CLOSED GEODESICS ON ORBIFOLDS OF NONPOSITIVE OR NONNEGATIVE CURVATURE

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Abstract. In this note we prove existence of closed geodesics of positive length on compact developable orbifolds of nonpositive or nonnegative curvature. We also include a geometric proof of existence of closed geodesics whenever the orbifold fundamental group contains a hyperbolic element and therefore reduce the existence problem to developable orbifolds with π_1^{orb} infinite and having finite exponent and finitely many conjugacy classes.

1. Introduction

An orbifold is perhaps the simplest case of a singular space generalizing the notion of manifold. Orbifolds are topological spaces which are locally modelled on quotients of open subsets in Euclidean space by linear actions of finite groups. Each point of an orbifold carries additional data, that of a finite isotropy group, and the orbifold structure encodes this information.

Orbifolds arise naturally throughout geometry and one reason for the interest in orbifolds is that they exhibit geometric properties similar to manifolds. The specific problem that we take up in this paper has to do with the existence of closed geodesics on compact Riemannian orbifolds. Every compact manifold contains a closed geodesic [16], and a beautiful and elegant proof of this fact is presented in [18], where this is proved by applying Morse theory to the energy functional on the loop space. The corresponding problem for orbifolds was studied by Guruprasad and Haefliger in [14], where they adapted the Morse theoretic approach to the orbifold setting and proved existence of closed geodesics on compact orbifolds whenever (1) the orbifold is not developable, or (2) the orbifold fundamental group is finite or contains an element of infinite order. Despite this progress, the general problem of existence of closed geodesics on compact orbifolds remains open.

In this note we investigate this problem and use a geometric approach that recovers and extends part (b) of Theorem 5.1.1 in [14]. We focus exclusively on developable orbifolds and establish our results by using the geometry of the universal covering space and properties of the orbifold fundamental group. Our approach provides an elementary proof of the following theorem (see [2, Proposition 2.16]):

Theorem 3.1 *A compact developable orbifold $\mathcal{Q}$ admits a closed geodesic of positive length whenever the orbifold fundamental group is finite or contains a hyperbolic element.*

Because elements of infinite order in $\pi_1^{orb}(\mathcal{Q})$ are hyperbolic, Theorem 3.1 improves on the result [14, Theorem 5.1.1 (b)] for developable orbifolds, and consequently, allows us to reduce the existence problem to compact orbifolds $\mathcal{Q}$ with

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$\pi_1^{orb}(\mathcal{Q})$ infinite and containing only elliptic elements. As shown in Proposition 3.4, this latter condition forces the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q})$ to have finitely many conjugacy classes and finite exponent.

An interesting question related to all of this is whether an infinite torsion group as above can act geometrically on a simply connected complete Riemannian manifold M (Question 3.5). While we are not able to rule out such actions in the general case, we are able to show that such actions cannot occur if M is assumed to carry a metric of nonpositive or nonnegative sectional curvature (Proposition 4.1 and 4.3). As a consequence we obtain the following:

Corollary 4.4 *Any compact Riemannian orbifold having nonpositive or nonnegative curvature admits a closed geodesic of positive length.*

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2. The Setup

We begin by setting up the notations and briefly recalling some of the background needed throughout the subsequent sections. The reader is encouraged to look up the references for the basics on orbifolds. Besides the original work of Satake [21] and Thurston [24], other good introductions to the classical theory of orbifolds include [1, Chapter 1], [6, Chapter III.$\mathcal{G}$] and [15, Chapter 6].

Throughout this note, unless explicitly stated otherwise, by 'compact orbifold' we mean a compact connected effective orbifold without boundary. We also assume all Riemannian orbifolds to be smooth of class C^r with $r \geq 4$ and state our results for this category of orbifolds (see also Remark 3.2).

2.1. **Orbifolds.** Let Q denote a Hausdorff topological space and let $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ be an open cover of Q which is closed under finite intersections. Let n be a positive integer.

A Riemannian n-orbifold structure $\mathcal{Q}$ on the space Q is given by an atlas of uniformizing charts $\{(\widetilde{U}_i, \Gamma_i, \varphi_i)\}_{i\in\mathcal{I}}$, where each $\widetilde{U}_i$ is a Riemannian n-manifold without boundary, Γ_i is a finite group of isometries of $\widetilde{U}_i$, and $\varphi_i : \widetilde{U}_i \to U_i$ is a continuous Γ_i-invariant map that induces a homeomorphism from $\widetilde{U}_i/\Gamma_i$ onto the open set U_i. The change of charts are Riemannian isometries. The orbifold $\mathcal{Q}$ is said to be effective if the action of each Γ_i on $\widetilde{U}_i$ is effective. Note that if all the groups Γ_i are trivial, or if they act freely on the $\widetilde{U}_i$'s then $\mathcal{Q}$ is a Riemannian manifold.

