ORBIFOLDS OF NON POSITIVE CURVATURE AND THEIR LOOP SPACE

By

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a project entitled "**Orbifolds of non positive curvature and their loop space**" by **George Dragomir** in partial fulfillment of the requirements for the degree of **Master of Science**.

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Chapter 1

Introduction

The notion of orbifold was first introduced by Satake in [**?**] under the name of V-manifold. Later it was rediscovered by Thurston in [T] as a tool for studying the topology of 3-manifolds and it is then when the term of orbifold was coined. The point of view of their definition was that the orbifold is a natural generalization of that of manifold by allowing some mild singularities. They are locally modeled by open sets in $\mathbb{R}^n$ modulo an action of some finite group of homeomorphisms or diffeomorphisms. Moreover the group is not fixed and can be changed as we pass from one point of the orbifold to another. An isomorphism of coordinate neighborhoods corresponds to equivariant actions of the same group on $\mathbb{R}^n$. A difference between their definitions consists in that Satake required the group action to have a fixed point set of codimension at least two while Thurston did not. Thurston's definition of orbifold allows group actions such as reflections through hyperplanes. If such a requirement is made, the orbifold is often referred as a *codimension two* orbifold. However, both of them used only faithful actions to define their orbifolds which corresponds to the so called now reduced orbifolds. This intuitive restriction appears to be very unnatural if for instance we want to study suborbifolds.

This perspective was basically motivated by the fact that orbifolds often arise as quotient spaces of manifolds by proper actions of discrete groups. However, orbifolds often occur in other branches of mathematics. For instance, in algebraic geometry, it was developed the concept of stack in order to deal with moduli problems. As it happens orbifolds arise quite naturally from the very same moduli problems and didn't take long to realize that the theory of stacks provide another way of understanding the category of orbifolds (and vice-versa). For example, the Deligne-Mumford moduli stack $\mathscr{M}_g$ for genus g curves is in fact an orbifold. So a reason for the importance of studying orbifolds is that many moduli spaces are better understand as orbifolds. This also motivates the modern approach of studying orbifolds, where the underlying idea is that an orbifold is best understood as a stack. As a manifold is completely determined by an open cover and the corresponding gluing maps, in the same way a stack will be completely determined by a groupoid representing it. There may be more that one such groupoid, but we use the notion of Morita equivalence to deal with this. In [**?**] the authors describe the procedure to go from an orbifold to a stack in a such way that the category of orbifolds constructed in chapter 2 turns out to be a full subcategory of the category of stacks; and also the procedure of going from a stack to a groupoid, producing again an embedding of categories. However there is a direct way of passing directly from an orbifold to a groupoid as we will see in chapter 3.

A motivation for the groupoid approach to the theory of orbifolds comes also from the theory of foliations where the groupoids play a fundamental role as they provide the main tool of studying the space of leaves, by means of the holonomy groupoid of the foliation. We recommend the Mooerdijk, haefliger... for a detailed exposure of this issue.

We will refer to this perspective as the traditional approach of the theory of orbifolds.

Chapter 2

Orbifolds

2.1 Group actions

In this section we briefly recall some facts from the theory of group actions on a topological or smooth manifold and we will mainly focus on the *properly discontinuous* actions (see below and also Proposition 2.1.2). A useful special case of a discontinuous action is the action of a finite group on a Hausdorff topological space and as we will see in the next section this will play an important role in understanding local properties of orbifolds. For a more detailed introduction to actions of discrete groups the reader may consult [B] and also [T1].

We will begin with some formal definitions. Suppose Γ is a group and M is a topological space or a smooth manifold.

Definition 2.1.1. *An* action *of* Γ *on* M *is a map* $\Gamma \times M \to M$, $(\gamma, x) \mapsto \gamma.x$ *such that, for all* $x \in M$

(i) $(\gamma \cdot \delta).x = \gamma.(\delta.x)$ *for all* $\gamma, \delta \in \Gamma$, *and*

(ii) $1.x = x$, *where* $1 \in \Gamma$ *is the identity element.*

Thus the first rule says that the identity of the group acts as identity, and the second rule says that two elements of Γ, acting successively, act as the product of two elements.

There are some standard notions associated with such an action. For a point $x \in M$, the set $\Gamma(x) = \{\gamma.x |\ \gamma \in \Gamma\} \subseteq M$ is called the *orbit* of x. We can introduce a relation on M by $x \sim y$ if and only if x and y are on the same orbit. It is easy to check that this is an equivalence relation. The space of equivalence classes of points of M will be called the *space of orbits* and will be denoted M/Γ (we agree to denote it like this even if the action here is considered from the right).

The elements of Γ which leave an element x fixed by their action form a subgroup $\Gamma_x = \{\gamma \in \Gamma |\ \gamma.x = x\}$ called the *isotropy group* at x. It is easy to see that if x and y are on the same orbit, say $y = \gamma.x$, then their isotropy groups are conjugate, $\Gamma_y = \gamma.\Gamma_x.\gamma^{-1}$, and in fact any conjugate subgroup to Γ_x occurs as an isotropy group Γ_y to some element $y \in \Gamma(x)$. If $\Gamma_x = \Gamma$, then x is said to be a *fixed point* of the action. The set of fixed points of the action is often denoted M^Γ. A subset $N \subset M$ is called Γ-*invariant* if it is fixed by the action of Γ, i.e. $\gamma.N = N$ for every $\gamma \in \Gamma$ (or with the notation above if $N \subset M^\Gamma$).

We say that Γ acts by *homeomorphisms* (*diffeomorphisms*) if there is a homomorphism $\rho : \Gamma \to \mathrm{Homeo}(M)$ ($\mathrm{Diffeo}(M)$), where $\mathrm{Homeo}(M)$ ($\mathrm{Diffeo}(M)$) denotes the group of homeomorphisms (diffeomorphisms) of M with the group law given by composition of maps. In what follows we will consider this kind of actions.

Here are some basic properties of group actions:

(i) The action of Γ on M is called *effective* if no element of the group, besides the identity element, fixes all the elements of the space, or equivalently if

$$\bigcap_{x \in M} \Gamma_x = \{1\}.$$

In this case the representation $\rho : \Gamma \to \mathrm{Homeo}(M)(\mathrm{Diffeo}(M))$ is faithful and we can regard Γ as a group of homeomorphisms (diffeomorphisms).

(ii) The action of Γ on M is called *free* if no point of M is fixed by an element of Γ other than the identity, or equivalently if the map

$$\Gamma \times M \to M \times M,\ (\gamma, x) \mapsto (\gamma.x, x) \text{ is injective}$$

such that $\gamma.x = x$ implies $\gamma = 1$ for any γ and any x.

(iii) The action of Γ on M is called *discrete* if Γ is a discrete subgroup of the group of homeomorphisms (diffeomorphisms), with the compact-open topology.

(iv) The action of Γ on M is said to have *discrete orbits* if every $x \in M$ has a neighborhood U such that the set $\{\gamma \in \Gamma |\ \gamma.x \in U\}$ is finite.

(v) The action of Γ on M is called *discontinuous* if every $x \in M$ has a neighborhood U such that the set $\{\gamma \in \Gamma |\ \gamma.U \cap U \neq \varnothing\}$ is finite.

(vi) Assume now that M is locally compact. The action of Γ on M is called *properly discontinuous* if for any compact sets $K_{1,2}$ in M, the set $\{\gamma \in \Gamma |\ \gamma.K_1 \cap K_2 \neq \varnothing\}$ is finite, or equivalently if the map

$$\Gamma \times M \to M \times M,\ (\gamma, x) \mapsto (\gamma.x, x) \text{ is proper.}$$

Recall that a map is proper if the preimages of compact sets are compact. Here Γ is assumed endowed with the discrete topology. Recall also that a proper map between locally compact Hausdorff spaces is closed.

Note that on a locally compact space any properly discontinuous action is discontinuous and any discontinuous action has discrete orbits, but the converses are not true in general.

The following characterization of properly discontinuously actions will be useful.

Proposition 2.1.2. *The action of a group Γ on a locally compact space X is* properly discontinuous *if and only if all of the following hold:*

(i) the space of orbits M/Γ is Hausdorff with the quotient topology;

(ii) each $x \in M$ has finite isotropy group;

(iii) each $x \in M$ has a Γ_x-invariant neighborhood U such that $\Gamma_x = \{\gamma \in \Gamma |\ \gamma.U \cap U \neq \varnothing\}$.

Proof. ($\Rightarrow$) (i) Let x and x' be points in M with distinct orbits of Γ and let K be a compact neighborhood of x. Since, by hypothesis, the set $\{\gamma \in \Gamma |\ \gamma.x' \in K\}$ is finite we can find a neighborhood U of x which is disjoint from the orbit of x'. Then $\bigcup_{\gamma\in\Gamma} \gamma.U$ is a Γ-invariant neighborhood of x which does not contain x' (so it does not intersect the orbit through x'). Similarly we can find a Γ-invariant neighborhood of x' which does not contain x, hence M/Γ is Hausdorff with the quotient topology.

(ii) is immediate from the definition by considering $K_1 = K_2 = \{x\}$.

(iii) Let $K_1 = \{x\}$ and K_2 be a compact neighborhood of x. Since the action is properly discontinuous, the set $\{\gamma \in \Gamma |\ \gamma.x \in K_2\}$ is finite and contains Γ_x. Thus we can find a compact neighborhood of x, say K_2' such that $\{\gamma \in \Gamma |\ \gamma.x \in K_2'\} = \Gamma_x$. Consider now the set $K = \bigcup_{\gamma\in\Gamma_x} \gamma.K_2'$ which is a compact and Γ_x-invariant neighborhood of x. Applying the definition for $K_1 = K_2 = K$, the set $\{\gamma \in \Gamma |\ \gamma.K \cap K \neq \varnothing\}$ is finite and contains Γ_x. Then we can find a neighborhood of x in K, say U', such that $\{\gamma \in \Gamma |\ \gamma.U' \cap U' \neq \varnothing\} = \Gamma_x$ and by taking $U = \bigcup_{\gamma\in\Gamma_x} \gamma.U'$ we obtain the Γ_x-invariant satisfying (iii).

($\Leftarrow$) By (i), for any $x, x' \in M$ such that x and x' are not on the same orbit there are neighborhoods U and U' such that $\gamma.U \cap U' = \varnothing$ for any $\gamma \in \Gamma$. If x and x' are on the same orbit, then we can find neighborhoods of x and x', say U and U', such that the set $\{\gamma \in \Gamma |\ \gamma.U \cap U' \neq \varnothing\}$ is finite and in fact that it has the cardinality equal to the order of the isotropy group of x (or x'). Indeed, assume that $x = \delta.x'$ for some $\delta \notin \Gamma_x$, and consider U to be a neighborhood of x as given by (iii) and U' to be $\delta.U$. Then $\{\gamma \in \Gamma |\ \gamma.U \cap U' \neq \varnothing\} = \{\gamma \in \Gamma |\ \gamma.U \cap \delta.U \neq \varnothing\} = \{\gamma \in \Gamma |\ U \cap (\gamma^{-1} \cdot \delta).U \neq \varnothing\} = \delta.\Gamma_x$ which by (ii) is finite. Thus, for any $x, x' \in M$ we can find neighborhoods U and U' such that $\{\gamma |\ \Gamma.U \cap U' \neq \varnothing\}$ is at most finite. Let now K be any compact in M. Then $K \times K$ is compact in $M \times M$ and so has a finite cover with sets of the form $U \times U'$ where $\{\gamma |\ \Gamma.U \cap U' \neq \varnothing\}$ is finite. Therefore the set $\{\gamma \in \Gamma |\ \Gamma.K \cap K \neq \varnothing\}$ is finite, i.e. the action is properly discontinuous.

$\square$

Remark 2.1.3. In the above proposition, if the action is free and the space M is a smooth manifold, then M/Γ is also a smooth manifold and the map $M \to M/\Gamma$ is a covering.

Consider now a smooth manifold M and a group Γ acting by diffeomorphisms on it. Note that the action of Γ on M induces an action on the tangent bundle TM defined by $\gamma.v := (d\gamma)_x.v$, for any $\gamma \in \Gamma$, $v \in T_xM$, $x \in M$. It is well known that if M is also connected and paracompact it admits a Riemannian metric. In the case when the action of Γ on M is proper one can prove that there exists a Γ-invariant Riemannian metric on M. We will include here the proof of this fact in the special case when Γ is a finite subgroup of Diffeo(M).

Lemma 2.1.4. *Let M be a connected paracompact differentiable manifold and Γ be a finite subgroup of Diffeo(M) acting on M. Then there exists a Γ-invariant Riemannian metric on M.*

Proof. Choose a Riemannian metric g on M and define a new one by averaging g over Γ:

$$\rho_x(v,w) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} g_{\gamma.x}(\gamma.v, \gamma.w)$$

for any $v, w \in T_xM$ and any $x \in M$. Then ρ is a Riemannian metric on M and it is Γ-invariant. $\square$

Remark 2.1.5. In the general case this can be done by averaging over Γ with respect to the restriction to Γ of a Haar measure μ defined on Diffeo(M) and using a Bruhat function. That is, a smooth function $\beta \in C^\infty(M, \mathbb{R})$ such that

$$\int_\Gamma \beta(\gamma.x)d\mu(\gamma) = 1$$

for all $x \in M$. Moreover β can be chosen in a such way that $\Gamma.K \cap \mathrm{supp}\beta$ is compact for all compact subsets $K \subset M$.

Let $\gamma \in \Gamma$ and denote $\Sigma_\gamma = \{x \in M |\ \gamma.x = x\}$ the subset of M which is fixed by γ and

$$\Sigma_\Gamma = \{x \in M |\ \Gamma_x \neq 1\} = \bigcup_{\gamma\in\Gamma, \gamma\neq 1} \Sigma_\gamma.$$

We say that a subset S of M is Γ-*stable* if it is connected and if for any $\gamma \in \Gamma$ we have either $\gamma.S = S$ or $\gamma.S \cap S = \varnothing$. The isotropy group of the Γ-stable set S is $\Gamma_S = \{\gamma \in \Gamma | \ \gamma.S = S\}$. Note that the Γ-stable subsets of M are exactly the connected components of Γ-invariant subsets of M and the following holds.

Proposition 2.1.6. *If Γ is finite, for any $x \in M$ there exists an arbitrarily small open Γ-stable neighborhood S of x such that $\Gamma_x = \Gamma_S$* (compare with (iii) in previous proposition). *Hence, the open Γ-stable subsets of M form a basis for the topology of M.*

Proposition 2.1.7. *Let M be a connected, paracompact smooth manifold and Γ a finite subgroup of Diffeo(M). Then Σ_Γ is a closed set with empty interior. Moreover, the homomorphism $d_x : \Gamma_x \to Aut(T_xM)$ is injective for every $x \in M$.*

Proof. The set

$$\Sigma_\Gamma = \bigcup_{\gamma\in\Gamma,\gamma\neq 1} \Sigma_\gamma = \{x \in M | \ \exists\gamma \in \Gamma, \gamma \neq 1 \text{ such that } \gamma.x = x\}$$

is obviously closed. It is the finite union of the sets of fixed points of a diffeomorphism, which are closed. An alternative way of seeing this is the following. Consider a sequence of points (x_n) in Σ_Γ which converges to a point x in M (we will prove that $x \in \Sigma_\Gamma$). Since each x_n is in Σ_Γ we can form a sequence of elements of Γ, γ_n such that $\gamma_n.x_n = x_n$ and $\gamma_n \neq 1$ for any n. But Γ is finite, so γ_n contains at least a constant subsequence which we will denote again γ_n and assume $\gamma_n = \gamma \neq 1$ for any n. The corresponding subsequence of x_n converges to x and moreover the sequence $\gamma_n.x_n = \gamma.x_n$ converges to $\gamma.x$. But $\gamma_n.x_n = x_n$ for any n so $\gamma_n.x_n$ converges to x also. Since the quotient space is Hausdorff we have that $\gamma.x = x$ and since $\gamma \neq 1$ we conclude that $x \in \Sigma_\Gamma$.

Let's prove now that it has empty interior. Consider the Γ-invariant Riemannian metric on M given by the above lemma, and consider the exponential map associated with this metric. Then, for each $x \in M$ there are $\varepsilon > 0$ and open neighborhood of x, U such that

$$\exp_x : B(0, \varepsilon) \to U$$

is a diffeomorphism from the ε-ball centered at the origin in the tangent space T_xM to U. Since the metric is Γ-invariant, the action of the isotropy group Γ_x on the tangent space at x is orthogonal, i.e. $(d\gamma)_x$ is an orthogonal transformation of T_xM, and

$$\exp_x \circ (d\gamma)_x = \gamma \circ \exp_x,$$

for any $\gamma \in \Gamma_x$. In particular, if $(d\gamma)_x = \mathrm{id}$ then the restriction $\gamma|_U = \mathrm{id}$. Since M is connected this implies $\gamma = 1$. To see this, consider the set

$$A = \{y \in M |\ \gamma.y = y \text{ and } (d\gamma)_y = \mathrm{id}\}.$$

Then, $A \neq \varnothing$ (since $x \in A$) and it is obviously closed. It is also open. If we assume that $y \in A$ then the condition $\gamma.y = y$ implies that $\gamma \in \Gamma_y$ and the condition that $(d\gamma)_y = \mathrm{id}$ implies that the restriction of γ to an open neighborhood of y is 1. Hence the whole neighborhood of y is contained in A, i.e. A is open. The connectedness of M implies $A = M$ and so, $\gamma = 1$ on M.

This proves that $d_x : \Gamma_x \to \mathrm{Aut}(T_xM)$ is injective, which in particular implies that Σ_γ has empty interior, for any $\gamma \neq 1$. Indeed, let $x \in \Sigma_\gamma$ and assume that there exists an open neighborhood U of x in Σ_γ. Then the restriction $\gamma|_U = 1$ and as we have seen this implies $\gamma = 1$ on M. Since Γ is finite, Σ_Γ has empty interior also and the proof is complete. ☐

Remark 2.1.8. The above proposition implies that the only diffeomorphism of finite order on a connected manifold which fixes an open set is the identity.

Proposition 2.1.9. *Let M be a connected, paracompact smooth manifold, Γ a finite subgroup of Diffeo(M) and let φ denote the natural projection $M \to M/\Gamma$. Let V be a nonempty, open, connected subset of M and $f : V \to M$ a diffeomorphism onto its image such that $\varphi \circ f = \varphi|_V$. Then there exists a unique $\gamma \in \Gamma$ such that $f = \gamma|_V$.*

Proof. Consider on M a Γ-invariant Riemannian metric given by Lemma 2.1.4. By the previous result, the set $M_0 = M \setminus \Sigma_\Gamma$ is open and dense in M. For any $x \in V \cap M_0$, the condition $\varphi \circ f = \varphi|_V$ implies that there is a unique $\gamma \in \Gamma$ such that $f(x) = \gamma.x$ and on a sufficiently small connected neighborhood of x in $V \cap M_0$ we have $(df)_x = (d\gamma)_x$.

Since the metric is Γ-invariant $(d\gamma)_x$, this means that $(df)_x$ preserves the metric at x. So, the restriction of f to $V \cap M_0$ is a Riemannian isometry. Since $V \cap M_0$ is dense in V, by continuity f is a Riemannian isometry on V and $f = \gamma$ on a neighborhood of x in V. But V is connected, therefore the two isometries f and $\gamma|_V$ are equal. $\square$

Remark 2.1.10. The last two results are still true in the case where M is assumed to be a connected topological manifold and Γ a finite subgroup of Homeo(M). This fact is a consequence of a result of Newman (see [**?**]) which states that a nontrivial homeomorphism of a manifold which fixes an open set cannot have finite order (compare with Remark 2.1.8).

2.2 Traditional approach to orbifolds

In the following sections we present the definitions and the basic properties of orbifolds in a traditional approach. The point of view of this chapter is that an orbifold structure generalizes the manifold topological (differentiable) structure. For more information the reader is referred to the original work of Satake in [Sa1] and also to that of Thurston in [T]. Other introductions for the classical theory of orbifolds include chapter 6 in [K] and section 2.4 in [MM]. A different formulation of most of the concepts presented here as well as a great deal of information on the differential geometry of orbifolds can be found in the appendix of [CR]. A detailed presentation of the geometric structure of 2-dimensional orbifolds is provided in [Sc].

Definition 2.2.1. *A topological (differentiable) n-dimensional* orbifold Q *consists of a Hausdorff space denoted* $|Q|$ *and called the underlying space of* Q*, together with an additional structure given by the following*

(i) a countable basis of open sets $\{U_i\}_{i\in I}$ *which is closed under finite intersections and such that* $|Q| = \bigcup_{i\in I} U_i$*;*

(ii) to each U_i *is associated a finite group* Γ_i*, an action of* Γ_i *on some open subset* $\widetilde{U}_i$ *of* $\mathbb{R}^n$ *by homeomorphisms (diffeomorphisms) and a homeomorphism* $\varphi_i : \widetilde{U}_i/\Gamma_i \to U_i$*;*

(iii) whenever $U_i \subseteq U_j$, there is a monomorphism

$$\lambda_{ij} : \Gamma_i \to \Gamma_j$$

defined only up to conjugation with elements of Γ_j and such that λ_{ij} induces an isomorphism between $\{\gamma \in \Gamma_i |\ \gamma.\widetilde{U}_i = \widetilde{U}_i\}$ and $\{\gamma \in \Gamma_j |\ \gamma.\widetilde{U}_j = \widetilde{U}_j\}$, and a λ_{ij}-equivariant (smooth) embedding

$$\widetilde{\varphi}_{ij} : \widetilde{U}_i \to \widetilde{U}_j$$

(i.e., for any $\gamma \in \Gamma_i$, $\widetilde{\varphi}_{ij}(\gamma.\widetilde{x}) = \lambda_{ij}(\gamma).\widetilde{\varphi}_{ij}(\widetilde{x})$ for all $\widetilde{x} \in \widetilde{U}_i$) defined up to composition with elements of Γ_j such that the following diagram commutes

Note that the actions of the Γ_i's on $\widetilde{U}_i$'s can be always assumed to be effective. Indeed, the set of elements in the group which act trivially form a normal subgroup and the action of the quotient is effective with the same space of orbits. Then each Γ_i can be regarded as a finite subgroup of $\mathrm{Homeo}(\widetilde{U}_i)$ in the topological case and respectively of $\mathrm{Diffeo}(\widetilde{U}_i)$ in the differentiable case.

The triple $(U_i, \widetilde{U}_i/\Gamma_i, \varphi_i)$ as in (ii) is called an *orbifold coordinate chart over U_i* or *uniformizing system of U_i* and the pair $(\lambda_{ij}, \widetilde{\varphi}_{ij}) : (U_i, \widetilde{U}_i/\Gamma_i, \varphi_i) \to (U_j, \widetilde{U}_j/\Gamma_j, \varphi_j)$ as in (iii) an *injection* between charts.

Remark 2.2.2. It easy to see that the composition of two injections is again an injection. The well definition of the maps $\widetilde{\varphi}_{ij}$ up to compositions with elements of the group Γ_j is a consequence of Proposition 2.1.9 in the smooth case and of Remark 2.1.10 in the topological case. Hence if $(\lambda_{ij}, \widetilde{\varphi}_{ij})$ and $(\lambda'_{ij}, \widetilde{\varphi}'_{ij})$ denote two injections between the same orbifold charts, then there is a unique $\gamma \in \Gamma_j$ such that $\widetilde{\varphi}'_{ij} = \gamma.\widetilde{\varphi}_{ij}$ and in this case $\lambda'_{ij} = \gamma \cdot \lambda_{ij} \cdot \gamma^{-1}$. In particular if $i = j$, $\widetilde{\varphi}_{ii}$ is an element $\gamma_i \in \Gamma_i$ and then λ_{ii} is just conjugation with γ_i. In general it is not true that whenever $U_i \subset U_j \subset U_k$ we have $\widetilde{\varphi}_{jk} \circ \widetilde{\varphi}_{ij} = \widetilde{\varphi}_{ik}$, but there exists an element $\gamma \in \Gamma_k$ such that $\widetilde{\varphi}_{jk} \circ \widetilde{\varphi}_{ij} = \gamma.\widetilde{\varphi}_{ik}$ and $\lambda_{jk} \circ \lambda_{ij} = \gamma \cdot \lambda_{ik} \cdot \gamma^{-1}$.

As in the manifold case, the covering $\{U_i\}_i$ is not an intrinsic part of the orbifold structure. Two coverings will give the same orbifold structure if they can be consistently combined to give a covering which still satisfies the properties (ii) and (iii). By an *orbifold* one should understand the orbifold with the structure given by a such maximal cover.

We begin with the simplest examples of orbifolds, more interesting examples are given in 2.6.5.

Proposition 2.2.3. *Let Γ be a group acting properly discontinuously on a manifold M. Then the quotient space M/Γ has a natural orbifold structure.*

Proof. We have seen already, in Proposition 2.1.2 (i), that under these assumptions the quotient space M/Γ is Hausdorff. We will construct an orbifold atlas for M/Γ, $\mathscr{U}$ satisfying the conditions (i)-(iii) in definition.

Let $\pi : M \to M/\Gamma$ denote the quotient map and let $x \in M/\Gamma$. Choose $\widetilde{x} \in M$ such that $\pi(\widetilde{x}) = x$ and let $\Gamma_{\widetilde{x}} = \{\gamma \in \Gamma |\ \gamma.\widetilde{x} = \widetilde{x}\}$ denote the isotropy group of $\widetilde{x}$. By Proposition 2.1.2 (iii), there exists an open connected neighborhood of $\widetilde{x}$, $\widetilde{U}_{\widetilde{x}}$ which is invariant to $\Gamma_{\widetilde{x}}$ and disjoint from its translates by elements of Γ not in $\Gamma_{\widetilde{x}}$. Then the restriction

$$\pi|_{\widetilde{U}_{\widetilde{x}}} : \widetilde{U}_{\widetilde{x}} \to U_x := \widetilde{U}_{\widetilde{x}}/\Gamma_{\widetilde{x}}$$

is a homeomorphism. Let $\widetilde{\mathscr{U}}$ be a maximal atlas on M. By eventually shrinking $\widetilde{U}_{\widetilde{x}}$, we can assume that $\widetilde{U}_{\widetilde{x}} \in \widetilde{\mathscr{U}}$ and so there is a homeomorphism $\varphi : \widetilde{U}_{\widetilde{x}} \to \varphi(\widetilde{U}_{\widetilde{x}}) \subset \mathbb{R}^n$. Then the composition

$$\varphi_x := (\varphi/\Gamma_{\widetilde{x}})^{-1} \circ \pi|_{\widetilde{U}_{\widetilde{x}}} : \varphi(\widetilde{U}_{\widetilde{x}})/\Gamma_{\widetilde{x}} \to U_x$$

is again a homeomorphism.

Hence $\{U_x|\ x \in M/\Gamma\}$ form an open cover for M/Γ and each U_x has an uniformizing system by $(U_x, \varphi(\widetilde{U}_{\widetilde{x}})/\Gamma_{\widetilde{x}}, \varphi_x)$. In order to get a suitable cover of M/Γ we should augment the above cover by adjoining finite intersections. Let now $x_1, x_2, \dots\ x_k \in M/\Gamma$ such that the corresponding sets $U_{x_1}, U_{x_2}, \cdots\ U_{x_k}$ as above satisfy $U_{x_1} \cap U_{x_2} \cap \cdots \cap U_{x_k} \neq \varnothing$. Then, since Γ acts by permutations on the set of connected components of $\pi^{-1}(U_{x_1} \cap U_{x_2} \cap \cdots \cap U_{x_k})$, there exist $\gamma_1, \gamma_2, \dots\ \gamma_k \in \Gamma$ such that $\gamma_1.\widetilde{U}_{\widetilde{x}_1} \cap \gamma_2.\widetilde{U}_{\widetilde{x}_2} \cap \cdots \cap \gamma_k.\widetilde{U}_{\widetilde{x}_k} \neq \varnothing$, where as above $\widetilde{U}_{\widetilde{x}_i}$ denote neighborhoods of $\widetilde{x}_i \in \pi^{-1}(x_i)$ invariant by $\Gamma_{\widetilde{x}_i}$. Then this intersection may be taken to be

$$U_{x_1} \widetilde{\cap \cdots\ \cap} U_{x_k}$$

which is obviously invariant to the action of the finite subgroup

$$\gamma_1 \cdot \Gamma_{\widetilde{x}_1} \cdot \gamma_1^{-1} \cap \cdots\ \cap \gamma_k \cdot \Gamma_{\widetilde{x}_k} \cdot \gamma_k^{-1}.$$

In this way we obtain a cover $\mathscr{U}$ of M/Γ which satisfies (i) and (ii) of definition 2.2.1.

We will show that the condition (iii) in the definition is also satisfied. Consider U and U' in $\mathscr{U}$ such that $U' \subset U$ and let $x \in U'$ and $\widetilde{x} \in M$ such that $\pi(\widetilde{x}) = x$. For x and U, consider $\widetilde{U}_{\widetilde{x}}$ and $\Gamma_{\widetilde{x}}$ as above and choose $\widetilde{U}'_{\widetilde{x}}$ (note that it should be the neighborhood for the same lift of x, $\widetilde{x}$). In order to prove that there is an embedding between the two charts, it suffices to prove that $\widetilde{U}'_{\widetilde{x}} \subset \widetilde{U}_{\widetilde{x}}$. To see this, assume it is not true and choose $\widetilde{y} \in \widetilde{U}'_{\widetilde{x}} \setminus \widetilde{U}_{\widetilde{x}}$. Then there should exist $\gamma \in \Gamma_{\widetilde{x}}$ such that $\gamma.\widetilde{y} \in \widetilde{U}' \cap \widetilde{U}$, since $\pi(\widetilde{y}) = y \in U' \subset U$. But both $\widetilde{U}'_{\widetilde{x}}$ and $\widetilde{U}_{\widetilde{x}}$ are $\Gamma_{\widetilde{x}}$-invariant and hence so is $\widetilde{U}'_{\widetilde{x}} \cap \widetilde{U}_{\widetilde{x}}$. This means that $\widetilde{y} \in \widetilde{U}'_{\widetilde{x}} \cap \widetilde{U}_{\widetilde{x}}$ which contradicts the fact that $\widetilde{y} \in \widetilde{U}'_{\widetilde{x}} \setminus \widetilde{U}_{\widetilde{x}}$, i.e. proves $\widetilde{U}'_{\widetilde{x}} \subset \widetilde{U}_{\widetilde{x}}$. $\square$

Note that the orbifold structure on M/Γ is *natural* in the sense that it depends only on the action of the group Γ and not on the choice of the atlas $\widetilde{\mathscr{U}}$ on M. It is called the *orbifold quotient of M by the properly discontinuous action of* Γ.

