to studying splittings?

2.3. **Determinantal facet ideals.** Let $X = (x_{i,j})$ be an $m \times n$ matrix of indeterminates where $m \leq n$, with $S = \mathbb{C}[X]$ the polynomial ring on those indeterminates. Let Δ be an $(m-1)$-dimensional pure simplicial complex on the vertex set $[n] = \{1, \ldots, n\}$ defined as follows: each facet $F = \{a_1 \leq a_2 \leq \cdots \leq a_m\}$ of Δ corresponds to a maximal minor of X denoted $\mu_F = [a_1, a_2, \ldots, a_m]$ obtained by taking the determinant of the submatrix of X with columns $a_1, a_2, \ldots, a_m$. Then define the *determinantal facet ideal* of Δ to be the ideal

$$J_\Delta = (\mu_F : F \in \mathcal{F}_\Delta),$$

where $\mathcal{F}_\Delta$ is the set of facets of Δ. These determinantal facet ideals have appeared in questions from algebraic statistics and combinatorial algebra. See [7] and [3] for more information.

Together with Ryan Edwards and Jenna Rajchgot, we showed that all J_Δ for semi-closed Δ are in fact Kazhdan-Lusztig varieties in the Grassmannian. As a result, the components in a primary decomposition for J_Δ must also be Kazhdan-Lusztig varieties which can be read off using the Bruhat poset. This translation allows us to import known results from the study of Schubert varieties to determinantal facet ideals. For instance, the Frobenius splitting on Kazhdan-Lusztig varieties shows that there is an ordering $<$ such that $\mathrm{init}(J_\Delta)$ is squarefree. Furthermore, S/J_Δ is F-pure. This offers a simplified proof of the main theorem from [3, Theorem 77].

3. Future Research

There are a number of natural extensions of the themes from the previous sections. Hessenberg varieties have already provided a fruitful testing ground for applying techniques from Schubert geometry to a less symmetric setting. Varying the Hessenberg function h, the linear operator A, or the permutation in $\mathcal{N}_w$ provides a diverse assortment of varieties that exhibit different behaviours. Even when A is nilpotent, recent research suggests that there are number of interesting variations in the remaining charts (from not GVD, to GVD without a fixed monomial ordering etc.). Characterizing when the $f_{i,j}^w$ form a Gröbner basis, when $I_{w,h}$ is GVD, and when $\mathrm{init}(I_{w_0,h})$ is vertex decomposable are some of the questions that still need to be answered.

Question 3.1. *Given a Hessenberg function h, and linear operator A, and a permutation w, find combinatorial rules that determine when $\mathrm{Hess}(A,h) \cap \mathcal{N}_w$ is geometric vertex decomposable or has a squarefree degeneration. Furthermore, when do the natural generators $f_{i,j}^w$ form a Gröbner basis?*

There are also a number of paths forward in the study of toric ideals of graphs. I helped show that toric ideals of graphs that possess a lexicographic squarefree degeneration are geometric vertex decomposable. It is known that these ideals are also Cohen-Macaulay. Put together, we have that these toric ideals of graphs are glicci. A central question in liaison theory is whether *every arithmetically Cohen-Macaulay subscheme of $\mathbb{P}^n$ is glicci*, and this result is one step forward towards a complete answer for toric ideals of graphs.

Question 3.2. *Let G be a finite simple graph which defines a Cohen-Macaulay toric ideal I_G. Is I_G geometric vertex decomposable? Is it glicci?*

Using geometric vertex decomposability has already simplified the techniques used in liaison theory. For example, the proof of results in [9] can be cut down by answering the previous question for bipartite graphs G.

References

[1] A. D'Alì, *Toric ideals associated with gap-free graphs*, J. of Pure and Applied Algebra 219 (2015), 3862–3872.

[2] H. Abe, L. DeDieu, F. Galetto and M. Harada, *Geometry of Hessenberg varieties with applications to Newton-Okounkov bodies*, Selecta Mathematica 24 (2018), 2129-2163.

[3] B.Benedetti, L. Seccia, and M. Varbaro, *Hamiltonian paths, closed complexes, and determinantal facet ideals*, (2021) arXiv: 2101.09243.

[4] E. Bierstone and P.D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. 128 (1997), 207–302.

[5] M. Brion, and S. Kumar, *Frobenius Splitting Methods in Geometry and Representation Theory*, Birkhäuser (2005).

[6] S. Da Silva, *On the Gorensteinization of Schubert varieties via Boundary Divisors* , (2018), arXiv: 1807.05416.

[7] V. Ene, J. Herzog, T. Hibi, F. Mohammadi, *Determinantal Facet Ideals*, Michigan Math J. 62 (2011).

[8] L. Escobar, B. Wyser and A. Yong, *K-orbit closures and Barbasch-Evens-Magyar varieties*, (2017), arXiv: 1708.06663.

[9] E. Gorla and J.C. Migliore and U. Nagel, *Gröbner bases via linkage*, J. Algebra 384 (2013), 110-134.

[10] P. Klein and J. Rajchgot, *Geometric Vertex Decomposition and Liaison*, (2020), arXiv: 2005.14289.

[11] A. Knutson, *Frobenius splitting, point-counting, and degeneration*, (2009), arxiv: 0911.4941.

[12] A. Knutson, *Schubert patches degenerate to subword complexes*, (2008), arXiv: 0801.4114.

[13] A. Knutson, E. Miller, A. Yong, *Gröbner geometry of vertex decompositions and of flagged tableaux*, Journal für die Reine und Angewandte Mathematik 630 (2009), 1–31.

[14] Bernd Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, 8. AMS (1996).