The Riemannian orbifold $\mathcal{Q}$ is said to be *developable* if it arises as the global quotient M/Γ of a Riemannian manifold M by the proper action of a discrete subgroup Γ of its group of isometries. Let $\pi : M \to Q$ denote the natural projection map. For a point $x \in Q$ we define the isotropy group of $x = \pi(\tilde{x})$ to be the finite group $\Gamma_x = \{\gamma \in \Gamma \mid \gamma\tilde{x} = \tilde{x}\}$. This group is uniquely determined up to conjugacy in Γ. The point $x \in Q$ is said to be a singular point if its isotropy group Γ_x is nontrivial and x is said to be a regular point otherwise. The singular set Σ of the orbifold $\mathcal{Q}$ is the collection of all the singular points in Q. If $\Sigma = \varnothing$, the orbifold $\mathcal{Q}$ is in fact a manifold.

Orbifold covering spaces are defined similarly to the ones for topological spaces. Thurston showed that each orbifold $\mathcal{Q}$ has a universal covering [24, Proposition 13.2.4] and also defined the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q})$ as the group of deck transformations of its universal orbifold covering.

In the case of a developable orbifold $\mathcal{Q} = M/\Gamma$, the quotient $M \to M/\Gamma$ can be regarded as an orbifold covering with Γ as the group of deck transformations. Any subgroup Γ' of Γ induces an intermediate orbifold covering $M/\Gamma' \to M/\Gamma$. On the other hand, any manifold covering $\widetilde{M} \to M$ gives an orbifold covering by composing with the quotient map $M \to M/\Gamma$. In particular, the universal covering space of M gives rise to the universal orbifold covering space of $\mathcal{Q}$, and the orbifold fundamental group belongs in a short exact sequence

$$1 \to \pi_1(M) \to \pi_1^{orb}(\mathcal{Q}) \to \Gamma \to 1.$$

Note that an orbifold is developable if and only if its universal covering space is a manifold.

From here on, $\mathcal{Q}$ will denote an effective n-dimensional compact connected developable Riemannian orbifold without boundary. We choose to write $\mathcal{Q}$ as the orbifold quotient M/Γ, where M is the orbifold universal covering space of $\mathcal{Q}$ and $\Gamma = \pi_1^{orb}(\mathcal{Q})$ is the orbifold fundamental group. Thus M is a connected, simply connected complete Riemannian n-manifold (with the natural Riemannian structure pulled back from $\mathcal{Q}$) and Γ is a discrete subgroup of the group $\mathrm{Isom}(M)$ acting properly and cocompactly by isometries on M. In short, we say that Γ acts *geometrically* on M.

2.2. **Metric structure.** The Riemannian structure on M induces naturally a length metric d on M: the distance $d(\tilde{x}, \tilde{y})$ between two points $\tilde{x}$ and $\tilde{y}$ in M is defined to be the infimum of the Riemannian length of all the piecewise continuously differentiable paths connecting $\tilde{x}$ and $\tilde{y}$. The topology of the metric space (M, d) coincides with the topology of the manifold M. Since M is complete, by the Hopf-Rinow theorem [6, Proposition I.3.7], the length metric space (M, d) is a geodesic space, i.e. any two points in M can be connected by a (metric) geodesic.

Since a Riemannian isometry is also a metric isometry, Γ acts by isometries on (M, d) and the quotient space Q of this action carries the quotient pseudo-metric associated to the length metric d on M. Because Q is Hausdorff, this pseudo-metric is actually a metric and induces the given topology of Q.

It is important to note, however, that unless the action of Γ on M is free (i.e. the orbifold $\mathcal{Q}$ is a manifold), the natural projection $\pi : M \to Q$ is neither a covering map nor a local isometry. In general, the fundamental group of the underlying topological space $\pi_1(Q)$ is isomorphic to the factor group Γ/Γ_0 where Γ_0 is the normal subgroup generated by all the elements in Γ which have fixed points in M (see [3]). It is an easy exercise to see that the projection π is a local isometry when restricted to $M \smallsetminus \pi^{-1}(\Sigma) \to Q \smallsetminus \Sigma$.

2.3. **Isometries.** Let now γ be an isometry of (M, d). That is, γ is a distance preserving self-homeomorphism of (M, d). Recall that the *displacement function* of an isometry γ is the function $d_\gamma : M \to \mathbb{R}_+$ defined by $d_\gamma(x) = d(x, \gamma x)$. The *translation length* of γ is the number $|\gamma| = \inf\{d_\gamma(x) \mid x \in M\}$. The *minimal set* of γ, denoted $\mathrm{Min}(\gamma) = \{x \in M \mid d_\gamma(x) = |\gamma|\}$, is the set of points where the displacement function attains this infimum. An isometry γ is called *semi-simple* if $\mathrm{Min}(\gamma)$ is non-empty.

Using the displacement function one has the following classification of the isometries of a metric space [6, Definition II.6.3]. An isometry γ is called:

(i) *elliptic* if γ has a fixed point,
(ii) *hyperbolic* if its displacement d_γ attains a strictly positive minimum,
(iii) *parabolic* if d_γ does not attain a minimum, i.e. $\mathrm{Min}(\gamma)$ is empty.

Each isometry of the metric space (M, d) belongs to one of the above disjoint classes, and an isometry is semi-simple if and only if it is elliptic or hyperbolic.