Definition 2.2.4. *An orbifold is called* good *if it arises as the global quotient by a discrete group acting properly discontinuously on a manifold. If the group can be chosen to be finite, then the orbifold is called* very good.

Remark 2.2.5. Sometimes, we refer to good orbifolds as being *developable*. In fact, another approach is to define orbifolds using charts for open sets U_i of the form (X_i, Γ_i), where each X_i is a manifold and Γ_i is a discrete group acting properly discontinuously on X_i with X_i/Γ_i homeomorphic to U_i. From this point of view, an orbifold Q is good (or developable) if it admits such an atlas with a single chart. As in [GH], we call the pair (X_i, q_i) a *uniformizing chart* of U_i, where $q_i : X_i \to U_i$ is a continuous map which induces a homeomorphism from X_i/Γ_i onto U_i. Then the compatibility condition (iii) in Definition 2.2.1 becomes: for all $x_i \in X_i$ and $x_j \in X_j$ such that $q_i(x_i) = q_j(x_j)$, there is a homeomorphism(diffeomorphism) h from an open connected neighborhood W of x_i to a neighborhood of x_j, such that $q_j \circ h = q_i|_W$. This time, we will call h a *change of charts*, and as we already saw it is defined up to composition with elements of Γ_j and in particular, if $i = j$ then h is the restriction of an element of Γ_i.

Similarly, we define *orbifolds with boundary* by taking as uniformizing systems open sets in the upper half plane

$$\mathbb{R}^n_{\geq 0} = \{(x_1, x_2, \ldots x_n)|\ x_n \geq 0\}$$

(or equivalently connected manifolds with boundary). The *boundary* of such an orbifold consists of points $x \in |Q|$ that correspond to $\mathbb{R}^n_0 = \{(x_1, x_2, \ldots, x_n)|\ x_n = 0\}$ (or to the boundary of $\widetilde{U}_i$). As in the manifold case, the boundary of an orbifold is an orbifold without boundary. A compact orbifold without boundary is called *closed.*

2.3 The singular set

Let Q be an orbifold and let $x \in Q$. Within an orbifold chart $(U_i, \widetilde{U}_i/\Gamma_i, \varphi_i)$ we can associate to x a group $\Gamma^{(i)}_x$ well defined up to isomorphism in the following way. Consider $\widetilde{x}, \widetilde{x}' \in \widetilde{U}_i$ such that $\varphi_i(\widetilde{x}) = \varphi_i(\widetilde{x}') = x$. Then their isotropy groups $\Gamma^{(i)}_{\widetilde{x}}$ and $\Gamma^{(i)}_{\widetilde{x}'}$ are conjugate to each other, since $\widetilde{x}$ and $\widetilde{x}'$ are on the same orbit. Denote this subgroup by $\Gamma^{(i)}_x$ and since it is independent of the choice of the lifts of x we will refer to it as the isotropy group of x in U_i.

Consider now $(U_i, \widetilde{U}_i/\Gamma_i, \varphi_i)$ and $(U_j, \widetilde{U}_j/\Gamma_j, \varphi_j)$ two orbifold charts containing x assume that $U_i \subset U_j$. Let $(\lambda_{ij}, \widetilde{\varphi}_{ij})$ denote the injection between them. Note that if x has non-trivial isotropy in U_i (i.e. $\Gamma_x^{(i)} \neq 1$) then it has non-trivial isotropy in U_j. Indeed, since the embedding $\widetilde{\varphi}_{ij}$ is λ_{ij}-equivariant, $\Gamma_x^{(j)}$ contains the subgroup $\lambda_{ij}(\Gamma_x^{(i)})$ which is not trivial since λ_{ij} is injective (the inclusion above is considered up to isomorphism, i.e. at least one of the isomorphic subgroups defining $\Gamma_x^{(j)}$ should contain $\lambda_{ij}(\Gamma_x^{(i)})$, and this is fine since both $\Gamma_x^{(j)}$ and λ_{ij} are defined up to conjugation with elements of Γ_j).

Define the *isotropy group at* x, Γ_x to be the smallest isotropy group of x corresponding to an orbifold chart containing x, i.e.

$$\Gamma_x = \bigcap_{i \in I} \Gamma_x^{(i)}.$$

It is obvious that Γ_x is always finite. (Equivalently the isotropy group at x can be defined as the *germ* of the action of $\Gamma_x^{(i)}$ at x.)

A point $x \in Q$ is called a *singular point* if it has non-trivial isotropy i.e. $\Gamma_x \neq \{1\}$ and it is called a *regular point* otherwise. Define the *singular set* of an orbifold to be

$$\Sigma_Q := \{x \in Q \mid \Gamma_x \neq \{1\}\}.$$

Then we say that an orbifold is a manifold if the singular set is empty. The following result concerning the singular set of an orbifold holds:

Proposition 2.3.1. *The singular set of an orbifold is closed and nowhere dense.*

Proof. Let $(U, \widetilde{U}/\Gamma, \varphi)$ be any orbifold chart that has nonempty intersection with the singular set. Then

$$\begin{aligned}
\Sigma_Q \cap U &= \{x \in U \mid \Gamma_x \neq 1\} \\
&= \{x \in U \mid \exists \gamma \in \Gamma, \gamma \neq 1 \text{ such that } \gamma.\widetilde{x} = \widetilde{x} \text{ for some } \widetilde{x} \in \varphi^{-1}(x)\} \\
&= \bigcup_{\widetilde{x} \in \widetilde{U}} \{\varphi(\widetilde{x}) \mid \exists \gamma \in \Gamma, \gamma \neq 1 \text{ such that } \gamma.\widetilde{x} = \widetilde{x}\} \\
&= \varphi\Big(\bigcup_{\widetilde{x} \in \widetilde{U}} \{\widetilde{x} \mid \exists \gamma \in \Gamma, \gamma \neq 1 \text{ such that } \gamma.\widetilde{x} = \widetilde{x}\}\Big) \\
&= \varphi(\Sigma_\Gamma)
\end{aligned}$$

Since Σ_Γ is closed and with empty interior (see Proposition 2.1.7 and Remark 2.1.10) and φ is a homeomorphism $\Sigma_\Gamma \cap U$ is closed and has empty interior. Hence Σ_Q is closed and since $|Q|$ is locally compact and Hausdorff, $\Sigma_Q = \bigcup_i \Sigma_Q \cap U_i$ has empty interior. $\square$

Note that in general the singular set is not a manifold and it may have several connected components of different dimension. From the definition we can see that any orbifold is locally compact. If we assume that in any uniformizing system the elements of the group have the fixed point set of codimension at least two (i.e. the singular set has the codimension at least two) then the orbifold is also locally path connected. In this case an orbifold is connected if and only if it is path connected.

2.4 Maps between orbifolds

Consider two orbifolds Q and Q' and a continuous map $f : |Q| \to |Q'|$ between their underlying spaces.

Let $x \in |Q|$ and $y = f(x) \in |Q'|$ and let V be an open neighborhood of y and U an open neighborhood of x such that $f(U) \subset V$. Let $(V, \widetilde{V}/\Gamma^*, \varphi^*)$ be an uniformizing system over V and $(U, \widetilde{U}/\Gamma, \varphi)$ an uniformizing system over U. Corresponding to these uniformizing systems, a *continuous (resp. smooth) lifting* of $f|_U : U \to V$ is a continuous (resp. smooth) map

$$\widetilde{f} : \widetilde{U} \to \widetilde{V}$$

such that $\varphi^* \circ \widetilde{f} = f \circ \varphi$ and for any $\gamma \in \Gamma$ there exists $\gamma^* \in \Gamma^*$ satisfying $\gamma^*.\widetilde{f}(\widetilde{x}) = \widetilde{f}(\gamma.\widetilde{x})$ for any $\widetilde{x} \in \widetilde{U}$.

For a different choice of uniformizing systems $(V, \widetilde{V}'/\Gamma^{*'}, \varphi^{*'})$ over V, respectively $(U, \widetilde{U}'/\Gamma', \varphi')$ over U, we say that the lifting $\widetilde{f}' : \widetilde{U}' \to \widetilde{V}'$ is *isomorphic* to $\widetilde{f}$ is there are bijections

$$(\lambda, \widetilde{\varphi}) : (U, \widetilde{U}/\Gamma, \varphi) \to (U, \widetilde{U}'/\Gamma', \varphi')$$

respectively

$$(\lambda^*, \widetilde{\varphi}^*) : (V, \widetilde{V}/\Gamma^*, \varphi^*) \to (V, \widetilde{V}'/\Gamma^{*'}, \varphi^{*'})$$

such that $\lambda^* \circ \widetilde{f} = \widetilde{f}' \circ \lambda$.

Let now $x_0 \in U$ and let $(U_0, \widetilde{U}_0/\Gamma_0, \varphi_0)$ be a uniformizing system over an open neighborhood $U_0 \subset u$ of x_0 and $(V_0, \widetilde{V}_0/\Gamma_0^*, \varphi_0^*)$ a uniformizing system over a neighborhood $V_0 \subset V$ of $f(x_0)$ such that $f(U_0) \subset V_0$. The lifting $\widetilde{f}$ will induce a lifting

$$\widetilde{f}_0 : \widetilde{U}_0 \to \widetilde{V}_0$$

of $f|_{U_0} : U_0 \to V_0$ in the following way. For any injection

$$(\lambda_0, \widetilde{\varphi}_0) : (U_0, \widetilde{U}_0/\Gamma_0, \varphi_0) \to (U, \widetilde{U}/\Gamma, \varphi),$$

consider the map $\widetilde{f} \circ \lambda_0 : \widetilde{U}_0 \to \widetilde{V}$ and note that $(\varphi^* \circ \widetilde{f} \circ \lambda_0)(\widetilde{U}_0) \subset V_0$ which implies $(\widetilde{f} \circ \lambda_0)(\widetilde{U}_0) \subset (\varphi^*)^{-1}(V_0)$. Therefore there is an injection

$$(\lambda_0^*, \widetilde{\varphi}_0^*) : (V_0, \widetilde{V}_0/\Gamma_0^*, \varphi_0^*) \to (V, \widetilde{V}/\Gamma^*, \varphi^*)$$

such that $(\widetilde{f} \circ \lambda_0)(\widetilde{U}_0) \subset \lambda_0^*(\widetilde{V}_0)$. Define now

$$\widetilde{f}_0 = (\lambda_0^*)^{-1} \circ \widetilde{f} \circ \lambda_0$$

which is the induced lifting of $f|_{U_0} : U_0 \to V_0$. Note that different choices of injections $(\lambda_0, \widetilde{\varphi}_0)$ give isomorphic liftings. We say that two liftings are *equivalent* at a point x if they induce isomorphic liftings on a smaller neighborhood of x.

A *continuous (resp. smooth) lifting* of $f : |Q| \to |Q'|$ is the following: given any point $y \in |Q'|$ and any uniformizing chart $(V, \widetilde{V}/\Gamma^*, \varphi^*)$ at y there exists a unifomizing chart $(U, \widetilde{U}/\Gamma, \varphi)$ at $x \in f^{-1}(y)$ and a continuous (resp. smooth) lifting $\widetilde{f}_x : \widetilde{U} \to \widetilde{V}$ of $f|_U : U \to V$ such that for any $x' \in U$ and any uniformizing chart $(U', \widetilde{U}'/\Gamma', \varphi')$

at x' and such that $U' \subset U$, the lifting $\widetilde{f}_{x'} : \widetilde{U}' \to \widetilde{V}$ of $f|_{U'} : U' \to V$ is *isomorphic* with the *induced* one on $\widetilde{U}'$ from $\widetilde{f}_x$. We say that two liftings of $f : |Q| \to |Q'|$ are equivalent if their local liftings are equivalent at each point in $|Q|$.

Definition 2.4.1. *A continuous (resp. smooth)* orbifold-map *between orbifolds $Q \to Q'$ is an equivalence class of continuous (resp. smooth) liftings of a continuous map between their underlying spaces $|Q| \to |Q'|$.*

We will denote by $\widetilde{f} : Q \to Q'$ an orbifold-map whose underlying continuous map is $f : |Q| \to |Q'|$. Note that two different (which are not isomorphic) orbifold-maps might have the same underlying continuous map (for an example see [CR] Example 4.1.6b in the Appendix)

We will define now a particular kind of orbifold-maps called *good maps* (see [CR] Definition 4.4.1). The advantage of using these maps is that we can define the pull-back bundles, fact which cannot be done by using general orbifold-maps. Also these good maps match the definition of a morphism in the category of groupoids (see section **??** and also **??**).

Let $\widetilde{f} : Q \to Q'$ be an orbifold-map with underlying continuous function $f : |q| \to |Q'|$. Suppose that there is an atlas $\mathscr{U}$ for Q and a collection of open subsets $\mathscr{U}'$ of Q' such that there is a one-to-oen correspondence between the elements of $\mathscr{U}$ and $\mathscr{U}'$, say $U \leftrightarrow U'$, with $f(U) \subset U'$ and $U_1 \subset U_2$ implies $U_1' \subset U_2'$. Moreover, there is a collection of liftings of f such that $\widetilde{f}_{UU'} : \widetilde{U} \to \widetilde{U}'$ satisfies that for each injection

$$(\lambda, \widetilde{\varphi}) : (U_1, \widetilde{U}_1/\Gamma_1, \varphi_1) \to (U_2, \widetilde{U}_2/\Gamma_2, \varphi_2)$$

there is another injection associated to it

$$\big((\nu(\lambda), \nu(\widetilde{\varphi})\big) : (U_1', \widetilde{U}_1'/\Gamma_1', \varphi_1') \to (U_2', \widetilde{U}_2'/\Gamma_2', \varphi_2')$$

such that

$$\widetilde{f}_{U_1U_1'} \circ \lambda = \nu(\lambda) \circ \widetilde{f}_{U_2U_2'},$$

and for any composition of injections $\lambda' \circ \lambda$ we have $\nu(\lambda' \circ \lambda) = \nu(\lambda') \circ \nu(\lambda)$. The collection of liftings $\{\widetilde{f}_{UU'}, \nu\}$ defines a lifting of f. If this lifting is in the same equivalence class as $\widetilde{f}$, then the collection $\{\widetilde{f}_{UU'}, \nu\}$ is called a *compatible system* of $\widetilde{f}$.

Definition 2.4.2. *An orbifold-map is called* good *if it admits a compatible system.*

The real line $\mathbb{R}$ as a smooth manifold is trivially an orbifold. The smooth orbifold-maps $f : Q \to \mathbb{R}$ are called *smooth functions* on the orbifold Q. Note that an orbifold function is smooth if and only if the map $f \circ \varphi$ is smooth for any orbifold chart $(U, \widetilde{U}/\Gamma, \varphi)$ in an orbifold atlas of Q. A (smooth) map from $\mathbb{R}$ (or an interval I) into an orbifold Q is called a *(smooth) path* in Q.

Similarly we can define *immersions* and *submersions* between orbifolds as differentiable maps between orbifolds that locally lift to immersions and submersions, respectively.

A *suborbifold* $Q' \subset Q$ is an orbifold Q' together with an orbifold embedding $i : Q' \hookrightarrow Q$.

2.5 The tangent space to an orbifold

Suppose now that $Q = M/\Gamma$ is a good orbifold. As we have seen in the previous section, we can extend the action of Γ on M to an action on TM by setting $\gamma.(\widetilde{x}, v) := (\gamma.\widetilde{x}, d(\gamma)_{\widetilde{x}} v)$, for all $\gamma \in \Gamma$ and $(\widetilde{x}, v) \in TM$. The quotient of TM by this action is the tangent bundle TQ of the orbifold Q. As in Proposition 2.2.3 it inherits a natural orbifold structure. For $x \in Q$, let $\widetilde{x} \in M$ denote one of its lifts. By taking the differentials $(d\gamma)_{\widetilde{x}}$ of the elements γ in the isotropy group Γ_x, we obtain a new group which acts on $T_{\widetilde{x}}M$. Since the group is independent of the choice of the lift, we will denote it by Γ_{x^*}. Hence the fiber in TQ above $x \in Q$ is $T_{\widetilde{x}}Q/\Gamma_{x^*}$ and is denoted T_xQ. Because T_xQ will not be a vector space at the singular points, we will call it the *tangent cone to Q at x*.

Since any orbifold is locally good, the construction above gives a local way to work with tangent cones to orbifolds. So, if we consider an orbifold atlas $(X_i, q_i)_{i \in I}$ (see Remark 2.2.5) for the orbifold structure on Q, by making a quotient space of the tangent bundle TX_i over X_i by Γ_i as above, we obtain a $2n$-dimensional orbifold Q_i. We can easily patch the orbifolds Q_i together to obtain a $2n$-dimensional orbifold TQ with a projection map $p : TQ \to Q$ such that the inverse image of a point in the orbifold is a vector space modulo a finite group action (the tangent cone). A full description of the tangent bundle, as well as general bundles over orbifolds will be given in the next chapter.

2.6 Riemannian orbifolds

Let Q be a differentiable orbifold and let $\mathscr{U} = \{(U_i, \widetilde{U}_i/\Gamma_i, \varphi_i)\}_{i\in I}$ be the maximal orbifold atlas on Q.

Definition 2.6.1. *A* Riemannian metric *on the orbifold Q is a collection $\rho = (\rho_i)$, where ρ_i is a Riemannian metric on $\widetilde{U}_i$ such that any embedding $\widetilde{\varphi}_{ij}$ coming from an injection between orbifold charts $(U_{i,j}, \widetilde{U}_{i,j}/\Gamma_{i,j}, \varphi_{i,j})$ is an isometry as a map between $(\widetilde{U}_i, \rho_i)$ to $(\widetilde{U}_j, \rho_j)$. An orbifold with such a Riemannian metric is called* Riemannian orbifold*.*

Remark 2.6.2. Note that in particular the Riemannian metrics ρ_i on $\widetilde{U}_i$ are Γ_i-invariant, so locally Riemannian orbifolds look like the quotient of a Riemannian manifold by a finite group of isometries. By a suitable choice of coordinate charts it can be assumed that the local group actions are by finite subgroups of $O(n)$ for a general n-dimensional Riemannian orbifold, and finite subgroups of $SO(n)$ for orientable Riemannian orbifolds. As for manifolds, the following proposition holds.

Proposition 2.6.3. *Any differentiable orbifold admits a Riemannian metric.*

Proof. Let Q be an orbifold and let $\{(U_i, \widetilde{U}_i/\Gamma_i, \varphi_i)\}_{i\in I}$ denote an orbifold atlas on it. Since $|Q|$ is paracompact we may assume that the cover $(U_i)_{i\in I}$ is locally finite. Then there is a smooth partition of unity $f_i : U_i \to \mathbb{R}$ subordinate to it. Indeed, we can choose on each $\widetilde{U}_i$ a Γ_i-invariant, non-negative smooth function $\widetilde{h}_i : \widetilde{U}_i \to \mathbb{R}$ such that $h_i = \widetilde{g}_i \circ \varphi_i$ can be extended over $|Q|$ by zeroes and such that $\{supp(h_i) \subset U_i : i \in I\}$

still covers $|Q|$. Denote $h(x) = \Sigma_{i\in I} h_i(x) \neq 0$ for all $x \in Q$ and consider $f_i := h_i/h$. Then $\{f_i : i \in I\}$ is a smooth partition of unity subordinate to the cover U_i.

Consider now an arbitrary Riemannian metric g_i on each $\widetilde{U}_i$. By Lemma 2.1.4 (see also Remark 2.1.5) there exists a Γ_i-invariant Riemannian metric α_i on each $\widetilde{U}_i$ obtained from g_i by averaging over Γ_i. For any $i \in I$, define a new Riemannian metric ρ_i on $\widetilde{U}_i$ as follows:

$$(\rho_i)_{\widetilde{x}}(v,w) := \Sigma_{j\in I} f_j(\varphi_i(\widetilde{x}))(\alpha_j)_{\widetilde{\varphi}_{ij}(\widetilde{x})}(d(\widetilde{\varphi}_{ij})_{\widetilde{x}}(v), d(\widetilde{\varphi}_{ij})_{\widetilde{x}}(w))$$

for any $\widetilde{x} \in \widetilde{U}_i$ and any $v, w \in T_{\widetilde{x}}\widetilde{U}_i$ and where $\widetilde{\varphi}_{ij}$ is an embedding coming from an injection between $(U_i, \widetilde{U}_i/\Gamma_i, \varphi_i)$ and any $(U_j, \widetilde{U}_j/\Gamma_j, \varphi_j)$, $j \in I$. Then, Lemma 2.1.4 together with the second part of the Remark 6 guarantee that the Riemannian metric defined in this way is well defined, i.e. it is independent of the choice of the embedding between the uniformizing charts. It is also easy to check that each embedding is an isometry, hence the collection $\rho = (\rho_i)$ is defines a Riemannian metric in the sense of the definition above. $\square$

Similar to the definition of the Riemannian metric on an orbifold, we can move on and define general tensor fields. So, in an uniformizing system $(U_i, \widetilde{U}_i/\Gamma_i, \varphi)_i)$, for any tensor field $\widetilde{\omega}_i$ on $\widetilde{U}_i$ by pre-composing with an element $\gamma \in \Gamma_i$ we obtain a new tensor field $\widetilde{\omega}_i{}^{\gamma}$ on $\widetilde{U}_i$. Then, by averaging, we obtain a Γ_i-invariant tensor field on $\widetilde{U}_i$,

$$\widetilde{\omega}_i{}^{\Gamma_i} := \frac{1}{|\Gamma_i|}\Sigma_{\gamma\in\Gamma_i}\widetilde{\omega}_i{}^{\gamma}.$$

Such a Γ_i invariant tensor field on $\widetilde{U}_i$ gives a tensor field ω on U_i.

Definition 2.6.4. *A* smooth tensor field *on an orbifold is one that lifts to smooth tensor fields of the same type in all uniformizing systems.*

Using the Riemannian metrics on the local uniformizing systems we can define the objects familiar from the Riemannian geometry of manifolds.

Example 2.6.5. (1) (*The cone*) Let $M = \mathbb{R}^2$ and let $\Gamma = \mathbb{Z}_n$ acting by rotations on it. The quotient space is topologically $\mathbb{R}^2$ but metrically is a cone with cone angle $2\pi/n$. It is a Riemannian manifold except at the cone point where the metric has a singularity. In this case, the singular locus consists of a single point (the cone point) and the isotropy group at any point is trivial except the singular point where it is $\mathbb{Z}_n$.

(2) (*The $\mathbb{Z}_n$-football*) Let $M = \mathbb{S}^2 \subset \mathbb{R}^3$ and $\Gamma = Z_n$ acting by rotations around the z-axis by an angle of $2\pi/n$. The quotient space is topologically $\mathbb{S}^2$ but metrically there are two singular points: the north and the south pole (the points N and S in figure 2.1).

(3) (*The pillow case*) Let $M = \mathbb{T}^2$ and $\Gamma = \mathbb{Z}_2$ acting by rotations around one of its axis. Then the quotient space is an orbifold whose underlying space is S^2 and has four singular points with nontrivial isotropy $\mathbb{Z}_2$. The sphere inherits a Riemannian metric of curvature 0 in the complement of the singular locus, and has curvature π at each of the four points.

(4) Let $M = \mathbb{R}^3$ and $\Gamma = \mathbb{Z}_2$ acting by the antipodal map $x \mapsto -x$. Since topologically $\mathbb{R}^3/\mathbb{Z}_2$ is a cone over $\mathbb{RP}^2$, the underlying space of this orbifold is not (topologically) a manifold.

(5) (*The $\mathbb{Z}_n$-teardrop*) The underlying space is $\mathbb{S}^2$ and the singular locus is a single point with isotropy group $\mathbb{Z}_n$, $n > 1$.

(6) (*The $\mathbb{Z}_p - \mathbb{Z}_q$-football*) The underlying space is again $\mathbb{S}^2$ and the singular locus is a pair of cone points (N and S in figure 2.4) with isotropy $\mathbb{Z}_p$, respectively $\mathbb{Z}_q$ ($p \neq q$).

Note that except the last two examples, the orbifolds considered above are good (actually they are very good). For a proof of the non-developability of the orbifolds in (5) respectively in (6) see (the fundamental group and Euler number...

Figure 2.1: The $\mathbb{Z}_n$-football is covered by $\mathbb{S}^2$ and its fundamental group is $\mathbb{Z}_n$. The tangent cone at N is the cone $\mathbb{R}^2/\mathbb{Z}_n$ of angle $2\pi/n$.

Figure 2.2: The pillowcase is covered by $\mathbb{T}^2$.

Figure 2.3: The $\mathbb{Z}_n$-teardrop is not developable. The orbifold atlas consists of two open sets U_1 and U_2 uniformized by $V_1 = D^2$ and $V_2 = D^2/\mathbb{Z}_n$ respectively.

2.7 Universal coverings and the fundamental group of orbifolds

We begin by briefly recalling the notion of coverings of topological spaces. A projection $\pi : \tilde{X} \to X$ is a covering map if every $x \in X$ has a neighborhood U such that $\pi^{-1}(U)$ is a disjoint union of sets V_i for each of which the restriction $\pi|_{V_i} : V_i \to U$ is a homeomorphism.

Orbifold covering spaces are defined similarly. A projection $p : Q' \to Q$ between orbifolds is called a covering map if it satisfies the condition that, for each $x \in Q$, there exists a neighborhood U uniformized by $(\widetilde{U}, \Gamma)$ such that for each connected component U_i of ${p'}^{-1}(U)$ in Q', the uniformizing systems of U_i is $(\widetilde{U}, \Gamma_i)$ for some subgroup $\Gamma_i \leq \Gamma$. Note that the underlying space $|Q'|$ is not generally a covering space of $|Q|$. The *universal covering* of a connected orbifold Q is a connected covering orbifold $p : \widetilde{Q} \to Q$ such that for any connected covering orbifold $p' : Q' \to Q$, $\widetilde{Q}$ is a covering orbifold of Q' with a projection $pr : \widetilde{Q} \to Q'$ factoring $p : \widetilde{Q} \to Q$ through $p' : Q' \to Q$.

$$\begin{tikzcd} \widetilde{Q} \arrow[r, "pr"] \arrow[d, "p"'] & Q' \arrow[dl, "p'"] \\ Q & \end{tikzcd}$$

Thurston proved existence of universal orbifold covers (see Proposition 13.2.4 in [T]) and used them to define the orbifold fundamental group in terms of deck transformations. We will present the details of this theory for the special case of 2-dimensional orbifolds, but before that we discuss the case of a global quotient $Q = M/\Gamma$. The quotient $M \to M/\Gamma$ can be regarded as an orbifold covering with Γ as the group of deck transformations. Similarly, any subgroup Γ' induces an intermediate orbifold covering $M/\Gamma' \to M/\Gamma$. On the other hand, any manifold covering $\tilde{M} \to M$ gives an orbifold covering by composing with the quotient map $M \to M/\Gamma$. In particular, the universal covering gives rise to a universal orbifold covering of Q, and the orbifold fundamental group belongs in a short exact sequence

$$1 \to \pi_1 M \to \pi_1^{orb} Q \to \Gamma \to 1.$$

For the remainder of this section, we take Q to be a 2-dimensional orbifold. For the purpose of proving the existence of universal covers, we can assume that Q has no boundary, for if it did, by doubling Q along its boundary, we get an orbifold without boundary that double covers Q and has the same universal cover as Q.

The singular locus of Q consists of cone points and corner reflectors, which are singularities modeled by R^2/D_n for the dihedral group of order $2n$. We can further assume Q does not contain any corner reflectors, for if it did, then by doubling Q along the reflector lines, we obtain an orbifold that covers Q with two cone points for each cone point in Q and one cone point for each corner reflector.

Denote by Σ the singular locus of Q and suppose $p : \hat{Q} \to Q$ is an orbifold covering. Note that $\widehat{Q} - p^{-1}(\Sigma)$ is a manifold cover for the regular set $Q \setminus \Sigma$. Let now $x_i \in \Sigma$ be a singular point with cone angle $2\pi/n_i$. That is, a neighborhood of x_i is uniformized by $(\mathbb{R}^2, \mathbb{Z}_{n_i})$ with x_i corresponding to the cone point in $\mathbb{R}^2/\mathbb{Z}_{n_i}$. Any point in $\widehat{Q}$ above x_i will have cone angle $2\pi/m_i$, where $m_i|n_i$. Denote by X the manifold obtained from Q by removing the interior of small cones centered at the singular points and denote by $\widehat{X} = p^{-1}(X) \subset \widehat{Q}$. Hence the regular set $Q \setminus \Sigma$ is just X with pointed discs attached to the ∂X. The covering $p|_{\widehat{X}} : \widehat{X} \to X$ induced by the orbifold covering $\widehat{Q} \to Q$ has the property that a circle C_i of ∂X bounding a cone of angle $2\pi/n_i$ has the pre-image consisting of circles which projects with degree which divides n_i. Thus $\pi_1(\widehat{X})$ contains all conjugates of $\alpha_i^{n_i}$, where $\alpha_i \in \pi_1(X)$ which represents the circle C_i. Define G to be the quotient of $\pi_1(X)$ by adding the relations $\alpha_i^{n_i} = 1$ and let H be the kernel of the natural homomorphism $\pi_1(X) \to G$. Clearly H is a subgroup of $\pi_1(\widehat{X})$. It follows that the covering $\widetilde{X}$ of X determined by H is universal among the covers of X which extend to a cover for Q. By adding to $\widetilde{X}$ the appropriate cones along $\partial\widetilde{X}$, we obtain an orbifold $\widetilde{Q}$, which is the universal orbifold covering of Q. As Q is the quotient of $\widetilde{Q}$ by the action of a group, in this case the group being G, we call G the fundamental group of Q.

Inspired by the construction above, we can easily give a presentation of the $\pi_1^{orb}(Q)$ in the case when Q is a closed 2-dimensional orbifold which has only cone points as singular points. We start with the fundamental group of the manifold obtained from Q by removing small neighborhoods of the cone points and we add the relations $\alpha_i^{n_i} = 1$. Thus, if the underlying space $|Q|$ is a closed orientable surface of genus g and if Q has m cone points of order n_i, $1 \leq i \leq m$, then a presentation for $\pi_1^{orb}(Q)$ is

$$\left\{a_1, b_1, \ldots, a_g, b_g, \alpha_1, \ldots, \alpha_m \;\middle|\; \alpha_i^{n_i} = 1, \prod_{i=1}^{g}[a_i, b_i]\alpha_1 \ldots \alpha_m = 1\right\}.$$

Using this presentation of the orbifold fundamental group, we can see that the orbifold considered in example (5) above is not covered by a manifold (i.e. is bad). Indeed, since the underlying space is $\mathbb{S}^2$, the orbifold fundamental group for the $\mathbb{Z}_n$-teardrop is obtained from the fundamental group of $\mathbb{S}^2$ minus a point (which is trivial) by adding a relation, so it is obviously trivial. Hence the $\mathbb{Z}_n$-teardrop has no covering (other than itself).