Because any Riemannian isometry of M is also a metric isometry of the associated length space (M, d), the above classification applies to the Riemannian isometries of M. Furthermore, since in this case the length space (M, d) is a complete simply connected geodesic space and Γ acts geometrically on M, the elements of Γ are semi-simple isometries (cf. [6, Proposition II.6.10(2)]). Thus we distinguish two classes of elements in Γ: the *elliptic* elements, which are the isometries with nonempty fixed point set in M; and the *hyperbolic* elements which are the semi-simple isometries that act on M without fixed points.

Remark 2.1. Our general assumption that the Riemannian orbifolds are of class at least C^4 can be relaxed to class C^2 for the present exposition (see also Remark 3.2). As noted in [6, Remark I.3.22], if the Riemannian structure on M is of class C^2, then a metric isometry of the length space (M, d) associated to M is also a Riemannian isometry (cf. [20]).

2.4. **Geodesics.** Recall that a *Riemannian geodesic* on M is a continuously differentiable path $\tilde{c} : I \to M$ which is locally distance-minimizing in the following sense: there exists a constant $v \geq 0$ such that for any interior point t of I there exists a neighbourhood $J \subseteq I$ of t such that $d(\tilde{c}(s), \tilde{c}(s')) = v|s - s'|$ for all $s, s' \in J$. We say that the geodesic $\tilde{c} : I \to M$ is *normalized* (or has *unit speed*) if $v = 1$, and that it is *minimal* if $d(\tilde{c}(s), \tilde{c}(s')) = |s - s'|$ holds for all $s, s' \in I$. In particular, the length of a minimal geodesic segment is equal to the distance between its endpoints.

It is important to notice that Riemannian geodesics need not be geodesics in the metric sense; in general they are only local geodesics and, a Riemannian geodesic is a metric geodesic if and only if it is minimal. Note also that given two points on a connected manifold M, it may be possible that there is no minimal geodesic connecting them or even no geodesic at all containing them. For example, in the Euclidean space with the origin removed there are no geodesics containing antipodal points. However, as we have mentioned before, if the manifold M is complete, by the Hopf-Rinow theorem, any two points in M can be connected through a minimal geodesic.

For the Riemannian orbifold $\mathcal{Q} = M/\Gamma$, given two points x and y in Q it is natural to ask for a geodesic connecting them to be a path $c : [0, 1] \to Q$ with $c(0) = x$ and $c(1) = y$ that lifts to a Riemannian geodesic in the universal cover M. However, in general such lift is not unique and an *orbifold geodesic* connecting x to y encodes the choice of a lift as we will now see.

We first recall the definition of an orbifold path in $\mathcal{Q} = M/\Gamma$. This is just a special case of the definition of a $\mathcal{G}$-path in the étale groupoid $\mathcal{G}$ of germs of change of charts of an orbifold. When $\mathcal{Q} = M/\Gamma$ is a developable orbifold, $\mathcal{G} = (\Gamma \ltimes M, M)$ is the étale groupoid associated to the action of Γ on M (see [6, Example III.$\mathcal{G}$.3.9(1)]).

Let $c : [0,1] \to Q$ be a continuous path in Q with $c(0) = x$ and $c(1) = y$. Let $\tilde{x} \in \pi^{-1}(x)$ and $\tilde{y} \in \pi^{-1}(y)$ and consider the pair $(\tilde{c}, \gamma)$, where

(*i*) $\tilde{c} : [0,1] \to M$ is a continuous path such that $\tilde{c}(0) = \tilde{x}$ and $\pi \circ \tilde{c} = c$;

(*ii*) $\gamma \in \Gamma$ is such that $\gamma\tilde{c}(1) = \tilde{y}$.

Let now $\tilde{x}' \in \pi^{-1}(x)$ and $\tilde{y}' \in \pi^{-1}(y)$ be a different choice of points in the orbit above x and y, and let $(\tilde{c}', \gamma')$ be such that $\tilde{c}' : [0,1] \to M$ is continuous, $\tilde{c}'(0) = \tilde{x}'$, $\pi \circ \tilde{c}' = c$ and $\gamma'\tilde{c}'(1) = \tilde{y}'$. We say that the pairs $(\tilde{c}, \gamma)$ and $(\tilde{c}', \gamma')$ are *equivalent* if there exists an element δ in Γ such that $\tilde{c}' = \delta.\tilde{c}$ and $\gamma' = \delta\gamma\delta^{-1}$. An orbifold path with underlying continuous path $c : [0,1] \to Q$ is an equivalence class of pairs $(\tilde{c}, \gamma)$ as above. Such an orbifold path is said to be smooth if $\tilde{c}$ is a smooth path in M.

Given two points x and y in Q, a Riemannian orbifold geodesic joining x to y is an equivalence class of a pair $(\tilde{c}, \gamma)$ where $\tilde{c} : [0,1] \to M$ is a Riemannian geodesic in M such that $\pi(\tilde{c}(0)) = x$ and $\pi(\tilde{c}(1)) = y$. Note that an orbifold geodesic represented by $(\tilde{c}, \gamma)$ is uniquely determined by the initial velocity vector $\dot{\tilde{c}}(0) \in T_{\tilde{c}(0)}M$ and the conjugacy class of γ in Γ. Note also that the underlying continuous path of an orbifold geodesic need not be a metric geodesic on Q with the quotient metric. Once a geodesic passes through a point with larger isotropy group, it stops being minimizing.