We will introduce first the Euler number of good compact 2-dimensional orbifold Q. First note that in general if $\widetilde{X} \to X$ is a k-fold covering space then $\chi(\widetilde{X}) = k\chi(X)$. As we have seen in the previous section every good compact 2-dimensional orbifold Q is finitely covered by a manifold N. Thus is natural to define the Euler number of Q by

$$\chi(Q) = \frac{1}{n}\chi(N),$$

where n is the degree of the covering $N \to Q$. Note that the Euler number of an orbifold is not in general an integer, but is always a rational number. We will compute $\chi(Q)$ directly from the description of the orbifold.

Consider first the case when the orbifold Q is a good compact 2-dimensional orbifold which has r cone points of order n_i, $1 \leq i \leq r$ as singular points. Let X be the closure of the manifold obtained from $|Q|$ by removing r disjoint 2-discs $D_1, D_2, \ldots, D_r$ centered at the cone points. Thus $|Q| = X \cup \left(\bigcup_{i=1}^{r} D_i\right)$ and

$$\chi(|Q|) = \chi(X) + r$$

since $\chi(D_i) = 1$ and $\chi(\mathbb{S}^1) = 0$. If $p : N \to Q$ is a finite manifold cover of Q as above and $\widetilde{X} = p^{-1}(X)$, then $p|_{\widetilde{X}} : \widetilde{X} \to X$ is a manifold covering of the same degree n, hence $\chi(\widetilde{X}) = n\chi(X)$. The pre-image in N of the discs D_i by p are only n/n_i 2-discs, so we have

$$\chi(N) = n\chi(X) + \sum_{i=1}^{r} n/n_i.$$

The formula for the orbifold Euler number, known also as the Riemann-Hurwitz formula, follows immediately

$$\chi(Q) = \chi(|Q|) - \sum_{i=1}^{r} \big(1 - \frac{1}{n_i}\big).$$

Consider now the case when Q is a good compact 2-dimensional orbifold which has r cone points of order n_i, $1 \leq i \leq r$ and s corner reflectors of order m_j, $1 \leq j \leq s$. By doubling Q along the reflector curves, we obtain a 2-fold orbifold cover of Q, denoted DQ, which has r pairs of cone points with cone angle $2\pi/n_i$, $1 \leq i \leq r$ and s cone points with angle $2\pi/m_j$, $1 \leq j \leq s$. Then by above formula we have

$$\chi(DQ) = \chi(|DQ|) - 2\sum_{i=1}^{r} \big(1 - \frac{1}{n_i}\big) - \sum_{j=1}^{s} \big(1 - \frac{1}{m_j}\big).$$

Since $\chi(DQ) = 2\chi(Q)$ and $\chi(|DQ|) = 2\chi(|Q|)$, we obtain in this case the following formula for the orbifold Euler number

$$\chi(Q) = \chi(|Q|) - \sum_{i=1}^{r} \big(1 - \frac{1}{n_i}\big) - \frac{1}{2}\sum_{j=1}^{s} \big(1 - \frac{1}{m_j}\big).$$

This formula works also for the bad orbifolds. The $\mathbb{Z}_p - \mathbb{Z}_q$-football has the Euler number $2 - (1 - \frac{1}{p}) - (1 - \frac{1}{q}) = -\frac{p+q}{pq}$ which proves that the orbifold cannot be covered by a manifold, hence is bad.

If the orbifold Q is equipped with a Riemannian metric (see Definition 2.6.1), we can extend to orbifolds the Gauss-Bonnet theorem (see [Sa2])

$$\int_Q K dA = 2\pi\chi(Q),$$

where K denotes the sectional curvature of Q (see **??**) and dA denotes the area element. Note that an orbifold has a well defined area which has the same naturality property as the euler number, i.e. if $\widetilde{Q} \to Q$ is a finite orbifold covering of degree n, then $A(\widetilde{Q}) = nA(Q)$. The argument used in the proof is similar to the one used above to determine the orbifold Euler number : we consider the manifold with boundary obtained by removing disjoint small discs containing the singular points and we apply the usual Gauss-Bonnet theorem for manifolds with boundary. We say that the orbifold Q has an elliptic, parabolic or hyperbolic structure if $\chi(Q)$ is respectively positive, zero or negative. If the Q is elliptic or hyperbolic then the area $A(Q) = 2\pi|\chi(Q)|$.

Figure 2.4: The $\mathbb{Z}_p - \mathbb{Z}_q$-football is not developable. The orbifold atlas consists of two open sets U_N and U_S uniformized by $V_N = D^2/\mathbb{Z}_p$ and by $V_S = D^2/\mathbb{Z}_q$ respectively.

Chapter 3

Groupoids

The purpose of this chapter is to describe orbifolds in terms of topological groupoids. As we will show in section 3.4 any orbifold structure can be represented in a natural way by an étale groupoid (the étale groupoid associated to the pseudogroup of change of charts of the orbifold). However, there might be more groupoids representing the same orbifold structure. In section 3.3 we introduce an equivalence relation between groupoids, called *Morita equivalence* and which is a week equivalence of categories. A theorem of Moerdijk and Pronc [**?**] states that the category of orbifolds is equivalent to a quotient category of the proper étale groupoids after inverting Morita equivalence. Hence, whenever we consider an orbifold, we can chose up to Morita equivalence a proper étale groupoid representing it. As we will see, the theory of groupoids provides a more convenient language for developing the foundation of the theory of orbifolds.

In the first section we begin by defining the groupoid and some standard notions associated to it and then we provide some examples of groupoids. In section 3.2 we investigate the relation between étale groupoids and the pseudogroups of local homeomorphisms of a topological space. We show there that to each étale groupoid we can associate a pseudogroup of local homeomorphisms of its space of objects and conversely that to any pseudogroup of local homeomorphisms of a topological space X there is associated an étale groupoid whose space of objects is X. However, this correspondence is not one-to-one as we can see in Remark 3.2.3. In section 3.3 we introduce the equivalence of étale groupoids and so we have all the ingredients for describing orbifolds in terms of étale groupoids which will be done in section 3.4.

In section 3.5 we define the notion of $\mathscr{G}$-space and describe the groupoid associated to the action of a groupoid on a topological space. We emphasis there how the language of groupoids leads to a uniform definition of structures over orbifolds, like covering spaces, vector bundles and principal bundles. A more detailed presentation of the results there is contained in [MM].

In section **??** we introduce morphisms from topological spaces to topological groupoids to be the groupoid homomorphism between the pair groupoid associated to the topological space and the topological groupoid. We give there a description of a morphism in terms of $\mathscr{G}$-maps and we see that if the groupoid represents an orbifold, this notion coincides with that of a good orbifold map (introduced in section 2.4.1) defined on a topological manifold (considered as an orbifold with trivial orbifold structure). Another way of describing morphisms from topological spaces to groupoids is in terms of principal $\mathscr{G}$-bundles. We show that there is a natural bijective correspondence between the set $H^1(K, \mathscr{G})$ of morphisms from the topological space K to the groupoid $\mathscr{G}$ and the set of isomorphism classes of principal $\mathscr{G}$-bundles over K. After we introduce the notion of relative morphisms, we describe the homotopy relative to a given morphism of a subspace and define the homotopy groups of a topological groupoid.

In section 3.7 we investigate the particular case of morphisms from $\mathbb{R}$ to a topological groupoid. In this case a morphism is described by an equivalence class of $\mathscr{G}$-paths ($\mathscr{G}$-maps from $\mathbb{R}$ or the interval $I = [0, 1]$ to the groupoid). This allows as to introduce the set of $\mathscr{G}$-paths between two points (objects) in the groupoid as well as the set of $\mathscr{G}$-loops based at a point. We give a description of the fundamental group of a groupoid based at a point and see that in the case of a connected ($\mathscr{G}$-connected) groupoid, up to isomorphism, the fundamental group is independent of the base point. In the last part we introduce the set of free loops on an orbifold and the set of free loops projecting to a constant loop, which is the base space of the inertia orbifold as we will see in chapter **??**.

We conclude this chapter with the section 3.8 on the classifying space of a groupoid. We see there that an equivalence between groupoids induces a homotopy equivalence between their classifying spaces. This allows us to define the homotopy type of an orbifold as being that of the classifying space of a groupoid representing it. In this way we recover the orbifold homotopy groups defined in section **??**. When Q is a Riemannian orbifold there is an explicit construction of the classifying space of the groupoid of germs of change of charts, which is independent of the particular atlas defining the orbifold structure. This construction is due to Haefliger (see [H] and also [GH]) and we present it here for the sake of completeness.

3.1 Definitions and examples

A groupoid can be thought of as a generalization of a group, a manifold and an equivalence relation. First as an equivalence relation, a groupoid has a set of relations that we will think of as arrows. It will be denoted here by $\mathscr{G}$. These elements arrows relate is a set X (that we will think as objects or points). Each arrow $g \in \mathscr{G}$ has a source $x = \alpha(g) \in X$ and a target $y = \omega(g) \in X$. Then we say $g : x \to y$, i.e. x is related to y. We want to have an equivalence relation. Then for the symmetry we need that each arrow to be invertible, for the reflexivity we need arrows which have any element in X as source and target at the same time and for transitivity we need a way to compose arrows. We also require $\mathscr{G}$ and X to be more than just sets. sometimes we want them to be locally Hausdorff, paracompact, locally compact topological spaces or smooth manifolds.

Here is the formal definition of the notion of groupoid we will use in this framework.

Definition 3.1.1. *A* groupoid *is a small category in which each arrow is invertible.*

We will denote the groupoid $(\mathscr{G}, X)$ and it consists of a set of *objects* X and a set of *arrows* $\mathscr{G}$, together with the following structure maps:

(i)

$$\alpha, \omega : \mathscr{G} \to X,$$

the *source* and the *target* maps, which assign to each arrow $g \in \mathscr{G}$ its initial object $\alpha(g)$, respectively its terminal object $\omega(g)$.

(ii)

$$m : \mathscr{G} \times_X \mathscr{G} := \{(h,g) \in \mathscr{G} \times \mathscr{G} |\ \alpha(h) = \omega(g)\} \ \rightarrow \ \mathscr{G},$$

the *composition* map, which assign to each pair of arrows (h,g) with $\alpha(h) = \omega(g)$ their composition hg with $\alpha(hg) = \alpha(g)$ and $\omega(hg) = \omega(h)$. This composition is required to be associative.

(iii)

$$u : X \rightarrow \mathscr{G},$$

the *unit* map, which identifies each object $x \in X$ with the unit arrow $1_x \in \mathscr{G}$. The unit arrow 1_x is a two-sided unit for the composition, i.e. $g1_x = g$ and $1_x h = h$ for any two arrows $g, h \in \mathscr{G}$ such that $\alpha(g) = x = \omega(h)$.

(iv)

$$i : \mathscr{G} \rightarrow \mathscr{G},$$

the *inverse* map, which assign to each arrow $g \in \mathscr{G}$ the inverse arrow $g^{-1} \in \mathscr{G}$ with $\alpha(g^{-1}) = \omega(g)$ and $\omega(g^{-1}) = \alpha(g)$. The inverse arrow g^{-1} is a two-sided inverse for the composition, i.e. $g^{-1}g = 1_{\omega(g)}$ and $gg^{-1} = 1_{\alpha(g)}$.

Let $(\mathscr{G}, X)$ be a groupoid and consider an object $x \in X$. The set

$$\mathscr{G}_x := \{g \in \mathscr{G} |\ \alpha(g) = \omega(g) = x\}$$

is a group and it is called the *isotropy group* of x. The subset

$$\mathscr{G}.x := (\omega \circ \alpha^{-1})(x) = \{y \in X |\ \exists g \in \mathscr{G}, \alpha(g) = x, \omega(g) = y\}$$

is called the $\mathscr{G}$*-orbit* of x. The $\mathscr{G}$-orbits form a partition of X and the sets of $\mathscr{G}$-orbits will be denoted by $X/\mathscr{G}$.

A *topological (smooth) groupoid* $(\mathscr{G}, X)$ is a groupoid for which the spaces $\mathscr{G}$ and X are endowed with topological (smooth) structure, such that the structure maps are continuous (smooth) maps and the unit map is homeomorphism (diffeomorphism) onto its image.

A topological groupoid is called *proper* if the map $(\alpha, \omega) : \mathscr{G} \to X \times X$ is a proper map. Note that in a proper groupoid every isotropy group is compact.

A topological (smooth) groupoid is called a *foliation groupoid* if each isotropy group $\mathscr{G}_x$ is discrete.

A topological (smooth) groupoid is called *étale* if the source and target maps are étale maps, i.e. are local homeomorphisms (diffeomorphisms). Note that any étale groupoid is a foliation groupoid and that any proper étale groupoid has finite isotropy groups.

A groupoid $(\mathscr{G}, X)$ of *local isometries* is an étale groupoid with a length metric on X that induces the given topology on X and is such that the elements of the associated pseudogroup are local isometries.

A groupoid of local isometries $(\mathscr{G}, X)$ is *Hausdorff* if $\mathscr{G}$ is Hausdorff as a topological space and for every continuous map $c : [0, 1) \to \mathscr{G}$, if $\lim_{t\to 1} \alpha \circ c$ and $\lim_{t\to 1} \omega \circ c$ exists, then $\lim_{t\to 1} c(t)$ exists.

A groupoid $(\mathscr{G}, X)$ of local isometries of X is *complete* if X is locally complete (i.e. each point of X has a complete neighborhood) and if the space of orbits with the quotient pseudometric is complete.

Example 3.1.2. (1) Any discrete group can be viewed as a groupoid, where the set of objects is the one-point space and the set of arrows is the group itself. In this case the composition of the groupoid is just the multiplication of the group. This groupoid is étale.

(2) Any topological space X can be viewed as an étale groupoid where all the arrows are units (or equivalently the unit map is a bijection). This is called the *unit groupoid* on X or the *trivial groupoid* X.

(3) Let M be a (smooth) manifold and let $\mathscr{U} = \{U_i\}_{i\in I}$ be an open cover. To the pair $(M, \mathscr{U})$ we associate a groupoid defined as follows. An object is a pair (m, i) such that $m \in U_i$ and endow the space of objects with the topology of $\coprod_{i\in I} U_i$.

The arrows are triples (m, i, j) such that $m \in U_i \cap U_j = U_{ij}$ and the topology of the space of arrows is that of $\coprod_{ij} U_{ij}$. Note that in this notation $U_{ij} \neq U_{ji}$. The structure maps are defined to be the natural maps: $\alpha|_{U_{ij}} : U_{ij} \to U_i$ the source, $\omega|_{U_{ij}} : U_{ij} \to U_j$ the target, $u|_{U_i} : U_i \to U_i = U_{ii}$ the unit map, $i|_{U_{ij}} : U_{ij} \to U_{ji}$ the inverse map and respectively $m|_{U_{ijk}} : U_{ijk} \to U_{ik}$ the multiplication map. The groupoid above is called the *covering groupoid* and is denoted $M_{\mathscr{U}}$.

(4) Let X be a topological space and Γ a group acting by homeomorphisms on it. One can define the *étale groupoid associated to this action* $(\Gamma \ltimes X, X)$ to be the category whose space of objects is the space X and whose space of arrows $\mathscr{G} := \Gamma \times X$, where Γ is endowed with the discrete topology. The source and the target map are defined to be: $\alpha(\gamma, x) = x$ the projection, and $\omega(\gamma, x) = \gamma.x$ the action. The composition is defined from the multiplication of the group by $(\gamma, x)(\gamma', x') = (\gamma \cdot \gamma', x')$, whenever $x = \gamma'.x'$, and the inverse of (γ, x) is $(\gamma^{-1}, \gamma.x)$.

(5) If a group Γ acts by isometries on a length space X, then the associated groupoid $(\Gamma \ltimes X, X)$ is a groupoid of local isometries and it is Hausdorff. It is complete if and only if X is complete as metric space.

3.2 Pseudogroups of local homeomorphisms

Of a particular interest is the correspondence between the étale groupoids $(\mathscr{G}, X)$ and the pseudogroup of local homeomorphisms of X.

First recall that a *pseudogroup* $\mathscr{H}$ of local homeomorphisms of a topological space X is a collection $\mathscr{H}$ of homeomorphisms $h : U \to V$ of open sets of X such that:

(i) the inverse and the composition of elements in $\mathscr{H}$ (whenever it is possible) are in $\mathscr{H}$;

(ii) the restriction of an element of $\mathscr{H}$ to any open subset of X belongs to $\mathscr{H}$;

(iii) the identity of X belongs to $\mathscr{H}$;

(iv) $\mathscr{H}$ is closed under union of elements in $\mathscr{H}$.

Proposition 3.2.1. *To each étale groupoid $(\mathscr{G}, X)$ there is associated a pseudogroup of local homeomorphisms of X.*

Indeed, if $(\mathscr{G}, X)$ is an étale groupoid, then each arrow $g \in \mathscr{G}$ with $\alpha(g) = x$ and $\omega(g) = y$, induces a well defined germ of a homeomorphism $\widetilde{g} : U_x \to V_y$ of the form $\widetilde{g} = \omega \circ s$, where $s : U_x \to \mathscr{G}$ is a continuous section of α over the (sufficiently small) open neighborhood U_x of x such that $s(x) = g$. It is easy to see now that the collection of all such local homeomorphisms induced by the arrows of $\mathscr{G}$ form a pseudogroup.

The groupoid $(\mathscr{G}, X)$ is called *effective* (or *reduced*) if the assignment $g \mapsto \widetilde{g}$ is faithful, i.e. for each point $x \in X$ the map $g \mapsto \widetilde{g}$ induces an injective group homomorphism between $\mathscr{G}_x$ the isotropy group of x, and the group $Hom_x(X)$ of homeomorphisms of X which fixes x. If X is a differentiable manifold and if the elements of the pseudogroup associated are diffeomorphisms, then the groupoid $(\mathscr{G}, X)$ is what we defined to be a *differentiable or a smooth étale groupoid.*

Proposition 3.2.2. *To each pseudogroup of local homeomorphisms we can associate an étale groupoid.*

Let $\mathscr{H}$ be a pseudogroup of local homeomorphisms of a topological space X. Let $f : U \to V$ be an element of $\mathscr{H}$ and consider (f, x), its germ at some $x \in U$. That is, the equivalence class of pairs (f, x), given by the equivalence relation $(f, x) \sim (f', x')$, if and only if $x = x'$ and f is equal to f' on some neighborhood of x. The point x is called the *source* and the point $f(x)$ is called the *target* of the germ of f at x.

Denote by $\mathscr{M}_{\mathscr{H}}(X)$ the space of germs of the local homeomorphisms of $\mathscr{H}$. This is an open subspace of the space $\mathscr{M}(X)$ of germs of continuous maps from open sets of X to X, endowed it with the germ topology (a basis of which consists of the subsets U_f which are the unions of germs of continuous maps $f : U \to X$ at various points of U).

Define the source and target maps

$$\alpha, \omega : \mathscr{M}_{\mathscr{H}}(X) \to X$$

associating to each germ its source, respectively its target. With respect to the subspace topology on $\mathscr{M}_{\mathscr{H}}(X)$, the maps α and ω are étale maps, i.e. they are continuous, open maps and their restriction to any sufficiently small open set is a homeomorphism into its image. We can also define a continuous composition map

$$m : \mathscr{M}_{\mathscr{H}}(X) \times_X \mathscr{M}_{\mathscr{H}}(X) \to \mathscr{M}_{\mathscr{H}}(X),$$

by associating to any two germs (f, x) and $(f', f(x))$ their (well defined) composition, the germ $(f' \circ f, x)$. The unit map is the natural inclusion of X in $\mathscr{M}_{\mathscr{H}}(X)$ which associates to each $x \in X$ the germ (id_X, x) and the inverse map $i : \mathscr{M}_{\mathscr{H}}(X) \to \mathscr{M}_{\mathscr{H}}(X)$ associates to each germ (f, x) the well defined germ $(f^{-1}, f(x))$.

Hence, we obtained an étale groupoid $(\mathscr{G}, X)$, with $\mathscr{G} = \mathscr{M}_{\mathscr{H}}(X)$, of all the germs of the elements oh $\mathscr{H}$, called the étale groupoid associated to $\mathscr{H}$.

Remark 3.2.3. From the étale groupoid $(\mathscr{G}, X)$ associated to a pseudogroup of local homeomorphisms $\mathscr{H}$ one can reconstruct $\mathscr{H}$ by considering the pseudogroup associated to $(\mathscr{G}, X)$. In general, the converse is not true. If $\mathscr{H}$ is the pseudogroup associated to an étale groupoid $(\mathscr{G}, X)$, then the étale groupoid associated to $\mathscr{H}$ is a quotient of the original groupoid $(\mathscr{G}, X)$. For instance one can consider the étale groupoid in Example 3.1.2 (1). In this case, the elements of the associated pseudogroup are the elements of the group and the space of germs of local homeomorphisms has only one element.

3.3 Morita equivalence

In this section we introduce the equivalence of étale groupoids and an equivalence relation among étale groupoids, namely the Morita equivalence. The reader should be warned that the exposition in this section is very succinct. For a more general definition of Morita equivalent groupoids (not necessarily étale) and also for the properties of groupoids which are invariant under the Morita equivalence, the reader is referred to [MM].

A *homomorphism* $(\psi, f) : (\mathscr{G}, X) \to (\mathscr{G}', X')$ between two étale groupoids consists of a continuous functor $\psi : \mathscr{G} \to \mathscr{G}'$ inducing a continuous map $f : X \to X'$. We say that (ψ, f) is an *étale homomorphism* if f is an étale map.

We say that the homomorphism $(\psi, f) : (\mathscr{G}, X) \to (\mathscr{G}', X')$ is an *equivalence* if it is étale and the functor ψ is an equivalence, that is:

(i) for each $x \in X$, ψ induces an group isomorphism from $\mathscr{G}_x$ onto $\mathscr{G}'_{f(x)}$,

(ii) $f : X \to X'$ induces a bijection between the orbits sets $X/\mathscr{G}$ and $X'/\mathscr{G}'$.

The equivalences between groupoids generate an equivalence relation among étale groupoids in the following way. We say that two étale groupoids $(\mathscr{G}_1, X_1)$ and $(\mathscr{G}_2, X_2)$ are *Morita equivalent* (or *weak equivalent*) if there exists a third étale groupoid $(\mathscr{G}, X)$ and two étale homomorphisms $(\psi_i, f_i) : (\mathscr{G}, X) \to (\mathscr{G}_i, X_i),\ i = 1, 2$ which are equivalences.

$$\begin{tikzcd} & (\mathscr{G}, X) \arrow[dl, "(\psi_1,f_1)"'] \arrow[dr, "(\psi_2,f_2)"] & \\ (\mathscr{G}_1, X_1) & & \mathscr{G}_2, X_2). \end{tikzcd}$$

The reflexivity and the symmetry of this relation between étale groupoids are obvious. Before we check the transitivity, note the following: if $(\varphi_i, h_i) : (\mathscr{G}_i, X_i) \to (\mathscr{G}', X')$ are two étale homomorphisms which are equivalences, then the *fiber product* $(\mathscr{G}, X) := (\mathscr{G}_1 \times_{\mathscr{G}'} \mathscr{G}_2, X_1 \times_{X'} X_2)$ is naturally an étale groupoid and the projections $(\mathscr{G}, X) \to (\mathscr{G}_i, X_i)$ are equivalences.

$$\begin{tikzcd} (\mathscr{G}_1 \times_{\mathscr{G}'} \mathscr{G}_2, X_1 \times_{X'} X_2) \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & (\mathscr{G}_1, X_1) \arrow[d, "(\varphi_1,h_1)"] \\ (\mathscr{G}_2, X_2) \arrow[r, "(\varphi_2,h_2)"'] & (\mathscr{G}', X'). \end{tikzcd}$$

Then the transitivity follows easily from the diagram

$$\begin{tikzcd}
 & & (\mathscr{G}, X) \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & & \\
 & (\mathscr{G}_1, X_1) \arrow[dl, "\varphi_1'"'] \arrow[dr, "\varphi_2'"] & & (\mathscr{G}_2, X_2) \arrow[dl, "\varphi_2"'] \arrow[dr, "\varphi_1"] & \\
(\mathscr{G}_1', X_1') & & (\mathscr{G}', X') & & (\mathscr{G}_2', X_2').
\end{tikzcd}$$

If $(\mathscr{G}, X)$ and $(\mathscr{G}', X')$ are differentiable étale groupoids, then a homomorphism $(\psi, f) : (\mathscr{G}, X) \to (\mathscr{G}', X')$ is called a *differentiable equivalence* if it is an equivalence and f is locally diffeomorphism. The equivalence generated by this relation is called differentiable equivalence.

In the case of groupoids of local isometries, the equivalence is generated by étale homomorphisms of groupoids $(\varphi, f) : (\mathscr{G}, X) \to (\mathscr{G}', X')$ which are equivalences and such that $f : X \to X'$ is locally an isometry.

Definition 3.3.1. *An étale groupoid* $(\mathscr{G}, X)$ *is* developable *if it is Morita equivalent to the groupoid* $(\Gamma \times \widetilde{X}, \widetilde{X})$ *associated to an action of a group* Γ *by homeomorphisms of a space* $\widetilde{X}$.

3.4 Orbifold groupoids

Recall the definition of a differentiable orbifold given in Remark 2.2.5 with the orbifold structure given by an atlas of uniformizing charts $\{(X_i, q_i)\}_{i\in I}$. Let $X = \bigsqcup_{i\in I} X_i$. We identify each X_i to a connected component of the differentiable manifold X and denote by $q : X \to |Q|$ the union of maps q_i, that is $q(x) = q_i(x)$ whenever $x \in X_i$. Any diffeomorphism h from an open subset U of X to an open subset of X such that $q \circ h = q|_U$ will be called a change of charts. The collection of change of charts form a pseudogroup $\mathscr{H}$ of local diffeomorphisms of X, called the pseudogroup of change of charts of the orbifold with respect to the atlas $\{(X_i, q_i)\}_{i\in I}$. We have seen that if $h : U \to V$ is a change of charts such that U and V are contained in the same X_i and

U is connected, then h is the restriction to U of an element of Γ_i, hence $\mathscr{H}$ contains in particular all the elements of the groups Γ_i. We say that two points $x, x' \in X$ are on the same orbit of $\mathscr{H}$ if there is an element $h \in \mathscr{H}$ such that $h(x) = x'$. This defines an equivalent relation on X whose classes are called the orbits of $\mathscr{H}$. We will denote $X/\mathscr{H}$ the set of orbits endowed with the quotient topology. The map $q : X \to |Q|$ induces a homeomorphism from $X/\mathscr{H}$ to $|Q|$.

Let now $\{(X_i^{(1)}, q_i^{(1)})\}_{i \in I_1}$ and $\{(X_i^{(2)}, q_i^{(2)})\}_{i \in I_2}$ be two atlases defining the same orbifold structure on $|Q|$, and let $\mathscr{H}_1$ respectively $\mathscr{H}_2$ be their pseudogroups of change of charts. We say that the two pseudogroups are *equivalent* if there is a pseudogroup $\mathscr{H}$ of local diffeomorphisms of the disjoint union $X = X_1 \sqcup X_2$ whose restriction to each X_j is equal to $\mathscr{H}_j$ and such that the inclusions of X_j into X induces homeomorphisms between the orbit spaces $X_j/\mathscr{H}_j \to X/\mathscr{H}$, $j = 1, 2$.

More generally, consider a pseudogroup of local diffeomorphisms $\mathscr{H}$ of a differentiable manifold X such that each point $x \in X$ has a neighborhood U such that the restriction of $\mathscr{H}$ to U is generated by a finite group of diffeomorphisms Γ_U of diffeomorphisms of U. Assume moreover that the space of orbits is Hausdorff. Then $X/\mathscr{H}$ has a natural orbifold structure (compare with Proposition 2.2.3).

To the pseudogroup $\mathscr{H}$ of change of charts of an atlas of uniformizing charts $\{(X_i, q_i)\}_{i \in I}$ defining an orbifold Q , we can associate the étale groupoid $(\mathscr{G}, X)$ of all the germs of change of charts, with the topology of germs. Then $q : X \to |Q|$ induces an homeomorphism between the space of orbits $X/\mathscr{G}$ to $|Q|$. As we have seen, we can reconstruct $\mathscr{H}$ from $(\mathscr{G}, X)$ and we will use the notation $Q = X/\mathscr{G}$ to denote the orbifold whose pseudogroup of change of charts is equivalent to the pseudogroup associated to the étale groupoid $(\mathscr{G}, X)$. Note that this groupoid is also proper.

An orbifold structure on a topological space Q could be defined as a differentiable equivalence class of (differentiable) proper étale groupoids $(\mathscr{G}, X)$ together with a homeomorphism from $X/\mathscr{G}$ to Q, such that

(i) $X/\mathscr{G}$ is Hausdorff, and

(ii) each point of X has a neighborhood U such that the restriction of $(\mathscr{G}, X)$ to U is the groupoid associated to an effective action of a finite group on U.

In fact, any proper étale groupoid $(\mathscr{G}, X)$ comes from an orbifold in this way. To see this, note first that any isotropy group of a proper étale groupoid is finite. Furthermore, any proper étale groupoid locally looks like the translation groupoid with respect to an action of the isotropy group. That is, for any $x \in X$ there exists an open neighborhood $U \subset X$ with an action of the isotropy group $\mathscr{G}_x$ such that the restriction $(\mathscr{G}|_U, U)$ is isomorphic to $(U \rtimes \mathscr{G}_x, U)$.

The étale groupoid of change of charts of an orbifold is developable if and only if the orbifold structure is developable.

Recall that an orbifold structure Q is said to be Riemannian if each X_i is a Riemannian manifold and if the changes of charts are Riemannian isometries. On $X = \sqcup X_i$ the Riemannian metric induces a length metric whose quotient gives a pseudometric on the space of orbits $|Q|$. In this case, the groupoid $(\mathscr{G}, X)$ of germs of changes of uniformizing charts is a groupoid of local isometries. The fact that $|Q|$ is Hausdorff implies that the quotient pseudometric is always a metric and induces the given topology on $|Q|$. Moreover the groupoid $(\mathscr{G}, X)$ is complete if and only if $|Q|$ is complete.

3.5 Structures over groupoids

Recall that if G be a topological group then topological space Y is called a (right) G-*space* if there is a continuous right action $Y \times G \to Y$, written $(y, g) = y.g$ satisfying $(y.g).g' = y.(gg')$ and $y.1_G = y$ for any $g, g' \in G$ and $y \in Y$.

Let now $(\mathscr{G}, X)$ denote a topological (smooth) groupoid with source and target projections $\alpha, \omega : \mathscr{G} \to X$. A (right) $\mathscr{G}$-*space* is a topological (smooth) manifold E together with a continuous (smooth) map $p_E : E \to X$ and a continuous (smooth) right action of $\mathscr{G}$ on E with respect to the map p_E. That is, a continuous (smooth) map from

$$E \times_X \mathscr{G} := \{(e, g) |\ p_E(e) = \omega(g)\}$$

to E, written $(e,g) = e.g$, and such that $\alpha(g) = p_E(e.g)$, $(e.g).g' = e.(gg')$ and $e.1_x = e$. Note that for any groupoid $(\mathscr{G}, X)$, the space X is trivially a $\mathscr{G}$-space.

For two such $\mathscr{G}$-spaces (E, p_E) and $(E', p_{E'})$, a map of $\mathscr{G}$-spaces is a continuous (smooth) map $h : E \to E'$ which commutes with the structure, i.e. $p_{E'} \circ h = p_E$ and $h(e.g) = h(e).g$ (here each action is to be considered in the corresponding $\mathscr{G}$-space). This defines a category $(\mathscr{G} - spaces)$. If

$$(\psi, f) : (\mathscr{G}', x') \to (\mathscr{G}, X)$$

is a homomorphism of groupoids, then there is a functor

$$\psi^* : (\mathscr{G} - spaces) \to (\mathscr{G}' - spaces)$$

which associates to each $\mathscr{G}$-space E the pullback $E \times_X X'$ with the induced action. If (ψ, f) is an equivalence of groupoids, then ψ^* is an equivalence of categories. Thus, up to equivalence of categories, the category $(\mathscr{G} - spaces)$ depends only on the Morita equivalence of the groupoid $\mathscr{G}$.

Similar to the group case, if E is a $\mathscr{G}$-space, one can define the groupoid associated to the action of $\mathscr{G}$ on E to be the groupoid $(E \times_X \mathscr{G}, E)$. Its arrows $g : e' \to e$ are arrows $g : p_E(e') \to p_E(e)$ in $\mathscr{G}$ with $e.g = e'$, and its source and target

$$\alpha_E, \omega_E : E \times_X \mathscr{G} \to E$$

are given by $\alpha_E(e,g) = e'$ the action and $\omega_E(e,g) = e$ the projection. We will denote this groupoid by $(E \rtimes \mathscr{G}, E)$, its dependance of the space X is to be understood.

There is an obvious homomorphism of groupoids

$$(\pi, p_E) : (E \rtimes \mathscr{G}, E) \to (\mathscr{G}, X)$$

and the fiber over $x \in X$ is $p_E^{-1}(x) \subset E$. Note that if $(\psi, f) : (\mathscr{G}', X') \to (\mathscr{G}, X)$ is a homomorphism of groupoids, then the diagram

$$\begin{tikzcd}
(\psi^*(E) \rtimes \mathscr{G}', \psi^*(E)) \arrow[r] \arrow[d] & (E \rtimes \mathscr{G}, E) \arrow[d, "(\pi, p_E)"] \\
(\mathscr{G}', X') \arrow[r, "(\psi, f)"'] & (\mathscr{G}, X).
\end{tikzcd}$$

is a week pullback up to Morita equivalence. We will define the quotient $E/\mathscr{G}$ to be the space of orbits of the groupoid $(E \rtimes \mathscr{G}, E)$. This in general is not a manifold and note that at the level of orbit spaces $E/\mathscr{G} \to X/\mathscr{G}$ the fiber above x is $p_E^{-1}(x)/\mathscr{G}_x$.

It is easy to see that if the groupoid $(\mathscr{G}, X)$ is an étale groupoid or is a foliation groupoid, then so is the groupoid $(E \rtimes \mathscr{G}, E)$. Moreover, if E is Hausdorff and $(\mathscr{G}, X)$ is proper, then $(E \rtimes \mathscr{G}, E)$ is also proper. In particular, if the groupoid $(\mathscr{G}, X)$ represents an orbifold $Q = X/\mathscr{G}$, then any Hausdorff $\mathscr{G}$-space E represents an orbifold $E/E \rtimes \mathscr{G} \to X/\mathscr{G}$ over Q.

A *covering space* over a groupoid $(\mathscr{G}, X)$ is a $\mathscr{G}$-space $\widehat{X}$ for which the the map $p : \widehat{X} \to X$ is a covering map. The covering spaces of an groupoid form a full subcategory of $(\mathscr{G} - spaces)$ denoted $Cov(\mathscr{G})$. If $(\psi, f) : (\mathscr{G}', X') \to (\mathscr{G}, X)$ is a groupoid homomorphism, the functor $\psi^* : (\mathscr{G} - spaces) \to (\mathscr{G}' - spaces)$ restricts to functor from $Cov(\mathscr{G})$ to $Cov(\mathscr{G}')$ and this is an equivalence of categories whenever (ψ, f) is an equivalence of groupoids.

Let Γ be a group acting by homeomorphisms on a simply connected topological space X. let Γ_0 be a subgroup of Γ. Let $\widehat{X} := X \times \Gamma/\Gamma_0$, where Γ/Γ_0 has the discrete topology. The group Γ acts naturally on $\widehat{X}$ by the rule $(x, \gamma'\Gamma_0).\gamma = (x.\gamma, \gamma'\Gamma_0\gamma)$. Let $p : X \times \Gamma/\Gamma_0 \to X$ be the natural projection. The functor $\pi : \widehat{X} \rtimes \Gamma \to X \rtimes \Gamma$ mapping $(\widehat{x}, \gamma) \mapsto (p(\widehat{x}), \gamma)$ gives the morphism $(\pi, p) : (\widehat{X} \rtimes \Gamma, \widehat{X}) \to (X \rtimes \Gamma, X)$ which can be considered as a covering. The natural inclusion $X \times \Gamma_0 \to \widehat{X} \times \Gamma$ sending $(x, \gamma_0) \mapsto ((x, \Gamma_0), \gamma_0)$ defines an étale homomorphism $(X \rtimes \Gamma_0, X) \to (\widehat{X} \rtimes \Gamma, \widehat{X})$ which is an equivalence. In fact the equivalence classes of connected coverings of the groupoid $(X \rtimes \Gamma, X)$ are in bijective correspondence with the conjugacy classes of subgroups of Γ.

In particular, for an orbifold Q, up to equivalence of categories, there is a well defined category $Cov(Q)$. In this case, $(\mathscr{G}, X)$ is the associated groupoid of germs of changes of charts of an atlas defining the orbifold structure on Q, then any covering $(\widehat{\mathscr{G}}, \widehat{X})$ of $(\mathscr{G}, X)$ is the groupoid of germs of changes of charts of an orbifold structure on the space of orbits $\widehat{Q} := \widehat{X}/\widehat{\mathscr{G}}$, and $\widehat{Q}$ is a covering orbifold.***

A *vector bundle* over a groupoid $(\mathscr{G}, X)$ is a $\mathscr{G}$-space E for which the projection map $p : E \to X$ is a vector bundle over the space of objects and the action of $\mathscr{G}$ on E is fiberwise linear. In particular each fiber E_x is a linear representation of the isotropy group $\mathscr{G}_x$. We denote $Vect(\mathscr{G})$ for the category of vector bundles over $(\mathscr{G}, X)$. If the groupoid $(\mathscr{G}, X)$ is Morita equivalent to another groupoid $(\mathscr{G}', X')$ then the category $Vect(\mathscr{G})$ is equivalent to $Vect(\mathscr{G}')$. In particular, if $(\mathscr{G}, X)$ represents an orbifold Q then up to equivalence of categories there is a well defined a category of vector bundles over Q denoted $Vect(Q)$. It is easy to see that the tangent bundle TX of the space of objects of a groupoid $(\mathscr{G}, X)$ has a natural structure of a vector bundle over $(\mathscr{G}, X)$. A metric on such a vector bundle is a metric in the usual sense which is preserved by the action of $\mathscr{G}$.

Let G be a topological group. A *principal G-bundle* over a groupoid $(\mathscr{G}, X)$ is a $\mathscr{G}$-space P with a left action $G \times P \to P$ which makes the projection $p : P \to X$ into a principal G-bundle over X and is compatible with the groupoid action in the following sense: for any $e \in P$, $\gamma \in G$ and $g \in \mathscr{G}$ such that $\omega(g) = p(e)$ we have $(\gamma e).g = \gamma(e.g)$.

3.6 Morphisms from spaces to topological groupoids. The homotopy groups

Let K be a topological space and $(\mathscr{G}, X)$ be a topological groupoid with source and target projections $\alpha, \omega : \mathscr{G} \to X$. A *morphism* from K to $(\mathscr{G}, X)$ is a homomorphism between the pair groupoid $(K \times K, K)$ (see examples of groupoids) to the groupoid $(\mathscr{G}, X)$. A more direct definition of a morphism from a topological space to a groupoid can be given using $\mathscr{G}$-maps (or cocycles. See [GH]).

Let $\mathscr{U} := \{U_i\}_{i \in I}$ be an open cover of the topological space K. A *$\mathscr{G}$-map* from K to $(\mathscr{G}, X)$ over the cover $\mathscr{U}$ is a collection of continuous maps $f_i : U_i \to X$ such that whenever $U_i \cap U_j \neq \varnothing$ there is a continuous map

$$f_{ij} : U_i \cap U_j \to \mathscr{G} \text{ with } \alpha(f_{ij}(x)) = f_i(x) \text{ and } \omega(f_{ij}(x)) = f_j(x)$$

and which satisfies the cocycle condition

$$f_{ik}(x) = f_{ij}(x) f_{jk}(x),$$

for any $x \in U_i \cap U_j \cap U_k$. Note that the cocycle condition implies in particular that each $f_{ii}(x)$ is a unit of $\mathscr{G}$ and also that $f_{ij}^{-1} = f_{ji}$. Moreover, since the maps f_i can be identified with the maps f_{ii} via the the natural inclusion $X \to \mathscr{G}$, the $\mathscr{G}$-map over $\mathscr{U}$ is completely characterized by the 1-cocycle f_{ij} over $\mathscr{U}$.

Two $\mathscr{G}$-maps over two open covers of K with value in $\mathscr{G}$ are equivalent if there is a $\mathscr{G}$-map with value in $\mathscr{G}$ on the disjoint union of those two covers extending the given ones on each of them. An equivalence class of $\mathscr{G}$-map is called a *morphism* from K to $\mathscr{G}$ (or when Q is an orbifold $\mathscr{G}\backslash X$ a "continuous map" from K to Q). The set of equivalence classes of $\mathscr{G}$-maps on K with value in $\mathscr{G}$ is denoted, according Haefliger, by $H^1(K, \mathscr{G})$. Any morphism from K to $\mathscr{G}$ projects, via $q : X \to \mathscr{G}\backslash X = |Q|$, to a continuous map from K to $|Q|$; note that two distinct morphisms may have the same projection.

If $\mathscr{G}$ and $\mathscr{G}'$ are the groupoids of germs of the changes of charts of two atlases defining the same orbifold structure on a space $|Q|$, then there is a natural bijection between the sets $H^1(K, \mathscr{G})$ and $H^1(K, \mathscr{G}')$. Any continuous map between topological spaces $f : K' \to K$ induces a map

$$f^* : H^1(K, \mathscr{G}) \to H^1(K', \mathscr{G}).$$

Two morphisms from K to $\mathscr{G}$ are homotopic if there is a morphism from $K \times [0, 1]$ to $\mathscr{G}$ such that the morphisms from K to $\mathscr{G}$ induced by the natural inclusions $k \mapsto (k, i)$, $i = 0, 1$, from K to $K \times [0, 1]$ are the given morphisms.

Another description of morphisms from K to $\mathscr{G}$ can be given in terms of isomorphism classes of principal $\mathscr{G}$-bundles over K (see [GH]).

A *principal* $\mathscr{G}$*-bundle* over K is a topological space E together with a surjective continuous map $p : E \to K$ and a continuous action $(e, g) \mapsto e.g$ of $\mathscr{G}$ on E with respect to a continuous map $q_E : E \to X$ such that $p(e.g) = p(e)$. Moreover we assume that the action is simply transitive on the fibers of p in the following sense. Each point of K has an open neighborhood U with a continuous section $s : U \to E$ with respect to p such that the map $U \times_X \mathscr{G} \to p^{-1}(U)$ mapping pairs $(u, g) \in U \times \mathscr{G}$ with $\omega(g) = q_E s(u)$ to $s(u).g$ is a homeomorphism. It follows that if $\mathscr{U} = (U_i)_{i\in I}$ is an open cover of K and if $s_i : U_i \to E$ is a local continuous section of p above U_i for each $i \in I$, then there are unique continuous maps $f_{ij} : U_i \cap U_j \to \mathscr{G}$ such that $s_i(u) = s_j(u)f_{ji}(u)$ for each $u \in U_i \cap U_j$. Thus $f = (f_{ij})$ is a 1-cocycle over $\mathscr{U}$ with value in $\mathscr{G}$.

If $f = (f_{ij})$ is a 1-cocycle over an open cover $\mathscr{U} = (U_i)_{i\in I}$ of K with value in $\mathscr{G}$, then we can construct a principal $\mathscr{G}$-bundle E over K by identifying in the disjoint union of the fiber products

$$U_i \times_X \mathscr{G} = \{(u, g) \in U_i \times \mathscr{G} : \omega(g) = f_{ii}(u)\}$$

the point $(u, g) \in U_i \times_X \mathscr{G}$, $u \in U_i \cap U_j$, with the point $(u, f_{ji}(u)g) \in U_j \times_X \mathscr{G}$. The projections $p : E \to K$ and $q_E : E \to X$ map the equivalence class of $(u, g) \in U_i \times_X \mathscr{G}$ to u and $\alpha(g)$ resp. and the action of g' on the class of (u, g) is the class of (u, gg'). A principal $\mathscr{G}$-bundle obtained in this way by using an equivalent cocycle is isomorphic to the preceding one, i.e. there is a homeomorphism between them projecting to the identity of K and commuting with the action of $\mathscr{G}$. This isomorphism is determined uniquely by a cocycle extending the two given cocycles.

Therefore we see that *there is a natural bijection between the set* $H^1(K, \mathscr{G})$ *and the set of isomorphism classes of principal* $\mathscr{G}$*-bundles over* K. This correspondence is functorial via pull back: if E is a principal $\mathscr{G}$-bundle over K and if $f : K' \to K$ is a continuous map, then the pull back f^*E of E by f (or the bundle induced from E by f) is the bundle $K' \times_K E$ whose elements are the pairs $(k', e) \in K' \times E$ such that $f(k')$ is the projection of e.

$\mathscr{G}$ itself can be considered as a principal $\mathscr{G}$-bundle over X with respect to the projection $\omega : \mathscr{G} \to X$, the map $q_{\mathscr{G}} : \mathscr{G} \to X$ being the source projection. Any continuous map $f : K \to X$ induces the principal $\mathscr{G}$-bundle $f^*\mathscr{G} = K \times_X \mathscr{G}$ over K.

Let K be a topological space, $L \subseteq K$ be a subspace and F be a principal $\mathscr{G}$-bundle over L. A morphism from K to $\mathscr{G}$ relative to F is represented by a pair (E, φ) where E is a principal $\mathscr{G}$-bundles E over K and φ is an isomorphism from F to the restriction $E_{|L}$ of E above L. Two such pairs (E, φ) and (E', φ') represent the same morphism from K to $\mathscr{G}$ relative to F if there is an isomorphism $\Phi : E \to E'$ such that $\varphi' = \Phi \circ \varphi$.

Two morphisms represented by (E_0, φ_0) and (E_1, φ_1) from K to $\mathscr{G}$ relative to F are *homotopic (relative to F)* if there is a bundle E over $K \times I$ and an isomorphism from $E_{|(K \times \partial I) \cup (L \times I)}$ to the bundle obtained by gluing $F \times I$ to $E_0 \times \{0\}$ and $E_1 \times \{1\}$ using the isomorphism φ_0 and φ_1.

Let $I^n = [0,1]^n$ be the n-cube, and let ∂I^n be its boundary. Fix a base point x in X. Let F be the bundle over ∂I^n induced from the bundle $\mathscr{G}$ by the constant map $\partial I^n \to X$ onto the point x. We define $\pi_n((\mathscr{G}, X), x)$ as the set of homotopy classes of principal $\mathscr{G}$-bundle over I^n relative to F. Similar to the case of topological spaces one proves that this set has a natural group structure, called the n^{th}- homotopy group of $(\mathscr{G}, X)$ based at x. In the case where $(\mathscr{G}, X)$ represents a connected orbifold Q, this group is called the n-th homotopy group of Q, and for n=1 the (orbifold) fundamental group of Q.

3.7 Paths and loops on orbifolds

Let $(\mathscr{G}, X)$ be an étale topological groupoid representing an orbifold Q. A $\mathscr{G}$-path on Q is a morphism from $\mathbb{R}$ to $(\mathscr{G}, X)$. If x and y are two points in X a $\mathscr{G}$-path between them parametrized by $[0,1]$ is just a morphism from $[0,1]$ as topological space to $(\mathscr{G}, X)$. Using the compactness of $[0,1]$, such a morphism can be represented over a finite subdivision of the unit interval.

Definition 3.7.1. *Let x and y be two points of X. A $\mathscr{G}$-*path *from x to y over a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ of the interval $[0,1]$ is a sequence $c = (g_0, c_1, g_1, \ldots c_k, g_k)$ where:*

(i) $c_i : [t_{i-1}, t_i] \to X$ are continuous maps,

(ii) g_i are elements of $\mathscr{G}$ such that $\alpha(g_i) = c_i(t_i)$ for $i = 1, 2, \ldots k$, $\omega(g_i) = c_{i+1}(t_i)$ for $i = 0, 1, \ldots k-1$ and $\alpha(g_0) = x$, $\omega(g_k) = y$.

If $(\mathscr{G}, X)$ is a groupoid of local isometries (for example if $(\mathscr{G}, X)$ represents a Riemannian orbifold Q) then we can define the *length* of the $\mathscr{G}$-path $c = (g_0, c_1, g_1, \ldots c_k, g_k)$, $l(c)$ as being the sum of the length of the paths c_i. The pseudodistance on the space of orbits $X/\mathscr{G}$ between the orbits of two points x and y is the infimum of the length of the paths joining x to y.

Among $\mathscr{G}$-paths from x to y parametrized by $[0,1]$ we can define an equivalence relation given by the following operations:

(i) Given a $\mathscr{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ over the subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$, we can add a new point $t' \in (t_{i-1}, t_i)$ together with the unit element $g' = 1_{c_i(t')}$ to get a new sequence, replacing c_i in c by c'_i, g', c''_i, where c'_i and c''_i are the restriction of c_i to the intervals $[t_{i-1}, t']$ respectively $[t', t_i]$.

(ii) Replace the $\mathscr{G}$-path c by a new one $c' = (g'_0, c'_1, g'_1, \ldots, c'_k, g'_k)$ over the same subdivision as follows: for each $i = 1, \ldots, k$ choose continuous maps $h_i : [t_{i-1}, t_i] \to \mathscr{G}$ such that $\alpha(h_i(t)) = c_i(t)$, and define $c'_i(t) := \omega(h_i(t))$, $g'_i := h_{i+1}(t_i) g_i (h_i(t_i))^{-1}$ for $i = 1, \ldots, k-1$ and $g'_0 := h_1(0) g_0$, $g'_k := g_k (h_k(1))^{-1}$.

Remark 3.7.2. (a) if two $\mathscr{G}$-paths on different subdivisions are equivalent, then we can pass from one to another first by considering their equivalent paths by (i) on a suitable common subdivision, and then by an operation similar to (ii).

(b) note that two equivalent $\mathscr{G}$-paths have the same initial and terminal point.

(c) for any $\mathscr{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ From x to y, we can find equivalent paths $c' = (g'_0, c'_1, g'_1, \ldots, c'_l, g'_l)$ such that $g'_0 = 1_x$ or $g'_l = 1_y$. For this reason, by abuse of notation we will denote sometimes the initial point of c by $c(0)$ and the terminal point by $c(1)$, even if g_0 or g_k may not be units.

(d) since the groupoid $(\mathscr{G}, X)$ is Hausdorff and étale, then the continuous maps h_i in (ii) above are uniquely defined by c and c'.

(e) if a $\mathscr{G}$-path c from x to y is such that all the c_i's are constant, then the equivalence class of c is completely characterized by an element $c \in \mathscr{G}$ with $\alpha(g) = x$ and $\omega(g) = y$.

The equivalence class of a $\mathscr{G}$-path c from x to y will be denoted by $[c]_{x,y}$, and the set of such equivalence classes will be denoted by $\Omega_{x,y}(\mathscr{G})$, or simply by $\Omega_{x,y}$.

As we have seen in the previous section, the set $\Omega_{x,y}$ corresponds bijectively to the set of isomorphism classes of principal $\mathscr{G}$-bundles E over $I = [0,1]$ relative to the bundle F over ∂I induced from $\mathscr{G}$ by the map $\partial I \to X$ sending 0 to x and 1 to y. The bundle E is obtained from c as the quotient of the union of the bundles $c_i^*(\mathscr{G})$ by the equivalence relation identifying $(t_i, g_i g) \in c_i^*(\mathscr{G})$ to $(t_i, g) \in c_{i+1}^*(\mathscr{G})$ for $i = 1, \ldots, k-1$. The isomorphism from $E|_{\partial I}$ to F maps $(0, g) \in c_1^*(\mathscr{G})$ to $(0, g_0 g)$ and $(1, g) \in c_k^*(\mathscr{G})$ to $(1, g_k^{-1} g)$.

If $x = y$, c is called a closed $\mathscr{G}$-path (based at x). Its equivalence class is called a $\mathscr{G}$-loop based at x and is denoted by $[c]_x$. The set of $\mathscr{G}$-loops based at x is denoted by $\Omega_x(\mathscr{G})$ or simply Ω_x. The set $\bigcup_{x \in X} \Omega_x$ of based loops will be denoted by Ω_X.

The set Ω_x is in bijective correspondence with the set of isomorphisms classes of principal $\mathscr{G}$-bundle over the circle $\mathbb{S}^1$ relative to the bundle F over $1 \in \mathbb{S}^1$ induced from $\mathscr{G}$ by the map sending 1 to x. The relative bundle $E_{[c]_x}$ corresponding to c is constructed as follows. For $j = 1, \ldots, k$, let $\mathbb{S}^1_j$ be the image of the interval $[t_{j-1}, t_j]$ by the map $t \mapsto e^{2i\pi t}$; let E_j be the pull back of the principal $\mathscr{G}$-bundle $\mathscr{G}$ by the map $\mathbb{S}^1_j \to X$ sending $e^{2i\pi t}$ to $c_j(t)$. The bundle $E_{[c]_x}$ is the quotient of the disjoint of the E_j by the equivalence relation identifying $(e^{2i\pi t_j}, g) \in E_{j+1}$ to $(e^{2i\pi t_j}, g_j g) \in E_j$ for

$j < k$ and $(1, g) \in E_1$ to $(1, g_k g_0 g) \in E_k$. The isomorphism from the restriction of $E_{[c]_x}$ over $\{1\}$ to F maps the equivalence class of $(1, g) \in E_1$ to $(1, g_0 g) \in F$.

Remark 3.7.3. If $c = (g_0, c_1, g_1, \dots, c_k, g_k)$ and $c' = (g'_0, c'_1, g'_1, \dots c'_k, g'_k)$ are two equivalent $\mathscr{G}$-paths from x to y (or closed $\mathscr{G}$-loops at x) over the same subdivision, then the maps h_i in (ii) above induces an isomorphism from the relative principal $\mathscr{G}$-bundle associated to c to the one associated to c'. By Remark 3.7.2 (d) this isomorphism is unique.

Let $x, y \in X$ and $c = (g_0, c_1, g_1, \dots, c_k, g_k)$ be a $\mathscr{G}$-path from x to y, defined over the subdivision $0 = t_0 < t_1 < \dots < t_k = 1$. We can define the *inverse of* c to be the $\mathscr{G}$-path from y to x given by $c^{-1} = (g'_0, c'_1, g'_1, \dots, c'_k, g'_k)$ defined over the subdivision $0 = t'_0 < t'_1 < \dots < t'_k = 1$, where for each $i = 0, \dots, k$ we have $t'_i = 1 - t_{k-i}$, $g'_i = g^{-1}_{k-i}$ and $c'_i(t) = c_{k-i}(1-t)$ for $t \in [t'_{i-1}, t'_i]$ and $i = 1, \dots, k$. It is easy to see that the inverses of equivalent $\mathscr{G}$-paths are equivalent.

If we are given two $\mathscr{G}$-paths $c = (g_0, c_1, g_1, \dots, c_k, g_k)$ over a subdivision $0 = t_0 < t_1 < \dots < t_k = 1$ and $c' = (g'_0, c'_1, g'_1, \dots, c'_{k'}, g'_{k'})$ over $0 = t'_0 < t'_1 < \dots < t'_{k'} = 1$ such that the initial point of c' is the terminal point of c, we can define their *composition*(or concatenation) $c * c'$ to be the $\mathscr{G}$-path $c'' = (g''_0, c''_1, g''_1, \dots, c''_k, g''_k)$ over a subdivision $0 = t''_0 < t''_1 < \dots < t''_{k+k'} = 1$, where

$$t''_i = t_i/2, \text{ for } \ i = 0, \dots, k \ \text{ and } \ t''_i = 1/2 + t'_{i-k}/2, \text{ for } \ i = k, \dots, k+k' \ ;$$

$$c''_i(t) = c_i(t/2), \text{ for } \ i = 1, \dots, k \ \text{ and } \ c''_i(t) = c'_{i-k}(2t-1), \text{ for } \ i = k+1, \dots, k+k' \ ;$$

$$\text{and } \ g''_i = g_i, \text{ for } \ 0, \dots, k-i, \quad g''_k = g'_0 g_k, \quad g''_i = g'_{i-k}, \text{ for } \ i = k+1, \dots, k+k'.$$

Again, if c is equivalent to $\overline{c}$ and c' is equivalent to $\overline{c}'$, then the composition $c * c'$ is equivalent to $\overline{c} * \overline{c}'$.

An *elementary homotopy* between two $\mathscr{G}$-paths $c = (g_0, c_1, g_1, \dots, c_k, g_k)$ over $0 = t_0 < t_1 < \dots < t_k = 1$ and $c' = g'_0, c'_1, g'_1, \dots, c'_k, g'_k)$ over $0 = t'_0 < t'_1 < \dots < t'_k = 1$, with the same end points, is a family of $\mathscr{G}$-paths parametrized by $s \in [s_0, s_1]$, $c^s = (g^s_0, c^s_1, g^s_1, \dots, c^s_k, g^s_k)$ over $0 = t^s_0 < t^s_1 < \dots < t^s_k = 1$ where t^s_i, c^s_i and g^s_i depend continuously on the parameter s, g^s_0 and g^s_k are independent of s and $c^{s_0} = c$, $c^{s_1} = c$.

We say that two paths are *homotopic (relative to their end points)* if one can be obtain from the other by a finite sequence of the following operations:

(i) equivalence of $\mathscr{G}$-paths,

(ii) elementary homotopies.

The homotopy class of a $\mathscr{G}$-path c will be denoted by $[c]$. If c and c' are two composable $\mathscr{G}$-paths, the homotopy class of $c * c'$ depends only on the homotopy classes of c and c' and will be denoted $[c * c'] = [c] * [c']$. If c, c' and c'' are composable $\mathscr{G}$-paths, then $[c * c'] * [c''] = [c] * [c' * c''] = [c] * [c'] * [c'']$.

With the operation of composition of $\mathscr{G}$-paths, the set of homotopy classes of $\mathscr{G}$-loops based at a point x_0 form a group called *the fundamental group* $\pi_1((\mathscr{G}, X), x_0)$ *of* $(\mathscr{G}, X)$ *based at* x_0. A continuous homomorphism of étale groupoids $(\psi, f) : (\mathscr{G}, X) \to (\mathscr{G}', X')$ induces a homomorphism $\pi_1((\mathscr{G}, X), x_0) \to \pi_1((\mathscr{G}', X'), f(x_0))$.

As we can see in the following proposition in the case of a $\mathscr{G}$-connected étale groupoid, up to isomorphism, the fundamental group $\pi_1((\mathscr{G}, X), x_0)$ is independent of the choice of the base point (see [BH]).

Proposition 3.7.4. *Let $(\mathscr{G}, X)$ be an étale groupoid and $x_0 \in X$ be a base point and let a be a $\mathscr{G}$-path joining x_0 to $x_1 \in X$. Then the map that associates to each $\mathscr{G}$-loop based at x_0 the $\mathscr{G}$-loop $a^{-1} * c * a$ based at x_1 induces an isomorphism from $\pi_1((\mathscr{G}, X), x_0)$ to $\pi_1((\mathscr{G}, X), x_1)$.*

If $(\psi, f) : (\mathscr{G}, X) \to (\mathscr{G}', X')$ is an equivalence of étale groupoids, then the induced homomorphism on the fundamental groups is an isomorphism.

A groupoid is called *simply connected* if it is $\mathscr{G}$ connected and its fundamental group is trivial.

In the case when the groupoid $(\mathscr{G}, X)$ represents a connected orbifold Q the fundamental group $\pi_1((\mathscr{G}, X), *)$ is denoted $\pi_1^{orb}(Q)$ and it is called the *orbifold fundamental group of* Q.

Let Q be a topological orbifold and $(\mathscr{G}, X)$ the étale groupoid associated to the pseudogroup of change of uniformizing charts. The set of free loops on Q is the set of equivalence classes of morphisms from the circle $\mathbb{S}^1$ to $(\mathscr{G}, X)$. Such a morphism can be represented by a closed $\mathscr{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ over a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ based at some point $x \in X$. This time the equivalence relation is generated by i) and ii) from the previous section together with

(iii) for any element $g \in \mathscr{G}$ such that $\alpha(g) = x$, then $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ is equivalent to $c.g := (g_0 g, c_1, g_1, \ldots, c_k, g^{-1} g_k)$.

The class of c under this equivalence relation is noted $[c]$ and is called a free loop on Q. It depends only of the orbifold structure Q and not on a particular atlas defining Q.

The groupoid $\mathscr{G}$ acts naturally on the right on the set Ω_X of based $\mathscr{G}$-loops with respect to the projection $p : \Omega_X \to X$ associating to a $\mathscr{G}$-loop based at x the point x: if g is an element of $\mathscr{G}$ with source x and target y, then $[c]_x.g \in \Omega_y$ is the $\mathscr{G}$-loop based at y represented by $c.g$. The action of $\mathscr{G}$ on Ω_X with respect to the projection assigning to a based $\mathscr{G}$-loop its base point is continuous. The quotient of Ω_X by this action is by definition the "space" of (continuous) free loops $|\Lambda(\mathscr{G})| = |\Lambda Q|$ on Q.