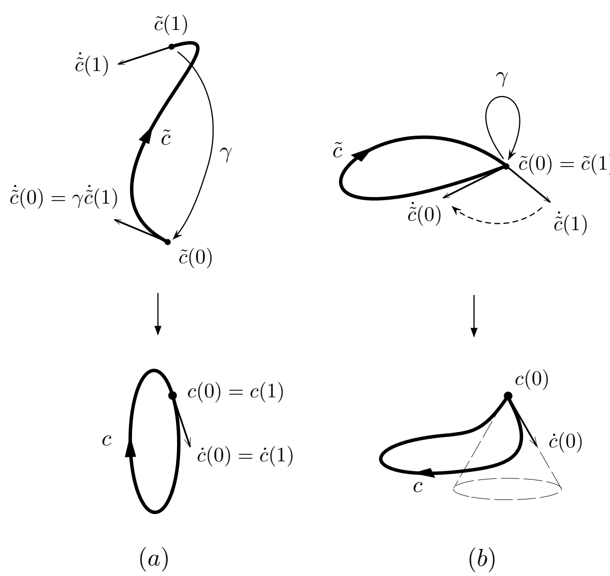

FIGURE 1. Examples of closed geodesics in $\mathcal{Q} = M/\Gamma$.

The closed geodesics of positive length on the developable orbifold $\mathcal{Q}$ are in one-to-one correspondence with the equivalence classes of pairs $(\tilde{c}, \gamma)$, where $\tilde{c} : [0,1] \to M$ is a non-constant geodesic segment in M and $\gamma \in \Gamma$ is an isometry such that:

$$\gamma\tilde{c}(1) = \tilde{c}(0) \text{ and } \gamma\dot{\tilde{c}}(1) = \dot{\tilde{c}}(0)$$

(see Figure 1). Two pairs $(\tilde{c}, \gamma)$ and $(\tilde{c}', \gamma')$ are equivalent if and only if there is an isometry $\delta \in \Gamma$ such that $\tilde{c}' = \delta.\tilde{c}$ and $\gamma' = \delta\gamma\delta^{-1}$.

3. A First Result

We now present an elementary proof of an existence result for closed geodesics on developable orbifolds that implies part (b) of the Theorem 5.1.1 of Guruprasad and Haefliger in [14]. As before, $\mathcal{Q}$ denotes a compact connected developable Riemannian orbifold of class C^4 and without boundary. Suppose $\mathcal{Q} = M/\Gamma$ with M simply connected and $\Gamma = \pi_1^{orb}(\mathcal{Q})$.

We begin by noticing that it follows easily from the definition of closed geodesics on developable orbifolds that any nontrivial closed geodesic $\tilde{c}$ in the universal cover M gives rise to a closed geodesic of positive length on $\mathcal{Q}$ represented by the pair $(\tilde{c}, 1)$, where 1 denotes the identity of Γ. The result of Lyusternik and Fet [17] on the existence of closed geodesics on compact manifolds can be used to show the existence of a closed geodesic of positive length in the case when the universal cover M of $\mathcal{Q}$ is compact. This is precisely the case when the orbifold fundamental group Γ is finite.

We will next show that the existence of closed geodesics of positive length in $\mathcal{Q}$ follows whenever the orbifold fundamental group Γ contains an element with acts without fixed points on M. The idea of the proof follows that of [4, Lemma 6.5].

Let $\gamma \in \Gamma$ be a hyperbolic isometry of M. By definition the minimal set $\mathrm{Min}(\gamma)$ is non-empty. Let $x \in \mathrm{Min}(\gamma)$. Since $\mathrm{Min}(\gamma)$ is γ-invariant, the translate γx of x belongs to $\mathrm{Min}(\gamma)$. Let now $\tilde{c} : [0,1] \to M$ be a minimizing geodesic in M connecting x to γx (such geodesic exists since M is complete) and let $y = \tilde{c}(1/2)$ be the midpoint of $\tilde{c}$. Then the translate $\gamma\tilde{c}$ is a minimizing geodesic connecting γx to $\gamma^2 x$, and γy is the midpoint of $\gamma\tilde{c}$ (see Figure 2 below). From the triangle

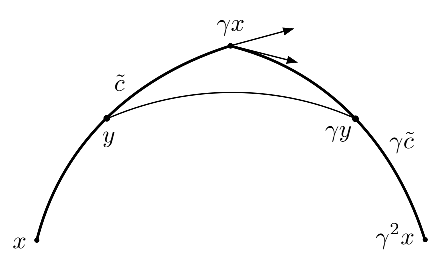

Figure 2. Midpoint argument.

inequality we have that

$$0 < d(y, \gamma y) \leq d(y, \gamma x) + d(\gamma x, \gamma y) = d(x, \gamma x) = |\gamma|,$$

which implies that $y \in \mathrm{Min}(\gamma)$ or equivalently that $d(y, \gamma y) = |\gamma|$. Since the distance between y and γy measured along $\tilde{c}$ and then $\gamma\tilde{c}$ is also $|\gamma|$, it follows that the concatenation of the two geodesics is a smooth geodesic, i.e. $\gamma.\dot{\tilde{c}}(0) = \dot{\tilde{c}}(1)$. Thus the pair $(\tilde{c}, \gamma^{-1})$ represents a closed geodesic of positive length in $\mathcal{Q}$.