Under the projection $q : X \to |Q|$, every free $\mathscr{G}$-loop is mapped to a free loop on $|Q|$. Therefore if $\Lambda|Q|$ is the space of free loops on the topological space $|Q|$, we have a map

$$|\Lambda Q| \to \Lambda|Q|.$$

We denote $|\Lambda^0 Q|$ the subset of $|\Lambda Q|$ formed by the free loops on Q projecting to a constant loop. An element of this subset is represented by a closed $\mathscr{G}$-path $c = (g_0, c_1, g_1)$, where g_0 is a unit 1_x, c_1 is the constant map from $[0, 1]$ to x and g_1 is an element of the subgroup $\mathscr{G}_x = \{\{g \in \mathscr{G} : \alpha(g) = \omega(g) = x\}$. The equivalence class $[c]$ of c correspond to the conjugacy class of g_1 in $\mathscr{G}_x$.

In the developable case, if Q is the quotient of a connected manifold X by a properly discontinuous action of a discrete subgroup Γ of its group of diffeomorphisms, then the free loops based at $x \in X$ are in bijective correspondence with pairs (c, γ), where $c : [0, 1] \to X$ is a continuous path with $c(0) = x$ and γ is an element of Γ mapping $c(1)$ to x. The free loops on Q are represented by classes of pairs (c, γ) like above, this pair being equivalent to $(c \circ \delta, \delta^{-1}\gamma\delta)$, where $\delta \in \Gamma$. Assuming X simply connected, the set of homotopy classes of elements of $|\Lambda Q|$ is in bijective correspondence with the set of conjugacy classes in Γ.

For instance let $X = \mathbb{R}^2$ and Γ be the group generated by a rotation ρ fixing 0 and of angle $2\pi/n$. Let $\mathscr{G}$ be the groupoid associated to the action of Γ on X. The orbifold $Q = X/\mathscr{G}$ is a cone. Consider the closed free $\mathscr{G}$-loop represented by the pair (c, ρ^k), where c is the constant path at 0. If we deform this loop slightly so that it avoids the origin, its projection to the cone $|Q|$ will be a curve going around the vertex a number of times congruent to k modulo n ; in particular, when $k = n$, it could also be a constant loop.

On free loop space, we have two operations, the first one defined by a change of parameter, and the second one by m-times iteration.

Since $|\Lambda Q| = H^1(\mathbb{S}^1, \mathscr{G})$, a homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ induces a bijection $h^* : H^1(\mathbb{S}^1, \mathscr{G}) \to H^1(\mathbb{S}^1, \mathscr{G})$. This gives a continuous action on $|\Lambda Q|$ of the group of homeomorphisms of $\mathbb{S}^1$ leaving invariant the subspace $|\Lambda^0 Q|$ of free loops of length 0. By restriction we get an action of the group $O(2)$ of isometries of $\mathbb{S}^1$ on $|\Lambda Q|$. The fixed points set of the action of $O(2)$ (or $SO(2)$) on $|\Lambda Q|$ is the subspace $|\Lambda^0 Q|$ of free loops of length zero.

For a positive integer m, the map $e^{2i\pi t} \mapsto e^{2i\pi mt}$ from $\mathbb{S}^1$ to $\mathbb{S}^1$ induces a map $H^1(\mathbb{S}^1, \mathscr{G}) \to H^1(\mathbb{S}^1, \mathscr{G})$. If $c = (g_0, c_1, \ldots, c_k, g_k)$ is a closed $\mathscr{G}$-path over a subdivision $0 = t_0 < \cdots < t_k = 1$, the image of $[c]$ by this map is equal to $[c^m]$, where c^m is the m-th iterate of c.

3.8 Classifying space

Given an groupoid $\mathscr{G}$ a very important construction is that of its *classifying space* $B\mathscr{G}$, the base space of a principal $\mathscr{G}$-bundle $E\mathscr{G} \to \mathscr{G}$. In the general case one possible construction is the geometric realization of the nerve of the groupoid representing the orbifold. This follows Segal's fat realization construction. An alternative method is Milnor's infinite join construction.

Let $(\mathscr{G}, X)$ denote a topological groupoid with source and target maps $\alpha, \omega : \mathscr{G} \to X$. Let $\mathscr{G}^{(n)}$ denote the iterated fiber product

$$\mathscr{G}^{(n)} := \{(g_1, g_2, \ldots, g_n) \mid g_i \in \mathscr{G}, \alpha(g_i) = \omega(g_{i+1}) \textit{ for } i = 1, \ldots, n-1\}$$

and let $\mathscr{G}^{(0)} = X$ denote the set of objects. It is worth thinking of $\mathscr{G}^{(n)}$ as the manifold of composable strings of arrows in $\mathscr{G}$:

$$x_0 \xleftarrow{g_1} x_1 \xleftarrow{g_2} x_2 \longleftarrow \cdots \xleftarrow{g_n} x_n$$

With this data we can form a simplicial set (see [Se]).

Definition 3.8.1. *A* semi-simplicial set (resp. group, space, manifold) $A_\bullet$ *is a sequence of sets (resp. groups, spaces, manifolds)* $\{A_n\}_{n\in\mathbb{N}}$ *together with maps*

$$A_0 \rightleftarrows A_1 \rightleftarrows A_2 \rightleftarrows \cdots \rightleftarrows A_m \rightleftarrows \cdots$$

$$\partial_i : A_m \to A_{m-1}, \quad s_j : A_m \to A_{m+1}, \quad 0 \leqslant i, j \leqslant m$$

called boundary *and* degeneracy *maps, satisfying*

$$\begin{aligned} \partial_i \partial_j &= \partial_{j-1}\partial_i \quad \textit{if } i < j \\ s_i s_j &= s_{j+1} s_i \quad \textit{if } i < j \\ \partial_i s_j &= \begin{cases} s_{j-1}\partial_i & \textit{if } i < j \\ 1 & \textit{if } i = j, j+1 \\ s_j \partial_{i-1} & \textit{if } i > j+1 \end{cases} \end{aligned}$$

The *nerve* of a category $\mathscr{C}$ (see [Se]) is a semi-simplicial set $N\mathscr{C}$ where the objects of $\mathscr{C}$ are vertices, the morphisms of $\mathscr{C}$ are the 1-simplexes, the triangular commutative diagrams are the 2-simplexes, and so on.

For a groupoid $(\mathscr{G}, X)$, the corresponding simplicial object $N\mathscr{G}$ is defined by $N\mathscr{G}_n = A_n = \mathscr{G}^{(n)}$ and the boundary maps $\partial_i : \mathscr{G}^{(n)} \to \mathscr{G}^{(n-1)}$:

$$\partial_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0 \\ (g_1, \ldots, m(g_i, g_{i+1}), \ldots, g_n) & \text{if } 1 \le i \le n-1 \\ (g_1, \ldots, g_{n-1}) & \text{if } i = n \end{cases}$$

and the degeneracy maps $s_j : \mathscr{G}^{(n)} \to \mathscr{G}^{(n+1)}$:

$$s_j(g_1, \ldots, g_n) = \begin{cases} (u(\alpha(g_1)), g_1, \ldots, g_n) & \text{for } j = 0 \\ (g_1, \ldots, g_j, u(\omega(g_j)), g_{j+1}, \ldots, g_n) & \text{for } j \ge 1 \end{cases}$$

Denote by Δ^n the standard n-simplex in $\mathbb{R}^n$. Let $\delta_i \colon \Delta^{n-1} \to \Delta^n$ be the linear embedding of Δ^{n-1} into Δ^n as the i-th face, and let $\sigma_j \colon \Delta^{n+1} \to \Delta^n$ be the linear projection of Δ^{n+1} onto its j-th face. Define the spaces $\Delta^n \times \mathscr{G}^{(n)}$, where the points of $\Delta^n \times (g_1, g_2, \ldots, g_n)$ are understood as the points in the simplex with longest sequence of edges being named as $(g_1, g_2, \ldots, g_n)$.

By gluing these spaces together along the simplicial operators we obtain the so called *geometric realisation* of the simplicial object $A_\bullet$.

Definition 3.8.2. *The* geometric realization $|A_\bullet|$ *of the simplicial object* $A_\bullet$ *is the space*

$$|A_\bullet| = \left(\coprod_{n \in \mathbb{N}} \Delta^n \times A_n \right) \Big/ \begin{array}{c} (z, \partial_i(x)) \sim (\delta_i(z), x) \\ (z, s_j(x)) \sim (\sigma_j(z), x) \end{array}$$

The semi-simplicial object $N\mathscr{C}$ determines $\mathscr{C}$ and its topological realization $B\mathscr{C}$ is called the *classifying space of the category.* Here $\mathscr{C}$ is a *topological category* in Segal's sense [Se].

For a groupoid $(\mathscr{G}, X)$ we will call $B(\mathscr{G}, X) = B\mathscr{G} = |N\mathscr{G}|$ the *classifying space of the groupoid.* Note that $B\mathscr{G}$ is an infinite dimensional space and the topology of $B\mathscr{G}$ is the quotient topology induced from the topology of $\coprod \Delta^n \times \mathscr{G}^{(n)}$ (hence the topologies of $\mathscr{G}^{(n)}$'s are relevant here).

An important basic property of the classifying space construction is that an equivalence between groupoids $(\psi, f) : (\mathscr{G}, X) \to (\mathscr{G}', X')$ induces a homotopy equivalence between their classifying spaces (see [Se], [M])

$$B\psi : B\mathscr{G} \to B\mathscr{G}'.$$

This means that for any point $x \in X$, the equivalence (ψ, f) induces an isomorphism of homotopy groups

$$\pi_n(B\mathscr{G}, x) \approx \pi_n(B\mathscr{G}', f(x)).$$

Thus, if Q is an orbifold whose orbifold structure is given by the groupoid $(\mathscr{G}, X)$ we can define its *homotopy type* as being that of the classifying space $B\mathscr{G}$:

$$\pi_n^{orb}(Q, x) = \pi_n(B\mathscr{G}, \tilde{x})$$

the definition being independent of the choice of the groupoid representing the orbifold and of the base point $x \in Q$ and of the lift $\tilde{x} \in X$ for which $q(\tilde{x}) \in |Q|$ is mapped to x by the induced homeomorphism $X/\mathscr{G} \to |Q|$.

Example 3.8.3. (1) For the groupoid $(*, G)$ the classifying space coincides with the classifying space BG of the group G. This space classifies the principal G-bundles.

(2) If $(M, \mathscr{U})$ is a manifold then the classifying space of the groupoid $M_{\mathscr{U}}$ is homotopy equivalent to M: $BM_{\mathscr{U}} \simeq M$. (see Se)

(3) For the groupoid $(G \ltimes M, M)$ associated to an action of a group G on a topological space M, the classifying space is the homotopy quotient

$$B(G \ltimes M) \simeq M_G = (EG \times M)/G$$

where the action of G is given by $g.(e, x) = (eg^{-1}, gx)$.

When $\mathscr{G}$ is the groupoid of germs of changes of charts of an atlas of uniformizing charts for a Riemannian orbifold Q of dimension n, there is an explicit construction of

$B\mathscr{G}$ which is independent of the particular atlas defining Q and which will be therefore be denoted BQ (see [GH]). Indeed consider the bundle of orthonormal coframes FX on X; an element of FX above $x \in X$ can be identified to a linear isometry from the tangent space T_xX at x to the Euclidean space $\mathbb{R}^n$; the group $O(n)$ of isometries of $\mathbb{R}^n$ acts naturally on the left on FX and this action commutes with the right action of the groupoid $\mathscr{G}$ on FX through the composition with the differential of the elements of $\mathscr{G}$. As the action of $\mathscr{G}$ on FX is free, the quotient $FX/\mathscr{G}$ is a smooth manifold depending only on Q and not on a particular atlas defining Q. The left action of $O(n)$ on FX gives a locally free action of $O(n)$ on $FX/\mathscr{G}$.

Choose a principal universal $O(n)$-bundle $EO(n) \to BO(n)$ for the orthogonal group $O(n)$ and take for $E\mathscr{G}$ the associated bundle $EO(n) \times_{O(n)} FX$, quotient of $EO(n) \times FX$ by the diagonal action of $O(n)$. The projection from FX to X gives a projection $q_{E\mathscr{G}} : E\mathscr{G} \to X$ with contractible fibers isomorphic to $EO(n)$. As the action of $O(n)$ on FX commutes with the natural right action of $\mathscr{G}$, we get a free action of $\mathscr{G}$ on $E\mathscr{G}$ with respect to the projection $q_{E\mathscr{G}}$, and $E\mathscr{G} \to E\mathscr{G}/\mathscr{G} = EO(n) \times_{O(n)} FX/\mathscr{G}$ is a universal principal $\mathscr{G}$-bundle whose base space $B\mathscr{G} = E\mathscr{G}/\mathscr{G}$ will be noted BQ. There is a canonical map $\pi : BQ \to |Q|$ induced from the map $q \circ q_{E\mathscr{G}} : E\mathscr{G} \to X/\mathscr{G} = |Q|$; it is the projection of the morphism from BQ to $\mathscr{G}$ associated to the principal $\mathscr{G}$-bundle $E\mathscr{G} \to BQ$; the fiber of π above a point $z = q(x)$ is an acyclic space with fundamental group isomorphic to the isotropy subgroup $\mathscr{G}_x$ of x.

Figure 3.1: A $\mathscr{G}$-path.

Figure 3.2: Equivalent $\mathscr{G}$-path over a subdivision with a new point $t' \in (t_{i-1}, t_i)$.

Figure 3.3: Equivalent $\mathscr{G}$-paths over the same subdivision.

Figure 3.4: An elementary homotopy of $\mathscr{G}$-paths.

Figure 3.5: A description of $\Delta^3 \times (g_1, g_2, g_3)$ and its boundary $\Delta^2 \times (g_1, g_2 g_3)$.

Chapter 4

Developability of orbifolds of non positive curvature

In this chapter we give a proof of developability for complete Riemannian orbifolds of non positive curvature. Apparently (we do not know a reference yet) the result was first proved by Gromov. However, the developability of complete orbifolds of non positive curvature appears as an exercise in [BGS] (see exercise (a) on page 16). This result is also a direct consequence of a result in [BH] (Theorem 2.15 and Corollary 2.16 in chapter III.$\mathscr{G}$). It is proved there that if $(\mathscr{G}, X)$ is a connected groupoid of local isometries which is Hausdorff and complete and such that the metric on X is locally convex, then it is developable. The proof there follows the Alexander-Bishop proof of the Cartan-Hadamard Theorem (see II.4 in [BH]). The chapter is structured as follows. In first section we introduce the notion of $\mathscr{G}$-geodesic path on a connected Riemannian orbifold, where $(\mathscr{G}, X)$ is the groupoid associated to the germs of change of charts of the orbifold structure. This is possible since in this case $(\mathscr{G}, X)$ is a groupoid of local isometries which is Hausdorff and is $\mathscr{G}$-connected. It is also complete if the orbifold is assumed to be complete. In section 4.2 we introduce an analogue of the exponential map from the manifold case, as being the map that associates to any (sufficiently short) vector $v \in T_xX$ the end point of a $\mathscr{G}$-geodesic issuing at x with velocity vector v. Note that this map is well defined (even as a map) up to the choice of the terminal point within its orbit. However, this gives a well defined map on the space of orbits which is the base map of a morphism from the tangent cone at the projection of x into Q. This is also the base map of the good C^∞ orbifold map Exp considered by Ruan and Chen in [CR].

In section 4.3 we investigate the relation between completeness and the geodesically completeness of the orbifold and we will prove a result similar to the Hoph-Rinow Theorem for manifolds that is, a connected Riemannian orbifold is complete if and only if it is geodesically complete. We also prove there that in a connected Riemannian orbifold which is complete any two points can be joined by a minimal geodesic $\mathscr{G}$-path.

In section 4.4 we introduce Jacobi fields and the notion of conjugate points along a $\mathscr{G}$-geodesic in an orbifold. Via the definition of Jacobi fields using the exponential map we relate the the conjugate points to the critical points of the exponential map.

In the last section we consider the case of complete Riemannian orbifolds of non positive curvature. Using Jacobi fields we prove that in this case there are no conjugate points along any $\mathscr{G}$-geodesic and using the results in section 4.4 we conclude that the exponential map is étale. Moreover, we can prove in this case that it is also a covering map. Then the morphism from the developable (by Lemma 4.5.3) groupoid associated to the action of $\mathscr{G}$ on T_xX into $(\mathscr{G}, X)$ whose base map is the exponential map is a covering. This proves that $(\mathscr{G}, X)$ is developable, i.e. the orbifold Q is developable. *The same proof works for the case when Q is a complete Riemannian orbifold which contains a pole by using the exponential map at the pole. The only thing to prove in that case is that the exp map is a covering map, without using the non positive curvature. This is possible since exp is étale and has the path lifting property. A different approach of proving this is by using the path space (at least in the simply connected case)?*

4.1 Geodesic $\mathscr{G}$-paths

Let Q be a Riemannian orbifold. Recall that an orbifold structure is said to be Riemannian if each uniformizing chart X_i is a Riemannian manifold and if the change of charts are local isometries. On $X = \sqcup X_i$ a Riemannian metric is defined as the union of the riemannian metrics on each X_i and this Riemannian structure induces a length metric whose quotient gives a pseudo-metric on the space of orbits $|Q|$. In this case, the étale groupoid of germs of change of uniformizing charts is a group of local isometries. The fact that the base space $|Q|$ is Hausdorff implies that the pseudometric is always a metric and induces the given topology on $|Q|$. Moreover the groupoid $(\mathscr{G}, X)$ is always Hausdorff and it is complete if and only if Q is complete.

In the case when Q is connected note that X as a disjoint union is not connected but it is *$\mathscr{G}$-connected* in the following way (it is *$\mathscr{G}$-path connected*). For any two points $x, y \in X$ there is a sequence of points $(x_1, y_1, \ldots, x_k, y_k)$ such that $x_1 = x$, $y_k = y$ and each x_i is in the orbit of y_{i-1} for all $i = 2, \ldots k$ and there is a path (in some X_i) joining x_i to y_i for all $i = 1, \ldots, k$ (or equivalently, there is a $\mathscr{G}$-path $c = (1_x, c_1, g_2, \ldots, c_k, 1_y)$ defined over some subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ connecting x to y). We will assume in what follows that Q is connected.

Definition 4.1.1. *A* geodesic $\mathscr{G}$-path *from x to y in a Riemannian orbifold is a $\mathscr{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ from x to y such that:*

(i) each $c_i : [t_{i-1}, t_i] \to X$ is a geodesic segment with constant speed $\dot{c}_i$;

(ii) the differential dg_i of a representative of g_i at $c_i(t_i)$, maps the velocity vector $\dot{c}_i(t_i)$ to the velocity vector $\dot{c}_{i+1}(t_i)$.

We say that a geodesic $\mathscr{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ is normalized if the all the geodesic segments c_i are normalized geodesics, i.e. they are parametrized by the arc-length. In what follows we will consider normalized geodesic $\mathscr{G}$-paths.

If c is a closed $\mathscr{G}$-path, it represents a closed geodesic $[c]$ (or a geodesic loop) on Q if moreover the differential of $g_0 g_k$ maps the velocity vector $\dot{c}_k(1)$ to the vector $\dot{c}_1(0)$. A free loop of length 0 is always a closed geodesic.

Note that the projection $q : X \to |Q|$ associates uniquely to each equivalence class of $\mathscr{G}$-geodesics $[c]$ parametrized over $[0,1]$ a continuous path $[0,1] \mapsto |Q|$. This is the base map of a smooth orbifold map which is also good in the sense of the Definition 2.4.2. Equivalently we could define a geodesic on an orbifold to be a good C^∞ orbifold map from $[0,1]$ to Q which locally lifts to geodesics. When there is no place of confusion, we will refer to the underlying map as of the geodesic path on the orbifold. Sometimes is convenient to consider the geodesic $\mathscr{G}$-paths issuing at a point which can be extended indefinitely. Then, one should consider (admissible) covers of $\mathbb{R}$ (or $[0,\infty)$) and the associated (possible infinite) sequence of elements g_i and geodesic paths c_i as in definition 4.1.

It is easy to see that if c is a geodesic $\mathscr{G}$-path from x to y, then the vector $d(g_0)^{-1}\dot{c}_1(0)$ at x is an invariant of the equivalence class $[c]_{x,y}$ and is called the initial vector of the $\mathscr{G}$-geodesic c.

In the developable case, if the orbifold Q is the quotient of a connected Riemannian manifold X by a discrete subgroup Γ of its group of isometries, then any closed geodesic on Q is represented by a pair (c,γ), where $c : [0,1] \to X$ is a geodesic and γ an element of Γ such that the differential of γ at $c(1)$ maps the velocity vector $\dot{c}(1)$ to $\dot{c}(0)$; another such pair (c',γ') represents the same geodesic if and only if there is an element $\delta \in \Gamma$ such that $c' = \delta.c$ and $\gamma' = \delta^{-1}\gamma\delta$.

As an example (see [GH]), consider the orbifold Q which is the quotient of the round 2-sphere S^2 by a rotation ρ of angle π fixing the north pole N and the south pole S. The quotient space $|Q|$ is a sphere with two conical points $[N]$ and $[S]$, images of N and S. There are two homotopy classes of free loops on Q. Closed geodesics homotopic to a constant loop are represented by a closed geodesic on S^2 (their length is an integral multiple of 2π). If they have positive length, their image in $|Q|$ is either the equator, a figure eight or a meridian. Closed geodesics in the other homotopy class are represented by a pair (c,ρ), where c is either the constant map to N or S, or a geodesic arc on the equator of length an integral odd multiple of π.

4.2 Exponential map

We would like to introduce on X a correspondent of the exponential map such that the geodesic $\mathscr{G}$-paths could be the image of a line segment containing the origin in T_xX. We will have to overcome the following difficulty. For any point $x \in X$ there is a uniquely defined exponential map for sufficiently small vectors $v \in T_xX$ using the Riemannian structure on the connected component X_i which contains x. In this case the geodesic obtained as the image by the exponential map of a segment containing the origin in the tangent space T_xX would lie in the connected component X_i, hence it would be just a particular type of $\mathscr{G}$-geodesic, namely one that can be represented over the entire interval. This is not enough since the $\mathscr{G}$-geodesics can have their terminal point in other connected components of X. Using the local existence of the exponential map and the groupoid structure we will construct an "exponential map" which is well defined up to a choice of an element in its fiber.

Let $x_0 \in X$. Then $x_0 \in X_i$ for some i and recall from the Riemannian geometry of manifolds that there exist a neighborhood $U \subset X_i$ of x_0 and $\epsilon > 0$ such that for each $x \in U$ and each tangent vector $v \in T_xX$ with length less then ϵ there is a unique geodesic $c_i : (-2, 2) \to X_i$ satisfying the conditions $c_i(0) = x$ and $\dot{c}_i(0) = v$. In this case there is a uniquely defined map denoted $\exp_x$, given by $\exp_x(v) = c(1)$ and the geodesic $c : [0, 1] \to X_i$ can be described by the formula $c(t) = \exp_x(tv)$.

Thus for every i and $x \in X_i$ the exponential map $\exp_x$ is defined throughout a neighborhood of $(x, 0)$ in T_xX and it is differentiable. Moreover there exists a neighborhood V of the origin in T_xX such that $\exp_x|_V$ is a diffeomorphism and the image $\exp_x(V) = U \subset X_i$ is called a geodesic neighborhood of x. If the open ball $B(0, \delta) \subset T_xX$ is such that $\overline{B}(0, \delta) \subset V$ then we call the image $e_x(B(0, \delta)) = B(x, \delta)$ the geodesic ball centered at x and of radius δ. The boundary of the geodesic ball is a submanifold of codimension one in X which is orthogonal to the geodesics rays issuing from x. Furthermore for any point $x \in X_i$ there exists a neighborhood W of x which is a geodesic neighborhood of each point point $y \in W$. That is, a neighborhood $W \subset X_i$ of x together with a $\delta > 0$, such that for every $y \in W$, the exponential map

$\exp_y$ is a diffeomorphism on $B(0,\delta) \subset T_yX$ and the geodesic ball $B(y,\delta)$ contains W. A such of pair (W,δ) is called a totally geodesic neighborhood for x. Note that any two points of W can be joined by a unique geodesic in X_i of length less than δ.

The following lemma (see [GH]) will allow us to extend the image of the exponential map to other connected components X_j of X.

Lemma 4.2.1. *Let $Q = X/\mathscr{G}$ be a Riemannian orbifold. Let $g \in \mathscr{G}$ with $x = \alpha(g)$ and $y = \omega(g)$. Let $\epsilon > 0$ be such that $B(x,\epsilon)$ and $B(y,\epsilon)$ are convex geodesic balls at x and y respectively. Then there is an element h of the pseudogroup of change of charts of Q which is an isometry from $B(x,\epsilon)$ and $B(y,\epsilon)$.*

Proof. Let $h : B(x,\epsilon) \to B(y,\epsilon)$ be the diffeomorphism which maps isometrically geodesic rays issuing from x to the geodesic rays issuing from y and such that the differential $dh_x = dg$. For $0 < r < \epsilon$ sufficiently small the restriction h_r of h to the $B(x,r)$ is an element of the pseudogroup of change of charts $\mathscr{H}$ (see 3.2 and also 3.1). It is sufficient to prove that the restriction of h to a neighborhood of the closure of $B(x,r)$ is also an element of $\mathscr{H}$. Let $z \in \partial B(x,r)$. Since the space of orbits is Hausdorff, the point z and its image $h(z)$ are in the same orbit of $\mathscr{H}$. Then there are sufficiently small neighborhoods U of z and V of $h(z)$ and an element $f : U \to V$ of $\mathscr{H}$ with $f(z) = h(z)$ and such that the restriction of $\mathscr{H}$ to the open neighborhood U is generated by a group Γ_U of diffeomorphisms of U and such that the germ of any element of $\mathscr{H}$ defined on U with the target in V is the germ of a the composition of f to an element of Γ_U. Since f is a Riemannian isometry, it must coincide with $h|_U$ on the geodesic rays issuing from x and hence on U. $\square$

The above lemma allows us to consider the following construction. Let $v \in T_xX$ such that the $\exp_x(v)$ is defined. Since the length of the geodesic $\exp_x(tv)$ is finite there exists an $\epsilon > 0$ and a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ of the interval $[0,1]$ such that $B(\exp_x(t_iv),\epsilon)$ are convex geodesic balls. By applying inductively the above lemma for $i = 1,\ldots,k$ it is easy to see that we obtain a $\mathscr{G}$-geodesic $c = (g_0, c_1, g_1, c_2, \ldots, c_k, g_k)$ equivalent with the initial one. Note that the g_i's may be units and that each $c_{i+1}(t_i)$ respectively $c_i(t_i)$ are in the orbit of $\exp_x(t_iv)$. In particular, the "end point" $\omega(g_k)$ of the $\mathscr{G}$-geodesic c is in the same orbit of $\exp_x(v)$. Hence we can associate to any pair (x, v) where x is a point in X and v is a vector T_xX of small enough length the orbit through $\exp_x(v)$. This gives a well defined map on the space of orbits which we will still denote $\exp_{\widetilde{x}} : |T_{\widetilde{x}}Q| \simeq T_xX/\mathscr{G}_x \to |Q|$ where $q(x) = \widetilde{x} \in |Q|$ associating to $\xi \in |T_{\widetilde{x}}Q|$ of small enough norm, the projection $q(\exp_x(v))$ where $v \in T_xX$ such that $v = dg_0(\xi)$ where g_0 is the element of $\Gamma_{\widetilde{x}}$ which lifts $\widetilde{x}$ to x. Note that this map is continuous (even more it is étale) and induces an étale groupoid homomorphism from the groupoid associated to the action of $\mathscr{G}_x$ on T_xX to the groupoid $(\mathscr{G}, X)$. Note also that this exponential map can be seen as the base map of the good orbifold map *Exp* considered by Ruan and Chen in [CR].

4.3 Geodesically complete orbifolds

Note that the construction above works for "short" vectors $v \in T_xX$ and we would like to investigate weather a such exponential map can be defined for vectors of arbitrarily length. Similar to the manifold case, we will call an orbifold *geodesically complete* if the exponential map $\exp_x : |T_xQ| \to |Q|$ is defined for all $|T_xQ|$. That is, (by the above argument) if for every $x \in X$ the exponential map $\exp_x$ is defined on all T_xX, i.e. any $\mathscr{G}$-geodesic starting at x can be extended infinitely. By analogy, we will say in this situation that X is $\mathscr{G}$-geodesically complete. We will see that this is always the case when the orbifold Q is complete.

In what follows, using the connectedness of Q (or equivalently the $\mathscr{G}$-connectedness of X) we will introduce a pseudodistance topology on X which agrees the induced topology on X by the Riemannian structure (this could be easily avoided, it just make the analogy with the manifold case simpler). Define the $\mathscr{G}$-distance between the two points by

$$d_{\mathscr{G}}(x, y) = \inf\{\sum_{i=1}^{k} d(x_i, y_i)\},$$

where the infimum is considered on all possible sequences $(x_1, y_1, \ldots, x_k, y_k)$ and for all k and $d(\cdot, \cdot)$ denotes the distance induced by the Riemannian metric on each connected component in X. Note that $d_{\mathscr{G}}$ defines a *pseudo-distance* on X (it is not a distance since any two distinct points in the same orbit have $d_{\mathscr{G}}$ distance zero) and that its restriction to each connected component $X_i \subset X$ is a distance which agrees with the induced by the Riemannian structure. This implies that the topology on each component is the same as the one induced (as the subspace topology) by the pseudo-distance topology given by $d_{\mathscr{G}}$ on X. Note that the pseudo-distance topology on X is not Hausdorff (it is not even Kolgomorov (T_0)). Moreover the projection $q : X \to |Q|$ becomes an isometry. With respect to this pseudo-distance topology, the association via the exponential map $(x, v) \mapsto y$, where $x \in X$ and $v \in T_xX$ of small norm and y is a point in the orbit of $\exp_x(v)$ (i.e. it is the 'end point' of the $\mathscr{G}$-geodesic given by $\exp_x(tv)$, $t \in [0, 1]$) is continuous.

We will say that a sequence $\{x^{(n)}\}_n$ of points in X is *$\mathscr{G}$-convergent* to a point $x \in X$ if it is convergent in the pseudo-distance topology, i.e. if for every $\epsilon > 0$ there is $n_{(\epsilon)}$ such that for any $n \geq n_{(\epsilon)}$ we have $d_{\mathscr{G}}(x^{(n)}, x) < \epsilon$. As expected the limit point of the sequence $x^{(n)}$ is not unique and obviously any other point in the orbit of x will also be the limit point. In a similar way we say that the sequence $x^{(n)}$ of points in X is *$\mathscr{G}$-fundamental* if for every $\epsilon > 0$ there is $n_{(\epsilon)}$ such that for any $n, m \geq n_{(\epsilon)}$ we have $d_{\mathscr{G}}(x^{(n)}, x^{(m)}) < \epsilon$. It is easy to see that the completeness of $|Q|$ is equivalent with the $\mathscr{G}$-completeness of X via the projection $q : X \to |Q|$.

The following proposition is an important property of geodesically complete orbifolds.

Proposition 4.3.1. *If a connected Riemannian orbifold $Q = X/\mathscr{G}$ is geodesically complete then any two points can be joined by a minimal geodesic $\mathscr{G}$-path.*

Proof. Let x and y be two points in X and let r denote the $\mathscr{G}$-distance between them. Consider a geodesic ball $B(x, \delta)$ at x (note that this ball is contained in the connected component of X containing x). The boundary of this ball is quasi-compact in the pseudometric topology and since the function

$$d_{\mathscr{G}}(y, \cdot) : X \to \mathbb{R}$$

is continuous, there exist a point in $x_0 \in \partial B(x, \delta)$ such that $d_{\mathscr{G}}(y, \partial B(x, \delta))$ attains its minimum. Then $x_0 = \exp_x(\delta v)$ for some $v \in T_x X$ with norm one.

Since X is $\mathscr{G}$-geodesically complete, there is an equivalence class of geodesic $\mathscr{G}$-path issuing from x with velocity v and which can be extended indefinitely. We will show that these geodesic $\mathscr{G}$ paths intersect the orbit through y and therefore there exists a $\mathscr{G}$-geodesic in this equivalence class which contain y. Moreover, we well prove that the length of this geodesic is equal to r.

Similar to the manifold case we will prove that a point which moves along one of the geodesic $\mathscr{G}$-paths as above must get closer (with respect to $d_{\mathscr{G}}$) to y. Let c be a such geodesic $\mathscr{G}$-path which is represented over an admissible cover of $[0, \infty)$ by a sequence $c = (g_0, c_1, g_1, \ldots, g_{k-1}, c_k, \ldots)$. For convenience we will denote $c(t)$ the point $c_i(t)$ for $t \in [t_{i-1}, t_i]$.

Consider the equation

$$d_{\mathscr{G}}(c(t), y) = r - t$$

and let A be the set of points in $[0, r]$ for which the equation holds.

A is not empty since the above equation is satisfied for $t = 0$ and it is clearly closed. Let $t^* \in A$. We will show that if $t^* < r$ then the equation holds for $t^* + \delta'$, where $\delta' > 0$ is sufficiently small. This implies that $\sup A = r$ and since A is closed we conclude that $r \in A$, i.e. $c(r)$ is in the orbit of y. Denote $x^* = c(t^*) \in X$ and consider a geodesic ball $B(x^*, \delta')$ at x^*. Let $x_0' \in \partial B(x^*, \delta')$ be the point which minimizes the $\mathscr{G}$-distance $d_{\mathscr{G}}(y, \partial B(x^*, \delta'))$. Note that by the previous lemma, we can assume without loss of generality that $[t^* - \delta', t^* + \delta'] \subset (t_{i-1}, t_i)$ for some i. Then it suffices to show that the point $x_0' = c(t^* + \delta')$.

Note first that

$$d_{\mathscr{G}}(c(t^*), y) = \inf_{z \in \partial B(x^*, \delta')} \big(d_{\mathscr{G}}(x^*, z) + d_{\mathscr{G}}(z, y)\big) = \delta' + d_{\mathscr{G}}(x_0', y)$$

and since $d_{\mathscr{G}}(c(t^*), y) = r - t^*$, it follows that

$$r - t^* = \delta' + d_{\mathscr{G}}(x_0', y) = d_{\mathscr{G}}(c(t^* + \delta'), y),$$

i.e.

$$d_{\mathscr{G}}(c(t^* + \delta'), y) = r - (t^* + \delta').$$

To prove now that $x_0' = c(t^* + \delta')$, note that using (*)

$$d_{\mathscr{G}}(x, x_0') \geq d_{\mathscr{G}}(x, y) - d_{\mathscr{G}}(y, x_0') = r - (r - t^* - \delta') = t^* + \delta'.$$

On the other hand, the $\mathscr{G}$-distance between x and x_0' measured along the $\mathscr{G}$-geodesic c up to x^* and then along the geodesic ray joining x^* to x_0' gives

$$d_{\mathscr{G}}(x, x_0') \leq d_{\mathscr{G}}(x, x^*) + d_{\mathscr{G}}(x^*, x_0') = t^* + \delta'.$$

Hence $d_{\mathscr{G}}(x, x_0') = t^* + \delta'$. Consider now $c(t_{i-1})$ which is a point in the same connected component of X as x^*. Clearly $d(c(t_{i-1}), x_0') \leq d(c(t_{i-1}), x^*) + d(x^*, x_0')$. We claim that we have equality in the inequality above. Indeed, if we assume that $d(c(t_{i-1}), x_0') < d(c(t_{i-1}), x^*) + d(x^*, x_0')$ then we have

$$\begin{aligned} t^* + \delta' = d_{\mathscr{G}}(x, x_0') &\leq d_{\mathscr{G}}(x, c(t_{i-1})) + d(c(t_{i-1}), x_0') \\ &< d_{\mathscr{G}}(x, c(t_{i-1})) + d(c(t_{i-1}), x^*) + d(x^*, x_0') \\ &= d_{\mathscr{G}}(x, x^*) + d_{\mathscr{G}}(x^*, x_0') \\ &= t^* + \delta' \end{aligned}$$

which cannot be true. This means that the broken path obtained from the geodesic segment $c_i|_{[t_{i-1}, t^*]}$ and the minimal geodesic joining $x^* = c_i(t^*)$ to x_0', is distance minimizing, and so it is a (unbroken) geodesic, i.e. it coincides with c_i. This proves that $x_0' = c(t^* + \delta')$ and completes the proof. $\square$

The following proposition relates the concepts of $\mathscr{G}$-completeness and $\mathscr{G}$-geodesically completeness of X. It is the similar of the Hoph-Rinow Theorem in the manifold case (or even more generally for length spaces (see [**?**]).

Theorem 4.3.2. *Let $Q = X/\mathscr{G}$ be a connected Riemannian orbifold. Then X is $\mathscr{G}$-complete if and only if it is $\mathscr{G}$-geodesically complete.*

Thus, a connected Riemannian orbifold Q is complete if and only if it is geodesically complete.

Proof. Assume that X is $\mathscr{G}$-complete and suppose that it is not $\mathscr{G}$-geodesically complete. That is, there exist a point $x \in X$ and a (normalized) $\mathscr{G}$-geodesic issuing from x

which is defined for $t < t^*$ and is not defined for t^*. Let $0 = t_0 < t_1 < \cdots < t_{k-1} < t^*$ be a subdivision of $[0, t^*]$ and $c = (g_0, c_1, g_1, \ldots, c_k)$ be a representative of this $\mathscr{G}$-geodesic over it such that c_k is defined for $t \in [t_{k-1}, t*)$ and not for t^*. Consider a sequence $(t^n)_n$ in $[0, t^*)$ which converges at t^*. We can assume without loss of generality that $t^n \in [t_{k-1}, t^*)$. Note that t^m is Cauchy and therefore for every $\epsilon > 0$ there exists n_ϵ such that if $n, m > n_\epsilon$ then $|t^n - t^m| < \epsilon$. Since c is a normalized $\mathscr{G}$-geodesic, this implies that $d_{\mathscr{G}}(c_k(t^n), c_k(t^m)) \leq |t^n - t^m| < \epsilon$, i.e. the sequence $c_k(t^n)$ is $\mathscr{G}$-fundamental. Hence it is $\mathscr{G}$-convergent to a point $y \in X$ (actually it is $\mathscr{G}$-convergent to any point in the orbit of y). Let (W, δ) be a totally geodesic neighborhood of y. For $\delta > 0$ there exists n_δ such that for $n \geq n_\delta$ we have $d_{\mathscr{G}}(c_k(t^n), c_k(t^m)) < \delta$. We can choose n_δ large enough such that (eventually passing to a subsequence) the orbits through $c_k(t^n)$ have nonempty intersection with W. Denote these intersections with x_n. It follows that for any $n, m \geq n_\delta$ we have $d(x_n, x_m) = d_{\mathscr{G}}(x_n, x_m) = d_{\mathscr{G}}(c_k(t^n), c_k(t^m)) < \delta$. Then, there exists a unique geodesic connecting x_n to x_m of length smaller than δ. Denote c' this geodesic. It is clear that this geodesic is equivalent to c_k whenever this is defined. Since $\exp_{x_n}$ is a diffeomorphism on $B(0, \delta) \subset T_{x_n}X$ and $W \subset \exp_{x_n}(B(0, \delta))$, the geodesic c' can be extended over t^*. By replacing in $c = (g_0, c_1, \ldots, c_k)$ the geodesic c_k with c' we obtain a $\mathscr{G}$-geodesic path issuing at x which is defined beyond t^* and whose restriction to $[0, t^*)$ is equivalent to c, i.e. c can be extended in its equivalence class. This leads to a contradiction and proves the claim.

Conversely, assume that X is $\mathscr{G}$-geodesically complete. Consider a $\mathscr{G}$-fundamental sequence $(x^n)_n$ in X. It is clearly that the set $A = \{x^n \mid n\}$ is bounded in the $\mathscr{G}$-distance. Thus there exist a ball $B_{\mathscr{G}}$ centered at a point $x \in X$ which contains its closure. By the above proposition, there exists a ball $B(0, r) \subset T_xX$ such that $B_{\mathscr{G}} \subset \exp_x(\overline{B}(0, r))$. As the map $\exp_x$ is continuous in the pseudo-distance topology and $\overline{B}(0, r)$ is compact in T_xX, the image $\exp_x(\overline{B}(0, r))$ is quasi-compact in X. Then the closure of A is also quasi-compact and so the sequence (x^n) contains a $\mathscr{G}$-convergent subsequence and being $\mathscr{G}$-fundamental, it converges. This proves that X is also $\mathscr{G}$-complete. $\square$

4.4 Jacobi fields

Let $x, y \in X$ and $c = (g_0, c_1, \ldots, c_k)$ be a geodesic $\mathscr{G}$-path connecting them over a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$.

Definition 4.4.1. *A* Jacobi field *along c is a sequence $J = (J_1, \ldots, J_k)$ of vector fields J_i along c_i and such that*

(i) dg_i maps $(J_i, \frac{D}{dt}J_i(t_i)) \mapsto (J_{i+1}, \frac{D}{dt}J_{i+1}(t_i))$, for $i = 1, \ldots, k-1$

(ii) each J_i is a Jacobi field along c_i, i.e. satisfies the Jacobi equation:

$$\frac{D^2 J_i}{dt^2}(t) + R(\dot{c}_i(t), J_i(t))\dot{c}_i(t) = 0,$$

for any $t \in [t_{i-1}, t_i]$ and $i = 1, \ldots, k$.

If c is a closed geodesic, a periodic Jacobi field *is a Jacobi field such that the differential $d(g_0 g_k)$ maps $(J_k(1), \frac{DJ_k}{dt}(1))$ to $(J_1(0), \frac{DJ_1}{dt}(0))$.*

Note that for each i the Jacobi equation (*), as a second order differential equation, has $2n$ linearly independent smooth solutions. Since each J_i along the geodesic c_i is uniquely determined by its initial conditions $(J_i(t_{i-1}), \frac{DJ_i}{dt}(t_{i-1}))$, by condition (i) in the definition we see that a Jacobi field J along a $\mathscr{G}$-geodesic c is uniquely determined by $(J_1(0), \frac{DJ_1}{dt}(0))$. So there are $2n$ linearly independent Jacobi fields, each of which can be defined throughout c. Moreover, note that the vector fields $\dot{c}(t)$ and $t\dot{c}(t)$ are Jacobi fields along c. The first one has the derivative zero and vanishes nowhere, and the second one is zero if and only if $t = 0$. Note also that the Jacobi fields do not depend on a particular choice of c in its equivalence class. That is, if $c' = (g_0', c_1', g_1', c_2', \ldots, c_k', g_k')$ is another representative of the geodesic $\mathscr{G}$-path c over the same subdivision then $J' = (J_1', J_2', \ldots, J_k')$ given by $J_i'(t) = \big(dh_i(t)\big)_{c_i(t)}(J_i(t))$ is a Jacobi field along c', where the h_i's are as in (ii), and J' is uniquely determined since the h_i's are unique by Remark 3.7.2 (d).

As in the manifold case, one can prove that every Jacobi field along a geodesic $\mathscr{G}$-path c may be obtained by a one-parameter variation of c through geodesics. That is, a sequence $\nu = (\tau_0, \nu_1, \tau_1, \nu_2, \ldots, \tau_{k-1}, \nu_k)$ associated to the division $0 = t_0 < t_1 < \cdots < t_k = 1$, where

(i) for each $i = 1, \ldots, k$, $\nu_i : (-\varepsilon, \varepsilon) \times [t_{i-1}, t_i] \to X$ are differentiable functions such that $\nu_i(0, t) = c_i(t)$ and each $\overline{\nu}_i(s)$, $s \in (-\varepsilon, \varepsilon)$ given by $\overline{\nu}_i(s)(t) = \nu_i(s, t)$ is a geodesic, and

(ii) each $\tau_i : (-\varepsilon, \varepsilon) \to \mathscr{G}$, $i = 0, \ldots, k$ are $\mathscr{G}$-valued differentiable functions such that $\tau_i(0) = g_i$ and such that $\alpha(\tau_i(s)) = \nu_i(s, t_i)$ and $\omega(\tau_i(s)) = \nu_{i+1}(s, t_i)$ for any $s \in (-\varepsilon, \varepsilon)$ and any $i = 0, \ldots, k-1$.

We will give a construction of Jacobi fields along a geodesic $\mathscr{G}$-path using the exponential map. Here we will consider only Jacobi fields which satisfy $J(0) = 0$, but analogous constructions can be obtain in the general case.

Let c be a geodesic $\mathscr{G}$-path on $Q = X/\mathscr{G}$ and J a Jacobi field along c with $J(0) = 0$. Denote $v = \dot{c}(0) \in T_{c(0)}X$ and $w = \frac{DJ}{dt}(0) \in T_v(T_{c(0)}X)$ and construct a (*) path $v(s)$ in $T_{c(0)}X$ with $v(0) = v$ and $\dot{v}(0) = w$. Put $\nu(s, t) = \exp_{c(0)}(tv(s))$ and define the Jacobi field $\overline{J}$ along c by $\overline{J}(t) = \frac{\partial \nu}{\partial s}(0, t)$.

Proposition 4.4.2. *With the notations above, $\overline{J} = J$ on $[0, 1]$.*

Proof. At $s = 0$ we have

$$\begin{aligned}\frac{D}{dt}\frac{\partial \nu}{\partial s} &= \frac{D}{\partial t}((d\exp_{c(0)})_{tv}(tw)) = \frac{D}{\partial t}(t(d\exp_{c(0)})_{tv}(w)) \\ &= (d\exp_{c(0)})_{tv}(w) + t\frac{D}{\partial t}((d\exp_{c(0)})_{tv}(w))\end{aligned}$$

Therefore, for $t = 0$

$$\frac{D\overline{J}}{dt}(0) = \frac{D}{\partial t}\frac{\partial \nu}{\partial s}(0, 0) = (d\exp_{c(0)})_0(w) = w.$$

Since the initial conditions are $J(0) = \overline{J}(0) = 0$ and $\frac{DJ}{dt}(0) = \frac{D\overline{J}}{dt}(0) = w$, from the uniqueness of the Jacobi fields, we conclude that $J = \overline{J}$. $\square$

Corollary 4.4.3. *Let c be a geodesic $\mathscr{G}$-path. Then a Jacobi field J along c with $J(0) = 0$ is given by*

$$J(t) = (d\exp_{c(0)})_{t\dot{c}(0)}(t\frac{DJ}{dt}(0)),\ t \in [0,1].$$

Definition 4.4.4. *A point $c(t') \in X$ is* conjugate *to $c(0) \in X$ along the geodesic $\mathscr{G}$-path c if there exists a non-zero Jacobi field along c such that $J(0) = 0 = J(t')$. The maximum number of such linearly independent fields is called the* multiplicity *of $c(0)$ and $c(t')$ as conjugate points.*

Note that the multiplicity of two conjugate points never exceeds $n-1$. Indeed, the dimension of the vector space consisting of all Jacobi fields which vanish at $t = 0$ has dimension at most n and the non-zero Jacobi field $J(t) = t\dot{c}(t)$ never vanishes for $t \neq 0$.

Note that this gives on the space of orbits $|Q|$ a well defined notion of conjugate points along a geodesic path and of their multiplicity. Thus two points $x, x' \in |Q|$ on a geodesic path $[c]$ are conjugate if their lifts to X are conjugate along a representative of $[c]$.

The relation between the conjugate points and the critical points of the exponential map is given by the following proposition.

Proposition 4.4.5. *Let c be a geodesic $\mathscr{G}$-path. Then $c(t')$ is conjugate to $c(0)$ along c if and only if the vector $t'\dot{c}(0)$ is a critical point for $\exp_{c(0)}$. Moreover, the multiplicity of $c(t')$ as conjugate to $c(0)$ is equal to the dimension of the kernel of the linear map $(d\exp_{c(0)})_{t'\dot{c}(0)}$.*

Proof. Let J be a non-zero Jacobi field along c which vanishes at 0 and t'. Denote $v = \dot{c}(0)$ and $w = J'(0)$. Then, from the Corollary 4.4 $J(t) = (d\exp_{c(0)})_{tv}(tw)$, $t \in [0,1]$ and since $(d\exp_{c(0)})_{tv}$ is linear, we have that $w \neq 0$, as J is not identically zero. But $0 = J(t') = (d\exp_{c(0)})_{t'v}(t'w)$ for $t' \neq 0$ and $w \neq 0$, which is possible if and only if $t'v$ is a critical point of the exponential map at $c(0)$.

For the second part of the proposition note that the Jacobi fields $J^{(1)}, J^{(2)}, \ldots, J^{(m)}$ along c which are zero at $t = 0$ are linear independent if and only if the vectors

$(\frac{D}{dt}J^{(1)})(0)$, $\frac{D}{dt}(J^{(2)})(0), \ldots, \frac{D}{dt}(J^{(m)})(0)$ are linear independent in $T_{c(0)}X$. Then, from the construction above, we can see that the maximal number of linear independent Jacobi fields along c which vanish also at $t = t'$, is equal to the maximal number of linear independent vectors such that the linear map $(d\exp_{c(0)})_{t'\dot{c}(0)}$ is zero, i.e. the dimension of its kernel. □ □

Let x be a point in an orbifold Q. We say that x is a *pole* if there are no conjugate points to it on any geodesic path starting at x (or actually on any geodesic path containing x). Note that if x is a pole, there are no Jacobi fields along any geodesic $\mathscr{G}$-path starting at x which vanish anywhere else than at x. In this case, by the proposition above, we see that the exponential map at any $\widetilde{x}$ with $q(\widetilde{x}) = x$ has no critical points, hence it is étale.

4.5 Orbifolds of non positive curvature

In the following proposition we will see that this is always the case if the orbifold has non positive curvature.

Proposition 4.5.1. *Let Q be a Riemannian orbifold with non positive sectional curvature. Then any point of Q is a pole.*

Proof. Let $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ be an arbitrary geodesic $\mathscr{G}$-path on X, over a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$, and $J = (J_1, J_2, \ldots J_k)$ a non-zero Jacobi filed along it which vanishes at zero. Then, for each $i = 1, \ldots, k$ we have

$$\frac{D^2 J_i}{dt^2}(t) + R(\dot{c}_i(t), J_i(t))\dot{c}_i(t) = 0,$$

for any $t \in [t_{i-1}, t_i]$, and

$$< \frac{D^2 J_i}{dt^2}(t), J_i > + < R(\dot{c}_i(t), J_i(t))\dot{c}_i(t), J_i > = 0.$$

Hence

$$< \frac{D^2 J_i}{dt^2}(t), J_i > = - < R(\dot{c}_i(t), J_i(t))\dot{c}_i(t), J_i > \geq 0,$$

for any $t \in [t_{i-1}, t_i]$. Therefore

$$\frac{d}{dt} < \frac{DJ_i}{dt}, J_i > = < \frac{D^2 J_i}{dt^2}, J_i > + \parallel \frac{DJ_i}{dt} \parallel^2 \geq 0$$

i.e. each function $< \frac{DJ_i}{dt}, J_i >$ is monotonically increasing on each $[t_{i-1}, t_i]$ and strictly increasing if $\frac{DJ_i}{dt} \neq 0$ on $[t_{i-1}, t_i]$.

Note that the condition (i) in the definition [..] together with the fact that $\mathscr{G}$ is a groupoid of local isometries imply that $< \frac{DJ_i}{dt}(t_i), J_i(t_i) > = < \frac{DJ_{i-1}}{dt}(t_i), J_{i-1}(t_i) >$, for any $i = 1, \ldots, k-1$. This defines a continuous function on the interval $[0, 1]$, which we will denote by $< \frac{DJ}{dt}, J >$. Moreover, this function is monotonically increasing on $[0, 1]$ and strictly increasing if $\frac{DJ}{dt} \neq 0$ on $[0, 1]$.

Suppose now that $c(0)$ has a conjugate points along c. Choose the first one. Then, there exists a Jacobi field as above and $t' \in (0, 1]$ such that $J_i(t') = 0$ for some $i \in \{1, 2, \ldots, k\}$, and so $< \frac{DJ_i}{dt}, J_i >$ vanishes at t'. Therefore $< \frac{DJ}{dt}, J >$ vanishes at both 0 and t', hence it has to vanish identically on $[0, t']$. This implies that $\frac{DJ_1}{dt}(0) = 0$ and since $J_1(0) = 0$, we have that $J \equiv 0$, which contradicts the fact that J is a non-zero Jacobi field. Hence there are no conjugate points to $c(0)$ along c, and since c is arbitrary, $c(0)$ is a pole and of course any point of Q is a pole. $\square$

In particular, on an orbifold with non positive curvature, the exponential map $\mathrm{Exp}_x : T_x Q \to Q$ has no critical points and is an orbifold local diffeomorphism. In what follows we will show that if the orbifold is also complete the exponential map is an orbifold covering map. This will give a proof of Gromov's developability theorem.

Proposition 4.5.2. *Let $Q = X/\mathscr{G}$ be a complete Riemannian orbifold with non positive sectional curvature. Then the exponential map $exp_x : T_x X \to X$ has the path lifting property. Moreover, it is a covering map.*

Proof. Since the orbifold is complete, the exponential map $\exp_x : T_x X \to X$ is defined for all the points $x \in X$ and is surjective. Since Q has non positive curvature, it is also a local diffeomorphism. This allows us to introduce a Riemannian metric on the tangent space $T_x X$ such that $\exp_x : T_x X \to X$ is a local isometry. Indeed, consider $u \in T_x X$ and put for any $v, w \in T_u(T_x X) \cong T_x X$

$$\mu_u(v, w) := \rho_{\exp_x(u)}(d(\exp_x)_u(v), d(\exp_x)_u(w)),$$

where $\rho_{\exp_x(u)}$ denotes the inner product metric on the tangent space at $\exp_x(u)$ to X induced by the Riemannian metric on the connected component of X containing $\exp_x(u)$. Note that since the Riemannian metric is invariant, the definition is good in the sense that it does not depend on a particular choice in the fiber above $\exp_x(u)$. Then μ_u defines an inner product in each tangent space $T_u(T_xX) \cong T_xX$ which clearly varies smoothly with respect to $u \in T_xX$, i.e. defines a metric on T_xX and the exponential map is a local isometry. Note that, by Prop 4.4, this metric is also complete, since the geodesics in T_xX through the origin are straight lines.

We will show now that $\exp_x$ has the path lifting property. Let $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ be an arbitrarily rectifiable $\mathscr{G}$-path in X, over a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$. Assume that $c(0) = x \in X$ and consider the origin $0 \in T_xX$. Then $\exp_x(0) = x$ and since $\exp_x$ is a local diffeomorphism at 0 we can lift c in a small neighborhoodof x. That is, there exists $\varepsilon > 0$ such that we can define $\widehat{c} : [0, \varepsilon] \to T_xX$ with $\widehat{c} = 0$ and $\exp_x(\widehat{c}) = c|_{[0,\varepsilon]}$. Denote by $A \subset [0, 1]$ the set of values for which the $\mathscr{G}$-path c can be lifted to a path starting from $0 \in T_xX$. Then A is nonempty and since $\exp_x$ is a local diffeomorphism on all T_xX, A is open and connected in $[0, 1]$. That is, $A = [0, t')$. If we show that t' is also in A, then A would be also closed and will follow that $A = [0, 1]$. This means that the $\mathscr{G}$-path c can be lifted throughout the whole interval $[0, 1]$, i.e. $\exp_x$ has the path lifting property.

To show that $t' \in A$, let $t^{(m)}, m = 1, \ldots$ be an increasing sequence in A such that $\lim t^{(m)} = t'$. Since $t' \in (0, 1]$, there is $i \in 1, 2, \ldots, k$ such that $t_{i-1} < t' \leq t_i$, and without loss of generality we can assume that $t_{i-1} < t^{(m)} \leq t' \leq t_i$ for all m. Note now that the set $\{\widehat{c}(t^{(m)})\}$ is contained in a compact subset $K \subset T_xX$. Indeed, otherwise, since T_xX is complete the set $\{\widehat{c}(t^{(m)})\}$ would be unbounded, and so the distance between $\widehat{c}(t^{(m)})$ and $\widehat{c}(t_{i-1})$ could be made arbitrarily large. However this is not possible since

$$d(\widehat{c}(t^{(m)}), \widehat{c}(t_{i-1})) \leq L|_{t_{i-1}}^{t^{(m)}}(\widehat{c}) = \int_{t_{i-1}}^{t^{(m)}} |\frac{d\widehat{c}}{dt}|dt = \int_{t_{i-1}}^{t^{(m)}} |d(\exp_x)_{\widehat{c}(t)}(\frac{d\widehat{c}}{dt})|dt$$
$$= \int_{t_{i-1}}^{t^{(m)}} |\frac{dc_i}{dt}|dt = L|_{t_{i-1}}^{t^{(m)}}(c_i)$$

and the length of c_i is finite.

The completeness of T_xX and the fact that $\{\widehat{c}(t^{(m)})\} \subset K$ imply that there is an accumulation point $v \in T_xX$ of $\{\widehat{c}(t^{(m)})\}$. Let V be a neighborhood of v such that $\exp_x|_V$ is a diffeomorphism onto an open neighborhood of $\exp_x(v) \in X_i$. Then $c_i(t') \in \exp_x(V)$ and by continuity of c_i there is a subinterval $I \subset [t_{i-1}, t_i]$ containing t' such that $c_i(I) \subset \exp_x(V)$. Since $\exp_x|_V$ is a diffeomorphism there is a lift of $c_i|_I$ through v, say $\overline{c}$. But $v \in V$ is an accumulation point for $\{\widehat{c}(t^{(m)})\}$, so there exists an index m such that $\widehat{c}(t^{(m)}) \in V$. Since $e_x|_V$ is bijective the lifts $\widehat{c}$ and $\overline{c}$ coincide on the interval $[t_{i-1}, t^{(m)}) \cap I$. Hence $\overline{c}$ is an extension of $\widehat{c}$ to I and so $\widehat{c}$ is defined at t', i.e. $t' \in A$. This completes the proof that $\exp_x$ has the path lifting property. Note that equivalent $\mathscr{G}$-paths in X lift to the same path in T_xX. In particular $\exp_x$ maps the terminal point of the lift $\widehat{c}$ to the terminal point of c. Since $\exp_x$ is étale and surjective, it is a covering map. □

For $x \in X$ there is a well defined continuous action of $\mathscr{G}$ on T_xX over $\exp_x : T_xX \to X$ defined in the following way. If $v \in T_xX$ and $g \in \mathscr{G}$ such that $\omega(g) = \exp_x(v)$, we define $v.g \in T_xX$ to be such that $\exp_x(v.g) = \alpha(g)$. It is well defined in the sense that if $y \in X$ is a different choice in the fiber above $\exp_x(v)$, i.e. there exists $h \in \mathscr{G}$ such that $y = e_x(v)$, then $\alpha(hg) = \alpha(g)$ and $\exp_x(v.(hg)) = \exp_x(v.g)$. Denote $(\mathscr{G}', T_xX)$ the groupoid associated to this action.

Lemma 4.5.3. *The space of orbits of the groupoid $(\mathscr{G}', T_xX)$ is Hausdorff and the natural projection $q' : T_xX \to T_xX/\mathscr{G}'$ is étale and induces an equivalence from $(\mathscr{G}', T_xX)$ to the groupoid $(T_xX \rtimes \mathscr{G}_x, T_xX)$. In particular, the groupoid $(\mathscr{G}', T_xX)$ is developable.*

Proof. First note that if $g \in \mathscr{G}$ is not an element in the isotropy group $\mathscr{G}_x$ and if the $\omega(g) = \exp_x(v)$ then $\exp_x(v.g)$ and $\exp_x(v)$ are not in the same fiber. In other words if $\omega(g) = y$ is the terminal point of a $\mathscr{G}$-geodesic issuing at x and with velocity vector $v \in T_xX$, then the $\mathscr{G}$-geodesic issuing at x and with velocity vector $v.g$ is not equivalent with the previous one and so its terminal point cannot be in the same fiber with y. More generally if $\widetilde{g} : B(\alpha(g), \epsilon) \to B(\omega(g), \epsilon)$ is a section of α as in lemma 4.2 and such that $\widetilde{g}(\alpha(g)) = g$, then $B_{\mathscr{G}}(\alpha(g), \epsilon) \cap B_{\mathscr{G}}(\omega(g), \epsilon) = \varnothing$. This proves that the projection $q' : T_xX \to T_xX$ is étale and that $T_xX/\mathscr{G}'$ is Hausdorff. The map $(v, g) \mapsto q'(v)$ gives an equivalence from $(\mathscr{G}', T_xX)$ to the groupoid $(T_xX \rtimes \mathscr{G}_x, T_xX)$. $\square$

Theorem 4.5.4. *Every connected complete Riemannian orbifold with non positive curvature is developable.*

Proof. Let $Q = X/\mathscr{G}$ be a such orbifold. Let $\pi : \mathscr{G}' \to \mathscr{G}$ be the map $(v, g) \mapsto g$. By Proposition 4.5.2 $(\pi, \exp_x) : (\mathscr{G}', T_xX) \to (\mathscr{G}, X)$ is a covering. By the lemma above $(\mathscr{G}', T_xX)$ is developable, therefore $(\mathscr{G}, X)$ is developable, i.e. the orbifold Q is developable. $\square$

Chapter 5

Loop spaces for orbifolds

We consider in this section a connected Riemannian orbifold $Q = X/\mathscr{G}$. Recall that $\Omega_X = \bigcup_{x \in X} \Omega_x$ is the union of the sets Ω_x of classes $[c]_x$ of continuous $\mathscr{G}$-loops based at x and the space $\Omega_{x,y}$ is the set of classes $[c]_{x,y}$ of continuous $\mathscr{G}$-paths connecting x to y.