We have thus proved the following ([2, Proposition 2.16], [11, Theorem 4.3]):

Theorem 3.1. *A developable compact Riemannian smooth orbifold $\mathcal{Q}$ of class at least C^4 has a closed geodesic of positive length if the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q})$ is finite or if it contains a hyperbolic element.*

We would like to mention that Theorem 3.1 is more general than [14, Theorem 5.1.1(b)] since semi-simple isometries of infinite order are hyperbolic, but the converse is not necessarily true. Therefore, by Theorem 3.1 the existence of closed geodesics of positive length follows whenever Γ has an element that acts without fixed point, which can be of finite order.

Remark 3.2. The differentiability condition of class at least C^4 in Theorem 3.1 is required by the variational argument used in [17]. When $\pi_1^{orb}(\mathcal{Q})$ contains a hyperbolic element, this condition can be relaxed to class C^2. In this case, the existence of a closed geodesic on $\mathcal{Q}$ is obtained without employing the result of Lyusternik and Fet [17] when the orbifold fundamental group is finite.

As a simple example, consider the compact orbifold $\mathcal{Q}$ obtained as the quotient of the round 2-sphere $\mathbb{S}^2$ by the action of the group generated by two elements $\langle\gamma, \delta\rangle$, where γ is the reflection in the horizontal equatorial plane and δ is the rotation of angle π around the vertical axis through the north pole N and the south pole S. Clearly $\gamma^2 = \delta^2 = (\gamma\delta)^2 = 1$ and $\pi_1^{orb}(\mathcal{Q}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ is the Klein four-group. The elements γ and δ are elliptic isometries with $\mathrm{Fix}(\gamma) = \{N, S\}$ and $\mathrm{Fix}(\delta) \simeq \mathbb{S}^1$, the equator. The element $\gamma\delta$ is the antipodal map on $\mathbb{S}^2$ and, since it acts without fixed points, $\gamma\delta$ is a hyperbolic isometry. It has translation length $|\gamma\delta| = \pi$ and minimal set $\mathrm{Min}(\gamma\delta) = \mathbb{S}^2$. The 'midpoint argument' applied to the element $\gamma\delta$ shows that for any point $x \in \mathbb{S}^2$ and any geodesic arc $\tilde{c} : [0,1] \to \mathbb{S}^2$ connecting x to its antipodal point $\gamma\delta(x) = -x$, the pair $(\tilde{c}, \gamma\delta)$ represents a closed geodesic of length π on the orbifold $\mathcal{Q}$.

Remark 3.3. It is easy to see that the translate $\gamma\tilde{c}$ of a geodesic $\tilde{c} : [0,1] \to M$ by an isometry $\gamma \in \Gamma$, is again a geodesic. Given a pair $(\tilde{c}, \gamma)$ representing a

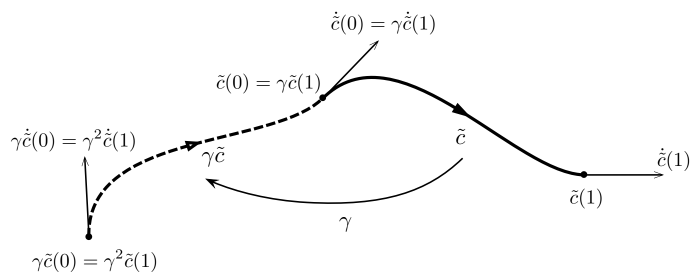

FIGURE 3. Collinear geodesics in M.

closed geodesic in $\mathcal{Q}$, the condition that $\gamma\tilde{c}(1) = \tilde{c}(0)$ implies that in M the two geodesic segments $\tilde{c}$ and $\gamma\tilde{c}$ have the point $\tilde{c}(0)$ in common; and the condition that $\gamma\dot{\tilde{c}}(1) = \dot{\tilde{c}}(0)$ implies that the union of the two geodesic segments is smooth at this point (Figure 3). Therefore, a closed geodesic of positive length exists on $\mathcal{Q} = M/\Gamma$ whenever there is a point $x \in M$ and an isometry $\gamma \in \Gamma$ that does not fix x, and the points $x, \gamma x,$ and $\gamma^2 x$ lie on a smooth geodesic in M.

Moreover, if $(\tilde{c}, \gamma)$ represents a closed geodesic with γ of finite order, say $\gamma^k = 1$, then the path $\tilde{c}^{(k-1)} := \tilde{c} * \gamma\tilde{c} * \ldots * \gamma^{k-1}\tilde{c}$, obtained by successively concatenating the translates of $\tilde{c}$ by γ, is a smooth closed geodesic in M (see Figure 4 below).

The only situation not covered by Theorem 3.1 is when Γ is infinite and each $\gamma \in \Gamma$ is elliptic. Since elliptic isometries have finite order, Γ is an *infinite torsion*

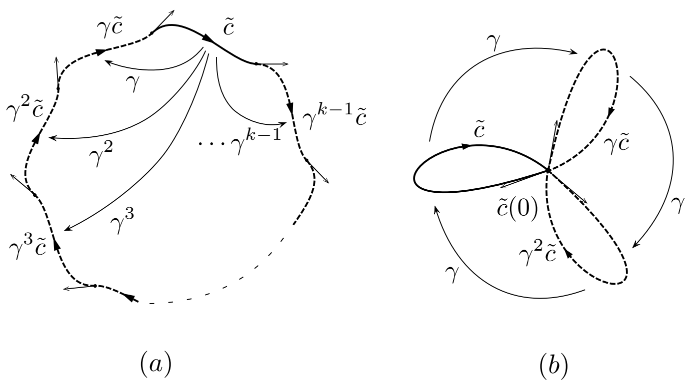

FIGURE 4. Closed geodesic in the universal cover M.

group. Moreover, since the action is proper and cocompact, Γ is finitely presented (cf. [6, Corollary. I.8.11]) and also has finitely many conjugacy classes of isotropy groups (cf. [6, Proposition I.8.5]). Another important property of groups which act exclusively by elliptic isometries is the following:

Proposition 3.4. *Suppose that the group Γ acts properly and cocompactly by elliptic isometries on a simply connected metric space. Then Γ has finite exponent.*

This follows from the fact that each element in Γ belongs to one of the isotropy groups, and these groups are finite. Then the least common multiple of the orders of the isotropy groups gives an upper bound for the exponent of Γ. Note that one cannot deduce that Γ has finite exponent from the results of [14].

While examples of infinite torsion groups that are finitely generated and even of finite exponent are known to exist, there are no examples known to be finitely presentable (as also noted in [14, Remark 5.1.2]). The existence problem for closed geodesics of positive length on compact orbifolds is therefore intimately related (but not equivalent) to the following question:

Question 3.5. *Can an infinite torsion group Γ act properly and cocompactly by elliptic isometries on a complete simply connected Riemannian manifold M?*

Clearly, a negative answer to this question would imply the existence of closed geodesics on all compact orbifolds. On the other hand, if such actions were to exist, then by Remark 3.3, the existence of a closed geodesic on M/Γ would be equivalent to the existence of a closed smooth geodesic in M. There are many examples of complete simply connected non-compact manifolds that are known to have no closed geodesics (e.g. simply connected manifolds of nonpositive curvature, or simply connected Riemannian manifolds without conjugate points). An interesting problem is then, whether any of these spaces can admit geometric actions as in Question 3.5, for an affirmative answer would give rise to a compact orbifold with no closed geodesics of positive length. We will tackle with this problem in next section.

4. Geometric Conditions

In this section we continue to denote by $\mathcal{Q}$ a compact connected Riemannian developable n-orbifold without boundary, obtained as the quotient M/Γ of a simply connected Riemannian manifold M by the geometric action of a discrete group $\Gamma \subset \mathrm{Isom}(M)$. We also continue to retain the assumption that the Riemannian structure on $\mathcal{Q}$ (resp. M) is of class C^4.

As noted at the beginning of the previous section, any closed geodesic of positive length in M projects to a closed geodesic of positive length in the quotient $\mathcal{Q}$. On the other hand, as we have seen in Remark 3.3, any closed geodesic of positive length $(\tilde{c}, \gamma)$ in $\mathcal{Q} = M/\Gamma$ for which $\gamma \in \Gamma$ has finite order gives rise to a closed geodesic in M.

A particularly interesting situation is when the orbifold fundamental group Γ is infinite torsion and the universal cover M is a manifold without closed geodesics. For such manifolds M, the existence of closed geodesics on any compact orbifold quotient of M would follow if one could show that infinite torsion subgroups of $\mathrm{Isom}(M)$ cannot act properly and cocompactly on M (see Question 3.5). Clearly, if a discrete infinite torsion group Γ acts on such a manifold geometrically, then the orbifold quotient $\mathcal{Q} = M/\Gamma$ does not contain a closed geodesic of positive length.

If one believes that compact orbifolds are similar to manifolds, then the existence of closed geodesics would suggest that infinite torsion groups do not act properly and cocompactly by isometries on simply connected manifolds without closed geodesics in all the dimensions. The purpose of this section is to study certain similar situations and to show that such actions cannot exist under the assumption of certain curvature conditions.

There are many examples of complete, connected, simply connected Riemannian non-compact manifolds that do not admit closed geodesics and, in general, there are no topological restrictions (like on homotopy or homology groups) that are independent of the dimension of the manifold and that can be forced upon a complete manifold to obtain the existence of closed geodesics with respect to all Riemannian metrics ([23]).

For instance, given any (non-compact) complete Riemannian manifold N, the product $M = \mathbb{R} \times N$ with the (complete) metric

$$\langle X, Y \rangle = xy + e^r \langle X^*, Y^* \rangle^*,$$

where $X = (x, X^*)$ and $Y = (y, Y^*)$ are in $T_{(r,p)}(\mathbb{R} \times N)$, and $\langle \cdot, \cdot \rangle^*$ denotes the metric on N, has no closed geodesics of positive length. This holds regardless of whether the factor N has closed geodesics or not.