Proposition 5.0.5. *The set Ω_X of based $\mathscr{G}$-loops, as well as the set $\Omega_{x,y}$ of equivalence classes of $\mathscr{G}$-paths from x to y, has a natural structure of Banach manifold.*

Proof. Let $c = (g_0, c_1, g_1, \dots, c_k, g_k)$ be a closed $\mathscr{G}$-path over the subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ based at x (the source of g_0 and the target of g_k). Let c^*TX be the induced vector bundle over $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ (see ... a previous section). That is the quotient of the disjoint union of the bundles c_i^*TX by the equivalence relation which identifies the point $(t_i, \xi_i) \in c_{i-1}^*TX$ with the point $(t_i, dg_i(\xi_i)) \in c_i^*TX$ for $1 < i \leq k$ and $(0, \xi_0) \in c_1^*TX$ with $(1, d(g_k^{-1}g_0^{-1})(\xi_0)) \in c_k^*TX$. The projection to the base space $c^*TX \to \mathbb{S}^1$ maps the equivalence class of (t, ξ) to t modulo 1. As in Remark 3.3 if we consider another closed $\mathscr{G}$-path c' based at x which is equivalent to c then there is a natural isomorphism between the vector bundles c^*TX and c'^*TX.

Consider now the vector space $C^0(\mathbb{S}^1, c^*TX)$ of continuous sections of c^*TX. Such a section is represented by a vector field v along c. That is a sequence $v = (v_1, v_2, \dots, v_k)$, where each v_i is a vector field along c_i such that compatibility conditions are satisfied at each of the points t_i (see Definition 4.2 (i)). This vector space can be viewed as the 'tangent space' of the 'manifold' Ω_X at the point $[c]_x$. Note now that the inner product on the fibers of TX given by the Riemannian metric on X induces an inner product on each fiber in c^*TX. Using this inner product, we can define the sup norm on $C^0(\mathbb{S}^1, c^*TX)$ to be

$$||v||_\infty = \Sigma_{i=1}^k sup_{t\in[t_{i-1},t_i]}|v_i|.$$

This norm makes $C^0(\mathbb{S}^1, c^*TX)$ into a Banach space.

To describe the natural atlas for Ω_X we will make use of the exponential map restricted to 'short' vector fields. Given a $\mathscr{G}$-path c based at x, choose $\epsilon > 0$ such that, for each $t \in [t_{i-1}, t_i]$, the exponential map $\exp_{c_i(t)}$ is defined on the ball of radius ϵ in $T_{c_i(t)}X$ and it is a diffeomorphism onto the geodesic ball $B(c_i(t_i), \epsilon)$ (see *exp map*). By a previous result each of the arrows $g_i : c_i(t_i) \to c_{i+1}(t_i)$ extends uniquely (see 2.1.5) to a section $\tilde{g}_i$ of α defined on the ball $B(c_i(t_i), \epsilon)$ for $0 < i \leq k$ respectively $B(x, \epsilon)$ for $i = 0$.

Let $\tilde{U}_c^\epsilon$ be the open ball centered at the zero section and of radius ϵ in $C^0(\mathbb{S}^1, c^*TX)$. We define the exponential map $\exp_c^\epsilon : \tilde{U}_c^\epsilon \to U_c^\epsilon \subseteq \Omega_X$ by mapping a section $v = (v_1, \ldots, v_k)$ to the equivalence class of the based $\mathscr{G}$-path $c^v = (g_0^v, c_1^v, g_1^v, \ldots, c_k^v, g_k^v)$ defined as follows: the paths $c_i^v(t) := \exp_{c_i(t)} v_i(t)$ and the arrows $g_i^v := \tilde{g}_i(c_i^v(t_i)) : c_i^v(t_i) \to c_{i+1}^v(t_i)$ for $0 < i < k$, $g_k^v := \tilde{g}_k(c_k^v(1))$ and $g_0^v := \tilde{g}_0(\omega(g_k^v))$. Note that, as expected, the $\mathscr{G}$-path c^v is not necessarily based at x, but at a point in $B(x, \epsilon)$. It is not hard to see that the construction above depends essentially on the equivalence class of c. So, if we consider an equivalent $\mathscr{G}$-path at x, say c' then the representative $\mathscr{G}$-path $c'^{v'}$, the image of the section v' by the map $\exp_{c'}^\epsilon : \tilde{U}_{c'}^\epsilon \to U_c^\epsilon$ has the same base point as c^v and they are equivalent. Above, v' is considered to be the correspondent of v via the natural isomorphism between the vector spaces $C^0(\mathbb{S}^1, c^*TX)$ and $C^0(\mathbb{S}^1, c'^*TX)$ induced by the vector bundles isomorphism between c^*TX and c'^*TX.

Note that the set U_c^ϵ consists precisely of the equivalence classes of continuous based $\mathscr{G}$-paths c' whose representative over the subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$, $c' = (g'_0, c'_1, g'_1, \ldots, c'_k, g'_k)$ is such that all $c'_i(t)$ are images $\exp_{c_i(t)}(v_i(t))$ for some vector field $v = (v_0, v_1, \ldots, v_k)$ along c with the norm smaller than ϵ, up to composition with maps $h_i : [t_{i-1}, t_i] \to \mathscr{G}$ with $h_i(t) : c'_i(t) \to \exp_{c_i}(v_i(t))$. Note that by Remark 3.2

(d) the maps h_i are uniquely determined by $\exp_{c_i(t)}(v_i(t))$ and hence by v since each $\exp_{c_i(\cdot)}$ is bijective. It is easy to see that the arrows g_i' defined in a similar way as above are also uniquely determined by v. Hence each map $\exp_c^\epsilon$ is a bijection.

At each point $[c]_x$ of Ω_X for a small enough $\epsilon > 0$ we define the chart $(U_c^\epsilon, (\exp_c^\epsilon)^{-1})$. It is easy to see that the images U_c^ϵ for various c and ϵ form then a basis for the topology on Ω_X and that the change of charts are differentiable. Therefore Ω_X is a Banach manifold. Recall that the topology of the based loop space Ω_X is given by the metric

$$d_\infty([c]_x, [c']_x) = \Sigma_{i=1}^k \sup_{t\in[t_{i-1},t_i]} d(c_i(t), c_i'(t)),$$

where both c and c' are representatives over the same subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$ and $d(\cdot, \cdot)$ is the metric derived from the Riemannian metric on X. Note that Ω_X^0 is a finite dimensional submanifold of Ω_X.

The Banach manifold structure on $\Omega_{x,y}$ is defined similarly. For a $\mathscr{G}$-path $c = (g_0, c_1, \ldots, c_k, g_k)$, the tangent space at $[c]_{x,y}$ is isomorphic to the space of vector fields $v = (v_1, \ldots, v_k)$ along c which vanish at 0 and 1. □ □

As we have already seen in a previous section, the groupoid $\mathscr{G}$ acts naturally on the right on the set Ω_X of based $\mathscr{G}$-loops with respect to the projection $p : \Omega_X \to X$ associating to a $\mathscr{G}$-loop based at x the point x. The action of $\mathscr{G}$ on Ω_X with respect to the projection assigning to a based $\mathscr{G}$-loop its base point is continuous. We defined the quotient of Ω_X by this action to be the "space" of free loops $|\Lambda(\mathscr{G})| = |\Lambda Q|$ on Q.

Proposition 5.0.6. *The space $|\Lambda Q|$ of free $\mathscr{G}$-loops has a natural orbifold structure noted ΛQ. The subspace $|\Lambda^0 Q|$ of free loops of length zero is a "suborbifold" $\Lambda^0 Q$ of ΛQ.*

Proof. The natural orbifold structure is given by the groupoid structure associated to the action of $\mathscr{G}$ on the Ω_X with respect to the projection $p : \Omega_X \to X$. The groupoid of germs of changes of chart is the groupoid $\overline{\mathscr{G}} := \mathscr{G} \times_X \Omega_X$, the subspace of $\mathscr{G} \times \Omega_X$ consisting of pairs $(g, [c]_x)$ with $\alpha(g) = x$. The source (resp. target) projection maps $(g, [c]_x)$ to $[c]_x$ (resp. $[c]_x.g$). The composition $(g', [c']_{x'})(g, [c]_x)$ is defined if $[c']_{x'} = [c]_x.g$ and is equal to $(gg', [c]_x)$.

The suborbifold structure on $\Lambda^0 Q$ is obtained by replacing Ω_X by Ω_X^0 and $\mathscr{G}\times_X\Omega_X$ by $\mathscr{G} \times_X \Omega_X^0$.□ □

The orbifold $\Lambda^0 Q$ is called in the mathematical literature the *inertia orbifold* of Q, and it is, according Chen and Ruan the classical geometrical manifestation on the twisted sectors of the orbifold string theory (see Chen and Ruan and also Lupercio..).

As in the section 3.6, we can associate a classifying space to orbifold ΛQ (resp. $\Lambda^0 Q$). It turns out a that the functor B "commutes" with the construction of loop spaces, in the sense that there is a homotopy equivalence

$$B\Lambda Q \simeq \Lambda BQ,$$

where $B\Lambda Q$ denotes the classifying space of ΛQ and ΛBQ denotes the loop space of the classifying space of Q.

We consider as before a classifying space BQ, base space of a universal principal $\mathscr{G}$-bundle $E\mathscr{G} \to B\mathscr{G} = BQ$. Let $E\mathscr{G} \times_X \Omega_X$ be the subspace of $E\mathscr{G} \times \Omega_X$ consisting of pairs $(e, [c]_x)$ such that $q_{E\mathscr{G}}(e) = x$. We note $E\mathscr{G} \times_{\mathscr{G}} \Omega_X$ its quotient by the equivalence relation identifying $(e.g, [c]_x)$ to $(e, g.[c]_x)$.

Proposition 5.0.7. (prop. 3.2.1 [GH]) *$E\mathscr{G} \times_X \Omega_X \to E\mathscr{G} \times_{\mathscr{G}} \Omega_X$ is a principal universal $(\mathscr{G} \times_X \Omega_X)$-bundle. The base space $E\mathscr{G} \times_{\mathscr{G}} \Omega_X$ will be noted $B\Lambda Q$.*

Similarly $E\mathscr{G} \times_X \Omega_X^0 \to E\mathscr{G} \times_{\mathscr{G}} \Omega_X^0$ is a principal universal $(\mathscr{G} \times_X \Omega_X^0)$-bundle. Its base space $E\mathscr{G} \times_{\mathscr{G}} \Omega_X^0$ is noted $B\Lambda^0 Q$.

The natural projection $B\Lambda Q \to BQ$ (i.e. $E\mathscr{G} \times_{\mathscr{G}} \Omega_X \to E\mathscr{G}/\mathscr{G} = B\mathscr{G} = BQ$) is a Serre fibration with fibers isomorphic to Ω_x.

Theorem 5.0.8. (Theorem 3.2.2 [GH]) *ΛBQ is the base space of a universal $(\mathscr{G} \times_X \Omega_X)$-bundle. This bundle is the pull back of $E\mathscr{G} \times_X \Omega_X$ by a map $\varphi : \Lambda BQ \to B\Lambda Q$ which is a weak homotopy equivalence and commutes with the projections to BQ. Therefore ΛBQ is a classifying space for the orbifold ΛQ.*

For points $z \in BQ$ and $x \in X$ projecting to the same point of $|Q|$, the map φ induces a weak homotopy equivalence from the space $\Omega_z BQ$ of loops on BQ based at z to Ω_x.

Remark 5.0.9. Composing $\varphi : \Lambda BQ \to B\Lambda Q$ with the natural projection $B\Lambda Q \to |\Lambda Q|$, we get an $O(2)$-equivariant map

$$\Lambda BQ \to |\Lambda Q|.$$

Remark 5.0.10. One can show that, for a space K, there is a canonical correspondance associating to a principal $\mathscr{G}$-bundle E over $K \times \mathbb{S}^1$ a principal $\overline{\mathscr{G}} - bundle$ over K, where $\overline{\mathscr{G}} = \mathscr{G} \times_X \Omega_X$, inducing a bijection on isomorphisms classes. The universal $\overline{\mathscr{G}}$-bundle over ΛBQ corresponds to the principal $\mathscr{G}$-bundle over $\Lambda BQ \times \mathbb{S}^1$ which is the pull back of $E\mathscr{G}$ by the evaluation map $\Lambda BQ \times \mathbb{S}^1 \to BQ$ sending (l, t) to $l(t)$.

5.1 The Riemannian orbifold $\Lambda' Q$ of free $\mathscr{G}$-loops of class H^1

We consider as above a Riemannian orbifold $Q = \mathscr{G} \backslash X$. Note that on these spaces of continuous paths we cannot define neither the length nor the energy of a path. Therefore, in differential geometry one considers the subspaces of picewise differentiable $\mathscr{G}$-paths (i.e. each c_i is picewise differentiable). Their infinitesimal approximation at a picewise differentiable $\mathscr{G}$-path c is given by the vector space of picewise differentiable vector fields along c, denoted $C'^{\infty}(c^*TX) = C'^{\infty}(\mathbb{S}^1, c^*TX)$.

As we just have seen, the space Ω_X, as well as the space $\Omega_{x,y}$, caries a natural structure of a Banach manifold given by the Banach structure of of the space $C^0(c^*TX) = C^0(\mathbb{S}^1, c^*TX)$ with the norm $|| \cdot ||_{\infty}$. This space can be seen as the completion of $C'^{\infty}(c^*TX)$ with respect to the sup norm.

We will consider a different norm on $C'^{\infty}(c^*TX)$ denoted $|| \cdot ||_1$, derived from the following scalar product

$$< v, w >_1 = < v, w >_0 + < \nabla v, \nabla w >_0,$$

where v and w are picewise differentiable vector fields along a $\mathscr{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ and ∇v denotes the covariant derivative with respect to the induced connection on

c^*TX. The scalar product $< \cdot, \cdot >_0$ is given by

$$< v, w >_0 = \Sigma_{i=1}^{k} \int_{t_{i-1}}^{t_i} < v_i(t), w_i(t) > dt.$$

Here $< \cdot, \cdot >$ denotes the inner product on fiber of c^*TX above t, induced by the Riemannian metric on X. The completion of $C'^{\infty}(c^*TX)$ with respect to the norm $|| \cdot ||_1$ will be denoted $H^1(c^*TX)$, the vector space of H^1 sections of c^*TX. By a classical result of Lebesgue, a continuous section $v = (v_1, v_2, \dots v_k)$ is a H^1 section if each v_i is absolutely continuous and its covariant derivative ∇v_i is square integrable. $H^1(c^*TX)$ with the scalar product $< \cdot, \cdot >_1$ is a separable Hilbert space.

Definition 5.1.1. *A $\mathscr{G}$-path $c = (g_0, c_1, g_1, \dots, c_k, g_k)$ over a subdivision $0 = t_0 < t_1 < \dots < t_k = 1$ is of* class H^1 *if each c_i is absolutely continuous and the velocity functions $t \mapsto |\dot{c}_i(t)|$ are square integrable.*

Note that any equivalent $\mathscr{G}$-path to c will satisfy these conditions. We denote $\Omega'_{x,y}$ (respectively Ω'_x) the set of equivalence classes of $\mathscr{G}$-paths of class H^1 from x to y (respectively $\mathscr{G}$-loops based at x). Let $\Omega'_X = \bigcup_{x \in X} \Omega'_x$ and $|\Lambda' Q|$ be the set of free $\mathscr{G}$-loops on Q represented by closed $\mathscr{G}$-path of class H^1. The energy function E is defined on all those spaces.

As in the case of continuous $\mathscr{G}$-paths one can prove the following.

Proposition 5.1.2. *The set Ω'_X as well as the set $\Omega'_{x,y}$ has a natural structure of Riemannian Hilbert manifold.*

The tangent space at Ω'_X at $[c]_x$ is canonically isometric to the space $H^1(c^*TX)$ of H^1 sections of the bundle c^*TX. The inner product $< \cdot, \cdot >_1$ defined above is independent of the choice of c in its equivalence class. Similarly the tangent space to $\Omega'_{x,y}$ at $[c]_{x,y}$ can be identified with the Hilbert space of H^1-sections of the bundle c^*TX over $[0, 1]$ which vanish at 0 and 1 with the inner product $< \cdot, \cdot >_1$.

The natural action of $\mathscr{G}$ on Ω_X restricts to an action by isometries on Ω'_X. Therefore on the quotient space $|\Lambda'(Q)|$ we get a structure of Riemannian Hilbert orbifold noted $\Lambda' Q$. It is complete if Q is complete. Like before we have a continuous action of the group of diffeomorphisms of $\mathbb{S}^1$ on $|\Lambda' Q|$, and by restriction an action of $O(2)$ by isometries. (The inertia orbifold is invariant by this action.) The groupoid of change of charts is $\overline{\mathscr{G}}' := \mathscr{G} \times_X \Omega'_X$. As in previous section one proves that the space $B\Lambda' Q := E\mathscr{G} \times_{\mathscr{G}} \Omega'_X$ is a classifying space for $\Lambda' Q$.

Proposition 5.1.3. *The natural inclusions*

$$\Omega'_X \to \Omega_X, \qquad \Omega'_{x,y} \to \Omega_{x,y}$$

are continuous and are homotopy equivalences. In particular, if Q is connected, Ω'_X has the same weak homotopy type as the space of loops on BQ based at a fixed point.

The induced inclusion

$$B\Lambda' Q \to B\Lambda Q$$

is a homotopy equivalence.

Proof. The topologies on Ω_X and Ω'_X can be described as the those induced by the corresponding distances given by the Riemannian metric. If $[c]$ and $[c']$ are points in Ω_X then the distance $d_\infty([c],[c'])$ is the infimum of the length $L_\infty(F)$ of curves $F : [0,1] \to \Omega_X$ from $[c] = F(0)$ to $[c'] = F(1)$. Here the length is defined by

$$L_\infty(F) = \int_0^1 ||\frac{dF}{ds}||_\infty ds,$$

where $|| \cdot ||_\infty$ is the sup norm on $C^0(F(s)^*TX)$. If $[c]$ and $[c']$ are in Ω'_X, then the distance induced by the Riemannian metric $d_1([c],[c'])$ is the infimum of the length

$$L_1(F) = \int_0^1 ||\frac{dF}{ds}||_1 ds$$

of curves $F : [0,1] \to \Omega'_X$, where the $|| \cdot ||_1$ is the norm derived from the inner product $< \cdot, \cdot >_1$ on $H^1(F(s)^*TX)$. Note that if $v = (v_1, v_2, \ldots, v_k)$ is a H^1 vector field along a $\mathscr{G}$-loop of class H^1, then $||v||_\infty \leq \sqrt{2}||v||_1$. Indeed, if we choose $t_i^* \in [t_{i-1}, t_i]$ such that $\sup_{t \in [t_{i-1}, t_i]} ||v_i(t)|| = ||v_i(t_i^*)||$, then

$$\Big(\sup_{t\in[t_{i-1},t_i]}||v_i(t)||\Big)^2 = ||v_i(t)||^2 + \int_t^{t_i^*}\frac{d}{d\sigma}||v_i(\sigma)||^2 d\sigma$$
$$\le ||v_i(t)||^2 + 2\int_{t_{i-1}}^{t_i}||v_i(\sigma)||\;||\nabla v_i(\sigma)||d\sigma$$
$$\le \int_{t_{i-1}}^{t_i}||v_i(\sigma)||^2 d\sigma + \int_{t_{i-1}}^{t_i}||v_i(\sigma)||^2 d\sigma + \int_{t_{i-1}}^{t_i}||\nabla v_i(\sigma)||^2 d\sigma$$
$$\le 2\int_{t_{i-1}}^{t_i}\big(||v_i(\sigma)||^2 + ||\nabla v_i(\sigma)||^2\big)d\sigma,$$

and the inequality follows. This implies in particular that if $F:[0,1]\to\Omega'_X$ then $L_\infty(F)\le\sqrt{2}L_1(F)$. Therefore, for $[c],[c']\in\Omega'_X$ we have

$$d_\infty([c],[c']) = \inf\{L_\infty(F)\mid F:[0,1]\to\Omega_X\} \le \inf\{L_\infty(F)\mid F:[0,1]\to\Omega'_X\}$$
$$\le \inf\{\sqrt{2}L_1(F)\mid F:[0,1]\to\Omega'_X\} = \sqrt{2}d_1([c],[c']),$$

i.e. the inclusion $\Omega'_X\to\Omega_X$ is continuous.

The map $r(s,c)=r_s(c)$ is continuous as function of two variables. Note that this inclusion also compact. If the orbifold Q is compact, then this is true for $\Omega'_X\to\Omega_X$. In particular Ω'_x is a closed submanifold of Ω'_X. If the orbifold Q is complete then the space $\Omega'_{x,y}$ with the distance induced by the Riemannian metric is also complete. The same is true for Ω'_X if Q is compact. (see Klingemberg, Theorem 2.7)

We will show now that the above inclusions are homotopy equivalences by following the argument of Milnor used in [.M.Th.]. Denote by P the spaces Ω_X or $\Omega_{x,y}$ and by P' the spaces Ω'_X or $\Omega'_{x,y}$.

For a positive integer k, let P_k (resp.P'_k) be the subspace of P (respectively P') formed by the equivalence classes represented by $\mathscr{G}$-paths $c=(g_0,c_1,g_1,\dots,c_{2^k},g_{2^k})$ defined over the subdivision $0=t_0<\cdots<t_{2^k}=1$, where $t_i=i/2^k$, and such each $c_i(t_{i-1})$ is the center of a convex geodesic ball containing the image of c_i. Then to each such representative $\mathscr{G}$-path c we can associate uniquely the $\mathscr{G}$-path $\overline{c}=(g_0,\overline{c}_1,g_1,\dots,\overline{c}_{2^k},g_{2^k})$, where $\overline{c}_i$ is the (unique) geodesic segment joining $c_i(t_{i-1})$ to $c_i(t_i)$. Note that as in the manifold case the $\mathscr{G}$-path $\overline{c}$ is a "broken $\mathscr{G}$-geodesic" with a braking point at each t_i, $i=1,\dots,2^k$. Denote by B_k the set of all such broken

$\mathscr{G}$-geodesics. We will show that B_k is a deformation retract of P_k (respectively of P'_k). Indeed, denote by $r : P_k \to B_k$ the map $c \mapsto \overline{c}$ (i.e. each $r(c)_i = \overline{c}_i$) and define for $s \in [0,1]$ the family $r_s : P_k \to P_k$ as follows. For $t_{i-1} \leq s \leq t_i$ let

$$\begin{aligned}
&r_s(c)_j = r(c)_j, \text{ for } 1 \leq j < i \\
&r_s(c)_{i|[t_{i-1},s]} = \text{minimal geodesic from } c_i(t_{i-1}) \text{ to } c_i(s), \\
&r_s(c)_{i|[s,t_i]} = c_{i|[s,t_i]} \text{ and} \\
&r_s(c)_j = c_j, \text{ for } i < j \leq 2^k
\end{aligned}$$

Then $r_0 = id_{P_k}$ and $r_1 = r$. The continuity of the minimal geodesic with respect to its end points implies the continuity of $r(s,c) = r_s(c)$ as function of two variables. This proves that B_k is a deformation retract of P_k and that the inclusion $B_k \to P_k$ is a homotopy equivalence. In a similar way the set P'_k retracts to the space of broken geodesics and that $B_k \to P'_k$ is a homotopy equivalence. This gives a continuous deformation of $i_{|P'_k} : P'_k \to P_k$ and implies that this inclusion is a homotopy equivalence.

We will prove now that P(respectively P') is the homotopy limit of the subspaces P_k (respectively P'_k). That is, the projection map $p : P_\Sigma \to P$ is a homotopy equivalence, where P_Σ is the infinite union $P_0 \times [0,1] \cup P_1 \times [1,2] \cup \cdots$, topologiezed as a subset of $P \times \mathbb{R}$ (see Milnor, Appendix).

First, following Milnor, we will define a continuous function $A : P \to (0, 1/2^k]$ such that for any $t, t' \in [\frac{i-1}{2^k}, \frac{i}{2^k}]$ with $|t - t'| < 2^{k+1}A(c_i)$ we have that $c_i(t)$ and $c_i(t')$ can be joined by a unique minimal geodesic which varies differentiably with the end points.

Let $f : X \to [0,\infty)$ be a continuous function such that $f^{-1}([0,a])$ is compact for every $a \in [0,\infty)$. Let $\epsilon_1(a) \leq \infty$ be the largest number such that any two points with distance smaller than $\epsilon_1(a)$ can be joined by a unique minimal geodesic which varies differentiably with the end points. Since ϵ_1 is positive, monotone decreasing function, we can choose $\epsilon_2 : [0,\infty) \to \mathbb{R}$ such that ϵ_2 is continuous and $0 < \epsilon_2(a) < \epsilon_1(a)$. Now define $\epsilon : P \to \mathbb{R}$ by $\epsilon(c_i) = \epsilon_2\Big(\max_{t\in[0,1]}\big(f(c_i(t))\big)\Big)$. Thus ϵ is continuous, and any two points of $c_i([\frac{i-1}{2^k}, \frac{i}{2^k}])$ of distance $\leq \epsilon(c_i)$ are joined by a unique minimal geodesic.

Define a continuous function

$$F : P \times [0,1] \to \mathbb{R}$$

by

$$F(c_i, s) = (s-1)\epsilon(c+i) + \max_{|t-t'|\le s} d\big(c_i(t), c_i(t')\big).$$

Then F, considered as a function of s, is strictly monotone, and $F(c_i, 0) < 0 \le F(c_i, 1)$. Hence for each $c \in P$ there is a unique $s \in (0, 1/2^k]$ with $F(c_i, s) = 0$. Define $A(c_i) = s/2$. Since F is continuous, the function $A : P \to (0,1]$ is continuous. If $|t-t'| \le s = 2^{k+1}A(c_i)$ then $d\big(c_i(t), c_i(t')\big) \le (1-s)\epsilon(c_i) \le \epsilon(c_i)$ hence $c_i(t)$ and $c_i(t')$ are joined by a unique minimal geodesic.

Now define a continuous function $h : P \to P_k$ as follows. For a $\mathscr{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_l)$ we can find an integer k and a subdivision $0 = t_0 < t_1 < \cdots < t_{2^k} = 1$with $t_i = i/2^k$ and a $\mathscr{G}$-path path over it equivalent to the initial one. Let $h(c)$ be the unique $\mathscr{G}$-path such that (i) $h(c)_i$ coincides with c_i for $t = (i-1)/2^k$, $((i-1) + A(c_i))/2^k$, $((i-1) + 2A(c_i))/2^k$, $\ldots$ $((i-1) + m_iA(c_i))/2^k$, $i/2^k$, where k is the largest integer in $1/A(c_i)$; and

(ii) $h(c)_i$ is a geodesic in each intermediate interval.

One can deform continuously such a $\mathscr{G}$-path c to the $\mathscr{G}$-path $h(c)$. Passing to equivalence classes, this gives a continuous deformation. As the spaces P and P' are the increasing union of the open subspaces P_k and P'_k for $k = 1, 2, \ldots$, it follows that i is a homotopy equivalence. (see [Mi])

The last assertion follows from the fact that the above deformation commutes with the projection to X and with the action of $\mathscr{G}$.

□

5.2 The energy function

The energy function E is well defined on $\Omega'_{x,y}$ and Ω'_X. As it is invariant by the action of $\mathscr{G}$, it gives a well defined function on $|\Lambda' Q|$ still noted E.

E is a differentiable function on Ω'_X or $\Omega'_{x,y}$. The gradient grad E of E at $[c]_x$ is given by the formula:

$$\text{grad } E(v) = < \dot{c}, \nabla v >_0 = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \langle \dot{c}_i(t), \nabla v_i(t) \rangle dt,$$

where $v = (v_1, \ldots, v_k) \in H^1(c^*TX)$ and where $< \cdot, \cdot >$ denotes the inner product in the fiber above t in c^*TX induced by the Riemannian metric on X (see Theorem 1.20 [K1]).

Now we can characterize the critical points of E in Ω'_X and $\Omega'_{x,y}$.

Proposition 5.2.1. *The critical points $[c]_{x,y}$ of E on $\Omega'_{x,y}$ are the geodesics $\mathscr{G}$-paths from x to y and the critical points $[c]_x$ of E on Ω'_X are either points in ${\Omega'}_X^0$ (i.e. based $\mathscr{G}$-loops of length zero) or closed $\mathscr{G}$-geodesics at x.*

Proof. By partial integration and using that the action of $\mathscr{G}$ is by isometries we get

$$\begin{aligned}
\text{grad} E(v) &= \sum_{i=1}^{k} \Big(-\int_{t_{i-1}}^{t_i} \langle \nabla \dot{c}_i(t), v_i(t) \rangle dt + \langle \dot{c}_i(t_i), v_i(t_i) \rangle - \langle \dot{c}_i(t_{i-1}), v_i(t_{i-1}) \rangle \Big) \\
&= -\Big(\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \langle \nabla \dot{c}_i(t), v_i(t) \rangle \Big) + \langle \dot{c}_k(1), v_k(1) \rangle - \langle \dot{c}_1(0), v_1(0) \rangle \\
&= -\langle \nabla \dot{c}, v \rangle_0 + \langle \dot{c}_k(1), v_k(1) \rangle - \langle \dot{c}_1(0), v_1(0) \rangle
\end{aligned}$$

Thus, if c represents a $\mathscr{G}$-geodesic from x to y, then $\nabla \dot{c} = 0$ (i.e. $\nabla \dot{c}_i = 0$ for all i) and since $v \in H^1(c^*TX)$ we have that $v(0) = v(1) = 0$, which implies that $\text{grad}E(v) = 0$ for any v, i.e. $[c]_{x,y}$ is a critical point. Conversely, if we assume that at $[c]_{x,y}$ we have $\text{grad}E(v) = -\langle \nabla \dot{c}, v \rangle_0 = 0$ for all $v \in H^1(c^*TX)$, then $\nabla \dot{c} = 0$ i.e. c represents a $\mathscr{G}$-geodesic from x to y.