However, not all simply connected manifolds can be realized as the universal covering space of a compact orbifold. In other words not all simply connected complete Riemannian manifolds admit geometric actions by discrete (infinite) groups of isometries. In particular, product manifolds as above cannot be the universal covering space of a compact orbifold. To see this, assume $\Gamma \subset \mathrm{Isom}(M)$ acts geometrically on $M = N \times \mathbb{R}$ and let $K \subset M$ be a fundamental domain for the action. Since Γ acts cocompactly, the set K is compact and of course $M = \Gamma.K$. Then any point $x \in M$ has a neighbourhood $U_x \in M$ which is isometric (via an element $\gamma \in \Gamma$) to a neighbourhood of the point $\gamma^{-1}x \in K$. If M has the above metric, we can see that two points (r_1, p) and (r_2, q) cannot be in the same orbit of an isometry unless $r_1 = r_2$.

This shows in particular that the universal covering space of a compact orbifold has in some sense bounded and uniform geometry.

4.1. **Curvature constraints.** Given a Riemannian orbifold $\mathcal{Q}$ and a point $x \in Q$, we define the sectional curvature κ_x at x to be the sectional curvature $\kappa_{\tilde{x}}$ at one of its lifts $\tilde{x}$ in a orbifold chart at x. Furthermore, we say that the orbifold $\mathcal{Q}$ is of positive (resp. nonnegative, zero, negative, nonpositive) sectional curvature if the sectional curvature κ_x at any point $x \in Q$ has the appropriate sign.

Proposition 4.1. *Suppose $\mathcal{Q}$ is a compact Riemannian n-orbifold.*

(a) *If $\mathcal{Q}$ is developable and the sectional curvature is everywhere positive, then the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q})$ is finite.*

(b) *If $\mathcal{Q}$ has negative sectional curvature, then it is developable and the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q})$ cannot be an infinite torsion group.*

Proof. (a) In the case when $\mathcal{Q}$ has positive sectional curvature, its universal cover M is a complete connected and simply connected manifold of positive sectional curvature. The classical Bonnet–Meyer theorem states that if the curvature of M is bounded from below by a positive constant $\varepsilon > 0$, then M is compact (see [19] for the stronger form involving the Ricci curvature). In this case the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q})$ is necessarily finite.

If the curvature is not bounded away from zero, M may not be compact. However, in this case, by a well known theorem of Gromoll and Meyer, the full group $\mathrm{Isom}(M)$ of isometries of M is compact (see [13, Theorem 3]). Any discrete group that acts by isometries on M is finite and the action is necessarily non-free. The latter claim follows from the fact that any complete open manifold of positive curvature is contractible (see [13, Theorem 2]). Thus any quotient by a group of isometries of a complete non-compact manifold of positive curvature is a non-compact orbifold (which is not a manifold, i.e. has nonempty singular locus).

(b) Orbifolds with negative sectional curvature are examples of orbifolds of nonpositive curvature and are therefore developable [6, Corollary III.$\mathcal{G}$.2.16]. The universal covering of an orbifold of negative curvature is diffeomorphic to $\mathbb{R}^n$, where n is the dimension of the orbifold. Thus, for a compact orbifold of negative curvature $\mathcal{Q}$ the fundamental group $\pi_1^{orb}(\mathcal{Q})$ is always infinite, and it admits an automatic structure (cf. [12, Theorem 3.4.1]). Since infinite torsion groups cannot be automatic (see [12, Example 2.5.12]), they cannot be realized as the fundamental group of a compact orbifold of negative curvature. □

Corollary 4.2. *If $\mathcal{Q}$ is a compact Riemannian orbifold with Riemannian metric of positive or negative sectional curvature, then $\mathcal{Q}$ admits a closed geodesic of positive length.*

Proof. If $\mathcal{Q}$ has positive sectional curvature, then either it is not developable or if developable, by (a) in Proposition 4.1 the fundamental group $\pi_1^{orb}(\mathcal{Q})$ is finite. In either case such an orbifold admits closed geodesics of positive length. This follows in the developable case from Theorem 3.1 and in the non-developable one by using part (a) of [14, Theorem 5.1.1].

If $\mathcal{Q}$ has negative sectional curvature, then by part (b) of Proposition 4.1 the fundamental group $\pi_1^{orb}(\mathcal{Q})$ is infinite and cannot be torsion. Thus $\pi_1^{orb}(\mathcal{Q})$ contains an element of infinite order, thus hyperbolic, and by Theorem 3.1 the orbifold $\mathcal{Q}$ has a closed geodesic of positive length. □

Proposition 4.3. *Suppose $\mathcal{Q}$ is a compact Riemannian n-orbifold.*

(*a*) *If $\mathcal{Q}$ has nonpositive sectional curvature, then it is developable and the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q})$ cannot be an infinite torsion group.*
(*b*) *If $\mathcal{Q}$ is developable and has nonnegative sectional curvature, then the orbifold fundamental group $\pi_1^{orb}(\mathcal{Q})$ cannot be an infinite torsion group.*