In the case of Ω'_X , if $[c]_x \in {\Omega'}_X^0$ then it is obvious a critical point for E. If $[c]_x$ is a closed $\mathscr{G}$-geodesic then $\nabla \dot{c} = 0$ and $\dot{c}_1(0) = d(g_0 g_k)\dot{c}_k(1)$. Then the conclusion follows again from the compatibility conditions for $v \in H^1(c^*TX)$, $v_1(0) = d(g_0 g_k) v_k(1)$ together with the fact that the action is by isometries. Conversely, if $\text{grad}E(v) = \langle \dot{c}, \nabla v \rangle_0 = 0$ for any $v \in H^1(c^*TX)$, determine H^1 vector fields u, u' along c by

$$\nabla u = \dot{c}, \quad u(0) = 0;$$

$$\nabla u' = 0,\ u'(1) = u(1)$$

and consider $w = u - tu'$. Then $w(0) = w(1) = 0$ and $\nabla w = \dot{c} - u'$. But,

$$0 = \mathrm{grad}E(w) = \langle \dot{c}, \nabla w \rangle_0 = \langle \dot{c}, \dot{c} - u' \rangle_0$$

and

$$\langle u', \dot{c} - u' \rangle_0 = \langle u', \nabla w \rangle_0 = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{d}{dt} \langle u_i', w_i \rangle dt = \langle u'(1), w(1) \rangle - \langle u'(0), w(0) \rangle = 0$$

imply that $||\dot{c} - u'||_0 = 0$, i.e. $\dot{c} = u'$ and so $\nabla \dot{c} = \nabla u' = 0$.

$\square$

In order to be able to extend the classical theory of functions and their critical points on an Euclidian manifold to a Hilbert manifold, we will make use of the so-called Palais-Smale condition (C). This condition is a substitute for the failure of a proper Hilbert manifold to be locally compact.

Condition 5.2.2. (The Palais-Smale condition) *Let c_m be a sequence of closed $\mathscr{G}$-paths (resp. of $\mathscr{G}$-paths from x to y) such that*

(i) the sequence $E(c_m)$ is bounded,

(ii) the sequence $|grad\ E(c_m)|$ tends to zero;

then the sequence $[c_m]$ (resp. $[c_m]_{x,y}$) has accumulations points and any converging subsequence converges to a geodesic.

For a compact orbifold (resp. a complete orbifold) the Palais-Smale condition (C) holds for the function E on $|\Lambda' Q|$ (resp. on $\Omega'_{x,y}$) (see Theorem 2.9 [K1]). This implies that, for $a \geq 0$, the set of critical points of E in the subspaces $E^{-1}([0, a])$ of $|\Lambda' Q|$ or $\Omega'_{x,y}$ is compact.

Note that on $\Omega'_{x,y}$ the smallest value for E is $d(x, y)^2/2$. Hence, for $a < d(x, y)^2/2$ the set $E^{-1}([0, a])$ is empty. On Ω'_X the smallest value of E is zero and it is attained on Ω'^0_X.

The vector field $-\text{grad}\, E$ generates a local flow φ_t on Ω'_X (respectively on $\Omega'_{x,y}$). As it is known from the theory of differential equations, for every $[c]_x \in \Omega'_X$ (resp. $[c]_{x,y} \in \Omega'_{x,y}$) there exists a maximal interval $J = J([c])$ containing $0 \in \mathbb{R}$ on which $\varphi_t,\ t \in J$, is defined. If $\text{grad} E = 0$ then $\varphi_t = c$ for all $t \in \mathbb{R}$, i.e. in this case $J = \mathbb{R}$. It can be proved that the local flow on Ω'_X is defined for all $t \geq 0$ when Q is compact (resp. on $\Omega_{x,y}$ if Q is complete) (see Theorem 2.15 Klingenberg). Here we essentially use the fact that $E(\varphi_t)$ is bounded from below by 0. If $E(\varphi_t)$ is also bounded from above then the same argument will show that φ_t is defined also for $t \leq 0$.

On Ω'_X, φ_t commutes with the action of $\mathscr{G}$ and gives a local flow on $|\Lambda' Q|$. If Q is compact, the local flow φ_t is $\overline{\mathscr{G}}'$-complete in the following sense, where $\overline{\mathscr{G}}' := \mathscr{G} \times_X \Omega'_X$ is the groupoid of germs of change of charts of the orbifold $\Lambda' Q$. Given $\overline{x} \in \Omega'_X$ and $\tau \geq 0$, one can find a $\overline{\mathscr{G}}'$-path $\overline{c} = (\overline{g}_0, \overline{c}_1, \overline{g}_1, \ldots, \overline{c}_k, \overline{g}_k)$ over a subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_k = \tau$ of the interval $[0, \tau]$ such that $\overline{\omega}(g_0) = \overline{x}$ and $\overline{c}_i(t) = \varphi_{t - t_{i-1}}(\overline{c}_i(t_{i-1}))$ for $i = 1, \ldots, k$. The image of this path in $|\Lambda' Q|$ is the φ_t trajectory of the projection of $\overline{x}$ for $t \in [0, \tau]$. When $\overline{x}$ remains in a small neighbourhood, such a $\overline{\mathscr{G}}'$-path exists over the same subdivision and varies continuously.

Given another such $\overline{\mathscr{G}}'$-path $\overline{c}' = (\overline{g}'_0, \overline{c}'_1, \overline{g}'_1, \ldots, \overline{c}'_k, \overline{g}'_k)$ issuing from $\overline{x}'$ defined over the same subdivision of the interval $[0, \tau]$ and an element $\overline{g} \in \overline{\mathscr{G}}'$ with source $\overline{x}$ and target $\overline{x}'$, then there are unique continuous maps $h_i : [t_{i-1}, t_i] \to \overline{\mathscr{G}}$ such that $\alpha(h_i(t)) = \overline{c}_i(t)$, $\omega(h_i(t)) = \overline{c}'_i(t)$, $\overline{g}'_0 h_1(0) = \overline{g}\overline{g}_0$ and $h_i(t_i)\overline{g}_i = \overline{g}'_i h_{i+1}(t_i)$ for $i = 1, \ldots, k-1$. This implies easily the following

Lemma 5.2.3. *Given a morphism from K to $\mathscr{G}$) and $\tau \geq 0$, there is a unique homotopy f_t of f parametrized by $t \in [0, \tau]$ whose projection to $|\Lambda' Q|$ is the flow φ_t applied to the projection of f.*

Let P be either $|\Lambda' Q|$ or $\Omega'_{x,y}$. In the first case we assume that $|Q|$ compact and in the second case that $|Q|$ is complete. Up to the end of this section, we assume that Q is compact (resp. complete) if $P = |\Lambda' Q|$ (resp. $P = \Omega_{x,y}$). For a number $a \in \mathbb{R}$, we denote P^a the set of point of P for which the value of the energy function is $\leq a$.

Definition 5.2.4. *A* φ-family *is a collection* $\mathscr{F}$ *of non-empty subsets* F *of* P *such that*

(i) E *is bounded on each* F *and*

(ii) $\mathscr{F}$ *is closed under the flow* φ_t*, i.e.* $F \in \mathscr{F}$*, then* $\varphi_t(F) \in \mathscr{F}$ *for all* $t \geq 0$.

Let $a \in \mathbb{R}$ *and chose sufficiently small* $\epsilon > 0$ *such that* E *has no critical values in* $(a, a+\epsilon]$*. A* φ-family of P mod P^a *is a* φ*-family* $\mathscr{F}$ *such that*

(iii) $F \in \mathscr{F}$ *implies that* $F \not\subset P^{a+\epsilon}$.

The value

$$a_{\mathscr{F}} = \inf_{F \in \mathscr{F}} \sup E_{|F}$$

is called the critical value of the φ-family $\mathscr{F}$ *of* P *mod* P^a.

Note that if $a < 0$ a φ-family is always a φ-family of P mod P^a. The importance of the concept of a φ-family lies in the fact that its critical value is a critical value for E as it can be seen in the following theorem (see Theorem 2.18 Klingenberg).

Theorem 5.2.5. *The critical value* $a_{\mathscr{F}}$ *of a* φ*-family* $\mathscr{F}$ *of* P *mod* P^a *is always* $> a$ *and there is a critical point of* E *with value* $a_{\mathscr{F}}$.

We will first prove two lemmas. The first one is an immediate consequence of condition (C).

Lemma 5.2.6. *Let* $a > 0$ *and denote by* Cr^a *the set of critical points of* E *in* $E^{-1}([0,a])$*. Let* U *be an open neighborhood of* U *in* P*. Then there exist* $\epsilon = \epsilon(Cr^a) > 0$ *and* $\delta = \delta(\epsilon) > 0$ *such that* $[c] \in \left(P^{a+\epsilon} - P^{a-\epsilon}\right) \cap \mathbf{C}U$ *implies* $|gradE| \geq \delta$.

Proof. Assume this is not the case. Then this would mean that there exists a sequence $[c_m]$ in $\mathbf{C}U$ with $\lim |\text{grad}E(c_m)| = 0$ and $\lim E(c_m) = a$. Then the Palais-Smale condition gives us the existence of a limit point in Cr^a of this sequence, which contradicts the choice of $[c_m]$. $\square$ $\square$

Lemma 5.2.7. *Let* a *be a noncritical value of* E*. Then* $a > 0$ *and there exist* $\epsilon > 0$ *and* $t^* > 0$ *such that* $\varphi_t(P^{a+\epsilon}) \subset P^{a-\epsilon}$.

Proof. From the previous lemma we have, for $U = \varnothing$, the existence of $\epsilon > 0$ and $\delta > 0$ such that $|\mathrm{grad}E([c])| \geq \delta$ for $[c]$ satisfying $a - \epsilon \leq E(c) \leq a + \epsilon$. If $E(c) < a - \epsilon$, then also $E(\varphi_t(c)) < a - \epsilon$, for all $t \geq 0$. Let $t^* = 2\epsilon/\delta^2$ and consider $[c]$ such that $a - \epsilon < E(c) \leq a + \epsilon$. Assume now that $E(\varphi_t(c)) > a - \epsilon$ for $0 \leq t \leq t^*$. Then

$$\begin{aligned} E(\varphi_t(c)) &= E(c) + \int_0^{t^*} \frac{d}{dt} E(\varphi_t(c)) dt = E(c) - \int_0^{t^*} |\mathrm{grad}E(\varphi_t(c))|^2 dt \\ &\leq a + \epsilon - \delta^2 t^* = a - \epsilon \end{aligned}$$

which is a contradiction.□ □

Proof. (*Proof of Theorem*) The definition of $a_{\mathscr{F}}$ implies that for every $\epsilon > 0$ there exists $F \in \mathscr{F}$ such that $\sup E_{|F} < a_{\mathscr{F}} + \epsilon$. Hence $a_{\mathscr{F}} > a$. Suppose $a_{\mathscr{F}}$ is not a critical value for E. Then by Lemma [..] there exists $\epsilon_0 > 0$ and $t^* > 0$ such that $\varphi_s(P^{a_{\mathscr{F}}+\epsilon_0}) \subset P^{a_{\mathscr{F}}-\epsilon_0}$ for all $t \geq t^*$. In particular this would imply $\sup E_{|F} < a_{\mathscr{F}} - \epsilon_0$ with F as above, which is not possible. Hence $a_{\mathscr{F}}$ is a critical value for E. □ □

Applying this theorem to the φ-family formed by the points of a connected component of P and for $a < 0$, we get the following corollary.

Corollary 5.2.8. *The energy function E restricted to a connected component of P assumes its infimum in some point, and such a point is a critical point of E.*

5.2.1 The second order neighborhood of a critical point

We study the second order neighborhood of a critical point c of E, i.e. the Hessian $D^2E(c)$. Consider a $\mathscr{G}$-path $c = (g_0, c_1, \ldots, c_k, g_k)$ over the subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$, representing a closed geodesic of positive length based at x or a geodesic from x to y. Let $v = (v_1, \ldots, v_k)$ and $w = (w_1, \ldots, w_k)$ are vector fields along c, identified to elements of the tangent space $T_{[c]_x}\Omega'_X$ or $T_{[c]_{x,y}}\Omega'_{x,y}$, The Hessian of the energy function at the critical point $[c]_x \in \Omega'_X$ of $[c]_{x,y}$ in $\Omega'_{x,y}$ is given by the following expression.

$$D^2E(c)(v, w) = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (\langle \nabla v_i(t), \nabla w_i(t) \rangle dt + \langle R(\dot{c}_i(t), v_i(t))\dot{c}_i(t), w_i(t) \rangle) dt$$

where R is the curvature tensor of the Riemannian metric on X. This is a symmetric and bilinear function of v and w.

The tangent space T_c at $[c]_x$ (or at $[c]_{x,y}$) splits as an orthogonal direct sum of the negative, the null and the positive space of the corresponding selfadjoint operator:

$$T_c = T_c^- \oplus T_c^0 \oplus T^+.$$

By an argument similar to that used in the construction of P_k' in the proof of Prop [...] we can choose a representative of c defined over a sufficiently fine subdivision of the unit interval such that each c_i is contained in an open set with the property that any of its two points can be connected by unique minimal geodesic which depends differentiably of the two points. Then we can define the finite dimensional space $T_c(t_0, t_1, \ldots, t_k) \subset T_c$ of broken Jacobi fields along c, i.e. consisting of vector fields $v = (v_1, v_2, \ldots, v_k)$ such that each v_i is a Jacobi field along c_i. As in the manifold case one can prove that the tangent space T_c splits as the direct sum

$$T_c = T_c(t_0, t_1, \ldots, t_k) \oplus T',$$

where T' denotes the vector space consisting of the vector fields along c which vanish simultaneously at the braking points $0 = t_0 < t_1 < \ldots < t_k = 1$. Moreover, the two subspaces are mutually perpendicular with respect to the inner product $D^2E(c)$ and the Hessian restricted to T' is positive definite (see Lemma 15.3 Milnor [...]). An immediate consequence of this fact is that the negative space T_c^- (as well as the null space T_c^0) is always a finite dimensional vector space. The dimension of the negative space is called the index of c. As in the manifold case one can prove that T_c^0 consists precisely of the Jacobi vector fields along c (this is an alternative way to see that the null space is always finite dimensional). In the case of $\Omega'_{x,y}$, its dimension is called the nullity of c and as we have seen it never exceeds $n-1$. In the case of Ω'_X, the nullity is defined to be $\dim T_c^0 - 1$ since in this case $\dot{c} = (\dot{c}_1, \dot{c}_2, \ldots, \dot{c}_k)$ belongs to T_c^0. We say that the Hessian of the energy function is *degenerate* if the nullity is positive.

5.2.2 Finite dimensional approximation

As above, for a real number $a \geq 0$, we denote by P^a (resp. $P^{<a}$) the subspaces of $P = \Omega'_X$ and $\Omega'_{x,y}$ where the energy function is $\leq a$ (resp. $< a$). In the first case we assume $|Q|$ to be compact, and complete in the second case. Similar to the manifold case we will investigate the topology of P^a by constructing a finite dimensional approximation to it.

Let $k > 0$ be an integer and $0 = t_0 < t_1 < t_2 < \ldots < t_k = 1$ be the subdivision of the unit interval with $t_i = i/2^k$. Consider the subspaces $P_{k,\epsilon}$ of P formed by elements represented by $\mathscr{G}$-paths of class H^1, $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ defined over the above subdivision and such that each $c_i(t_{i-1})$ is the center of a convex open geodesic ball of radius ϵ containing the image of c_i. Note that a similar subspace was used in the proof of Proposition 5.9. We have the following lemma concerning the space P_k.

Lemma 5.2.9. *There exists $\epsilon > 0$ and a sufficiently large integer k such that every element of P^a can be represented by an element of $P_{k,\epsilon}$.*

Proof. If the orbifold is compact then there exists $\epsilon > 0$ such that every point is the center of a convex geodesic ball of radius ϵ. In particular, if $z \in X$, the exponential map is defined on the ϵ-ball centered at the origin in T_zX and maps this ball diffeomorphically onto the ball $B(z, \epsilon)$ centered at z and of radius ϵ. Moreover, note that using the compactness of $|Q|$ there exists an integer $k_0 > 0$ such that every $\mathscr{G}$-path has a representative over the subdivision $0 = t_0 < t_1 < \ldots < t_{k_0} = 1$ with $t_i = i/2^{k_0}$.

Let now $[c] \in P^a$. From the inequality $L^2 \leq 2E$ we have that $L(c) \leq \sqrt{2a}$. Choose k such that $2^k > \min\{2^{k_0}, 2a/e^2\}$. If $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ is a representative of $[c]$ over the uniform subdivision with norm $1/2^k$ then for each i we have

$$\left(L_{|[t_{i-1},t_i]}(c_i)\right)^2 = 2(t_i - t_{i-1})E_{|[t_{i-1},t_i]}(c_i) \leq 2\frac{1}{2^k}E(c) \leq \frac{2a}{2^k} < \epsilon^2.$$

But $L(c_i) < \epsilon$ means that c_i is contained in the convex geodesic ball centered at $c_i(t_{i-1})$ and of radius ϵ, i.e. $c = (g_o, c_1, g_1, \ldots, c_k, g_k)$ is an element of $P_{k,\epsilon}$.

The above construction works also in the case when the orbifold is complete as follows. Let S denote the closed ball

$$\{\overline{z} \in |Q| \mid d(\overline{z}, \overline{x}) \leq \sqrt{2a}\},$$

where $\overline{x} = q(x)$ and $d(\cdot,\cdot)$ denotes the metric on the base space induced by the Riemannian metric on Q. Since $|Q|$ is complete, S is a compact set. Again the inequality $L^2 \leq 2E$ implies that the projection of any element of P^a lies entirely in this set. Then an ϵ and a k_0 as above can be defined for all the $\mathscr{G}$-paths with energy $\leq a$. □ □

Consider now the subspace $B_k^a \subset P^a$ consisting of broken $\mathscr{G}$-geodesics of energy smaller than a. This is a finite dimensional manifold. The tangent space $T_cB_k^a$ to B_k^a at a broken geodesic $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ can be identified with the space $T_c(t_0, t_1, \ldots, t_k)$ of broken Jacobi fields along c. The following result shows that the finite dimensional manifold B_k^a provides a faithful model for the infinite dimensional space P^a.

Proposition 5.2.10. *(Finite dimensional approximations)(Proposition 4.1.3. [GH])*

(i) The space B_k^a is a deformation retract of the space P^a. The restriction of the energy function to $P^{<a}$ and to $B_k^{<a}$ have the same critical points, and at such points the nullity and the index are the same.

(ii) The suborbifold $\Lambda' Q^a = \mathscr{G}\backslash\Omega'^a_X$ retracts by deformation onto the finite dimensional suborbifold $\Lambda Q^a(k) := \mathscr{G}\backslash\Omega'^a_X(k)$. The inclusion

$$B\Lambda Q^a(k) = E\mathscr{G} \times_{\mathscr{G}} \Omega'^a_X(k) \to B\Lambda' Q^a = E\mathscr{G} \times_{\mathscr{G}} \Omega'^a_{x,y}$$

is a homotopy equivalence.

Proof. (i)The proof of the first part follows the argument used in the proof of Prop. 5.9. For the second part note that every $\mathscr{G}$-geodesic is also a broken $\mathscr{G}$-geodesic. Thus every critical point of E on $P^{<a}$ will be in $B_k^{<a}$. Conversely, if c is a critical point for the energy function restricted to the space $B_k^{<a}$, then (see the proof of 5.4) it has to be an unbroken $\mathscr{G}$-geodesic. The fact that the index (resp. nullity) of $D^2E_{|P^{<a}}(c)$ is equal to the index (resp. nullity) of $D^2E_{|B_k^{<a}}(c)$ is a consequence of the fact that the index (resp. the nullity) localizes in the (finite dimensional) space $T_c(t_0, t_1, \ldots, t_k)$ of broken Jacobi fields along c.

(ii) The last claim follows from the observation that the deformation commutes with the action of $\mathscr{G}$. □

5.3 Existence of at least one closed geodesic with positive length

Theorem 5.3.1. *On every compact connected Riemannian orbifold which is bad there exists at least one closed geodesic with positive length.*

Proof. Let $Q = X/\mathscr{G}$ be a such orbifold. There is a point $x \in X$ and a nontrivial element g in its isotropy group $\mathscr{G}_x$ such that the closed loop based at x represented by the pair (c, g), where $c : [0.1] \to X$ is the trivial map to x, is homotopically trivial (***give a proof***). The homotopy gives us a continuous path in $|\Lambda' Q|$ joining the point $z \in |\Lambda^0 Q|$ represented by (c, g) to the point $z' \in |\Lambda^0 Q|$ represented by the constant loop. Those points are in different components of $|\Lambda^0 Q|$ by [..?] and these components are compact [..?]. By [Prop...?] the sets of points $|\Lambda'^{\epsilon} Q|$ of $|\Lambda' Q|$ for which the energy function is smaller than ϵ for various $\epsilon > 0$ form a fundamental system of neighbourhoods of $|\Lambda'^0 Q|$. Therefore the family of paths in $|\Lambda' Q|$ joining z to z' is a φ-family $\mod |\Lambda'^0 Q|$ and we can apply Theorem 5.8. ☐

Theorem 5.3.2. *Let $Q = M/\Gamma$ be a good compact connected Riemannian orbifold (i.e. M is a connected Riemannian manifold and Γ is a discrete group acting on M by isometries and such that the quotient space is compact). There is at least one closed geodesic on Q wit positive length in the following cases:*

(i) Γ is finite (i.e. the orbifold is very good) or it contains an element of infinite order;

(ii) M is simply connected and any of the higher homotopy groups $\pi_i(M)$, $i > 1$ is notrivial or M is a nontrivial $K(\pi, 1)$.

Proof. (i) If the group Γ is finite then M is compact and the classical result of Fet implies that there exists at least one closed geodesic on M with positive length. Its projection gives a closed geodesic on Q with positive length.

In the second case, the element of infinite order in Γ gives an element of infinite order in $\pi_1^{orb}(Q)$. The energy function restricted to the component of $|\Lambda' Q|$ corresponding to this element attains its minimum at some point which is critical for E

(by Corollary 5.11). This point is a closed geodesic. Since it represents an element of infinite order, it has positive length.

(ii) Suppose that there is $i > 1$ such that $\pi_i(M)$ is the first not trivial homotopy group. Then the standard loop fibration shows that $\pi_{i-1}(\mathscr{L}M)\approx\pi_i(M)$, where $\mathscr{L}M$ denotes the space of loops on M. By Hurewicz Theorem $H_j(\mathscr{L}M) \approx \pi_j(\mathscr{L}M)$ for the for the first nontrivial π_j. Hence $H_{i-1}(\mathscr{L}M) \approx \pi_i(M)$ is nontrivial. Applying the Fundamental Theorem of Morse Theory for the loop space and energy function, we obtain the existence of an index $i - 1 > 0$ critical point. This is a closed geodesic on M of positive length, since the index is positive. The projection of this geodesic will give us a geodesic of positive length on $Q = M/\Gamma$.

Consider now the case when all the higher homotopy groups $\pi_i(M)$, $i > 1$ are trivial. We will see that in this case $\pi_1(M)$ contains no element of finite order. Denote by $\widetilde{M}$ the universal cover of M. Then $\widetilde{M}$ is a complete manifold (see [..?] and $\pi_i(\widetilde{M}) = 0$ for all $1 \geq 0$. By a result of Witehead $\widetilde{M}$ is contractible and then the cohomology group $H^k(M)$ can be identified with the cohomology group $H^k(\pi_1(M))$ of the group $\pi_1(M)$. Assume now that $\pi_1(M)$ contains a nontrivial finite cyclic subgroup G. Then for a suitable covering space $\widehat{M}$ of M we have $\pi_1(\widehat{M}) = G$ and as above $H^k(G) = H^k(\widehat{M}) = 0$ for $k > n$. But since the cohomology groups of a finite cyclic group are nontrivial in arbitrarily high dimensions, this gives a contradiction. Hence, $\pi_1(M)$ contains no element of finite order. This implies that $\pi_1^{orb}(Q)$ contains an element of infinite order and again we can proceed as in the second part of (i). $\square$

Corollary 5.3.3. *On every compact connected 2-dimensional Riemannian orbifold there exists at least one closed geodesic with positive length.*

Remark 5.3.4. For the existence of a closed geodesic of positive length on compact orbifolds, the only case left open by the preceding theorem would be the following one. Let Γ be an infinite group all of whose elements are of finite order acting properly on a Riemannian manifold M by isometries with compact quotient $Q = \Gamma\backslash M$; does there exists a non constant geodesic $c : [0, 1] \to M$ and an element $\gamma \in \Gamma$ such that the differential of γ maps $\dot{c}(0)$ to $\dot{c}(1)$. Note that such a group Γ would be finitely presented, and no examples of such groups are known yet.

5.3.1 Developability of orbifolds with non positive curvature

Let $Q = X/\mathscr{G}$ be a connected Riemannian orbifold with non positive curvature. Consider $\widetilde{Q}$ its universal orbifold covering. Then $\widetilde{Q}$ is a complete Riemannian orbifold of non positive curvature. Let $(\widetilde{\mathscr{G}}, \widetilde{X})$ be the étale groupoid associated to the pseudogroup of change of charts of $\widetilde{Q}$. Note that this groupoid is Morita equivalent to the universal covering groupoid of $(\mathscr{G}, X)$. Since $\widetilde{Q}$ is simply connected, $\Omega'_{x,y}$ is connected for every $x, y \in \widetilde{X}$. Since $\widetilde{Q}$ has non positive curvature, every $\mathscr{G}$-geodesic from x to y has index 0 (there are no conjugate points along it). Thus, $\pi_i(\Omega'_{x,y}\widetilde{Q})$ are trivial for all $i \geq 1$. This implies that all the orbifold homotopy groups are trivial, i.e. $\widetilde{Q}$ is contractible. It follows that there is precisely one $\widetilde{\mathscr{G}}$-geodesic from x to y.

Let $T_x\widetilde{Q}$ denote the tangent cone to $\widetilde{Q}$ at x and let $(\widetilde{\mathscr{G}}_x \rtimes T_x\widetilde{X}, T_x\widetilde{X})$ denote the groupoid associated to the action of $\widetilde{\mathscr{G}}$ on $T_x\widetilde{X}$. Since up to equivalence of $\widetilde{\mathscr{G}}$-geodesics there is exactly one $\widetilde{\mathscr{G}}$-geodesic between any two points in the orbifold $\widetilde{Q}$, the exponential map $exp_x : T_x\widetilde{Q} \to \widetilde{Q}$ induces an equivalence (Morita) between the groupoids $(\widetilde{\mathscr{G}}_x \rtimes T_x\widetilde{X}, T_x\widetilde{X})$ and $(\widetilde{\mathscr{G}}, \widetilde{Q})$. Since $(\widetilde{\mathscr{G}}, \widetilde{X})$ is simply connected, the isotropy group $\mathscr{G}_x$ is trivial. Thus, $\widetilde{Q}$ is a manifold, i.e. Q is developable.

Note that in the above argument we used essentially the completeness of Q for the domain of the exponential map and also for the onto part and the fact that the orbifold has non positive curvature for the uniqueness of the geodesic connecting any two points (i.e. for the one-to-one part). The same argument works in that case when $\widetilde{Q}$ is a complete Riemannian orbifold which has a pole.

Bibliography

[B] A. F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, **91**, Springer-Verlag, New York, 1995.

[BGS] W. Ballmann, M. Gromov, V. Schröder, *Manifolds of nonpositive curvature*, Progress in Mathematics, **61**, Birkhäuser Boston, Inc., Boston, MA, 1985.

[BH] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der mathematischen Wissenschaften, **319**. Spinger-Verlag, Berlin, 1999.

[BL] J. E. Borzellino and B. G. Lorica, *The closed geodesic problem for compact Riemannian 2-orbifolds*, Pacific J. Math. **175** (1996), No. **1**, 39–46.

[CR] W. Chen and Y. Ruan, *Orbifold Gromov-Witten theory*, Orbifolds in mathematics and physics (Madison, WI, 2001), 25–85, Contemp. Math. Soc., **310**, AMS, Providence, RI, 2002.

[dC] M. P. do Carmo, *Riemannian geometry*, Mathematics: Theory and Aplications, Birkh auser Boston, Inc., Boston, MA, 1992.

[GH] K. Guruprasad and A. Haefliger, *Closed geodesics on orbifolds*, `math.DG/0306238`.

[H] A. Haefliger, *Groupoids d'holonomie et classifiants*, Structures transverses des feiulletages (Touluose 1982), Astérisque **116** (1984), 70–97.

[K] M. Kapovich, *Hyperbolic manifolds and discrete groups*, Progress in Mathematics, **183**, Birkhäuser Boston, Inc., Boston, MA, 2001.

[K1] W. Klingenberg, *Closed geodesics on Riemannian manifolds*, CBMS Regional Conference Series in Mathematics, No. **53**, AMS, Providence, RI, 1982.

[K2] W. Klingenberg, *Lectures on closed geodesics*, Grundlehren der mathematischen Wissenschaften, **230**. Spinger-Verlag, Berlin, 1978.

[Mi] J. Milnor, *Morse theory.* Annals of Mathematics Studies, No. **51**, Princeton University Press, Princeton, N.J. 1963.

[M] I. Moerdijk, *Orbifolds as groupoids: an introduction*, Orbifolds in mathematics and physics (Madison, WI, 2001), 205–222, Contemp. Math., **310**, AMS, Providence, RI, 2002.

[MM] I. Moerdijk and J. Mrčun, *Introduction to foliations and Lie groupoids.* Cambridge Studies in Advanced Mathematics, **91**. Cambridge University Press, Cambridge, 2003.

[Sa1] I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363.

[Sa2] I. Satake, *The Gauss-Bonnet theorem for V-manifolds*, J. Math. Soc. Japan **9** (1957), 464–492.

[Sc] P. Scott, *The geometries of* 3*-manifolds*, Bull. London Math. Soc. **15** (1993), No. **5**, 401–487.

[Se] G. Segal, *Classifying spaces and spectral sequences*, Publications mathématiques de l'I.H.É.S., **34**, (1968), 105–112.

[Ra] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts in Mathematics, **149**, Springer-Verlag, New York, 1994.

[T] W. P. Thurston, *The geometry and topology of 3-manifolds*, Chapter 13. Princeton University, 1978-1979.

[T1] W. P. Thurston, *Three-dimensional geometry and topology*, Volume 1, Princeton Mathematical Series, **35**, Princeton University Press, Princeton, NJ, 1997.