Proof. (a) It is well known that all orbifolds of nonpositive curvature are developable [6, Corollary III.$\mathcal{G}$.2.16]. If $\mathcal{Q}$ is such an orbifold, then its universal cover M is a simply connected complete manifold of nonpositive curvature. Equivalently, the space M with the metric induced by the Riemannian structure is a Hadamard space (i.e. a connected, simply connected, complete $CAT(0)$ space). By a result of Swenson in [22], any discrete group acting geometrically on a Hadamard space contains an element of infinite order, and therefore $\pi_1^{orb}(\mathcal{Q})$ cannot be infinite torsion.
(b) Assume now that $\mathcal{Q}$ is a compact developable orbifold of nonnegative sectional curvature. As before, its universal cover M is a complete simply connected manifold of nonnegative curvature. Unlike the positive curvature case, if M is non-compact, then its isometry group $\mathrm{Isom}(M)$ needs not be compact and there are examples of compact manifolds of nonnegative curvature whose universal covering space is not compact. Thus, if $\mathcal{Q}$ is a compact developable orbifold of nonnegative curvature, then it is possible for its universal covering space to be non-compact.

A key result concerning the manifolds of nonnegative curvature is the Toponogov Splitting Theorem [25], that states that any complete manifold M of nonnegative sectional curvature may be written uniquely as the isometric product $\overline{M} \times \mathbb{R}^k$, where $\mathbb{R}^k$ has the standard flat metric and $\overline{M}$ has nonnegative sectional curvature and contains no line (that is, a normal geodesic $\tilde{c} : (-\infty, \infty) \to M$, any segment of which is a minimal geodesic). Furthermore, Cheeger and Gromoll showed in [9] that if the isometry group of a manifold M of nonnegative sectional curvature is not compact, then M contains at least a line. In consequence, any complete manifold M of nonnegative curvature admits a unique isometric splitting $M = \overline{M} \times \mathbb{R}^k$ such that the isometry group of $\overline{M}$ is compact, and $\mathrm{Isom}(M) = \mathrm{Isom}(\overline{M}) \times \mathrm{Isom}(\mathbb{R}^k)$. Note that cf. [8] the same results hold in the more general case of manifolds with nonnegative Ricci curvature.

If Γ is a discrete group acting geometrically on M, then $\Gamma = \Gamma' \times \Gamma''$, where Γ' and Γ'' are discrete subgroups of $\mathrm{Isom}(\overline{M})$ and $\mathrm{Isom}(\mathbb{R}^k)$, respectively. Since $\mathrm{Isom}(\overline{M})$ is compact, the factor Γ' is necessarily finite. Note that if $\overline{M}$ is not compact then Γ cannot act cocompactly on M. On the other hand, the group Γ'' acts geometrically on the Euclidean factor $\mathbb{R}^n$ and, as before, it follows from Bieberbach's theorem that Γ'' contains elements of infinite order. This implies that Γ contains elements of infinite order, and so it cannot be torsion. □

Corollary 4.4. *If $\mathcal{Q}$ is a compact orbifold with Riemannian metric of nonpositive or nonnegative sectional curvature, then $\mathcal{Q}$ admits a closed geodesic of positive length.*

Proof. If $\mathcal{Q}$ is a compact orbifold of nonpositive curvature, then by part (a) in Proposition 4.3 the orbifold $\mathcal{Q}$ is developable and its fundamental group contains elements of infinite order, hence hyperbolic. The existence of a closed geodesic on $\mathcal{Q}$ follows from Theorem 3.1.

If $\mathcal{Q}$ is a compact orbifold of nonnegative sectional curvature, then either $\mathcal{Q}$ is not developable and the existence of a closed geodesic on $\mathcal{Q}$ follows from part

(a) of [14, Theorem 5.1.1]; or if $\mathcal{Q}$ is developable, by (b) in Proposition 4.3 the fundamental group $\pi_1^{orb}(\mathcal{Q})$ is either finite or contains elements of infinite order, and the existence of a closed geodesic on $\mathcal{Q}$ is given by Theorem 3.1. □

Note that if $\mathcal{Q}$ has zero curvature everywhere, the conclusion of Proposition 4.3 follows from a celebrated theorem of Bieberbach [5] (see also [7]). In this case Γ is a discrete group of isometries of the Euclidean space $\mathbb{E}^n$ and has compact fundamental domain. By Bieberbach's theorem, Γ is then virtually abelian, and since finitely generated abelian torsion groups are finite, Γ cannot be infinite torsion.

In the light of Theorem 3.1, in this note we obtained existence of closed geodesics on compact developable orbifolds $\mathcal{Q} = M/\Gamma$ by showing that Γ cannot be infinite torsion when the universal cover M satisfies certain curvature conditions. We point out that, via the uniformization theorem, these results help to establish the existence of closed geodesics for all compact developable orbifolds in dimension two. If $\mathcal{Q}$ is a 2-dimensional compact developable orbifold without boundary, then it has an elliptic, parabolic or hyperbolic structure (cf. [24, Theorem 13.3.6]). That is, $\mathcal{Q}$ has either positive, zero or negative curvature and, by Corollary 4.2 and 4.4, it admits a closed geodesic of positive length.

In [10] we take the opposite approach and, by showing that any compact orbifold of dimension 3, 5 or 7 admits a closed geodesic of positive length, we conclude that infinite torsion groups cannot act geometrically on simply connected complete Riemannian manifolds without closed geodesics and having dimension 3, 5 or 7 (see also [11, Corollary 4.10]).

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