THE QUASI-HYPERBOLICITY CONSTANT OF A METRIC SPACE

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ABSTRACT. We introduce the quasi-hyperbolicity constant of a metric space, a rough isometry invariant that measures how a metric space deviates from being Gromov hyperbolic. This number, for unbounded spaces, lies in the closed interval $[1, 2]$. The quasi-hyperbolicity constant of an unbounded Gromov hyperbolic space is equal to one. For a CAT(0)-space, it is bounded from above by $\sqrt{2}$. The quasi-hyperbolicity constant of a Banach space that is at least two dimensional is bounded from below by $\sqrt{2}$, and for a non-trivial $L_p$-space it is exactly $\max\{2^{1/p}, 2^{1-1/p}\}$. If $0 < \alpha < 1$ then the quasi-hyperbolicity constant of the $\alpha$-snowflake of any metric space is bounded from above by $2^{\alpha}$. We give an exact calculation in the case of the $\alpha$-snowflake of the Euclidean real line.

1. Introduction

Gromov hyperbolic spaces were introduced by Gromov in his seminal paper [Gro87] to study infinite groups as geometric objects. For a metric space $(X, d)$, we use the abbreviated notation $xy = d(x, y)$ where convenient. Recall that for three points $x, y, w$ in a metric space $(X, d)$, the Gromov product of $x$ and $y$ with respect to $w$ is defined as

$$(x \mid y)_w = \frac{1}{2} (xw + yw - xy).$$

Given a non-negative constant $\delta$, the metric space $(X, d)$ is said to be $\delta$-hyperbolic if

$$(x \mid y)_w \geq \min\{(x \mid z)_w, (y \mid z)_w\} - \delta$$

for all $x, y, z, w \in X$. A metric space $(X, d)$ is said to be Gromov hyperbolic if it is $\delta$-hyperbolic for some $\delta$. Any $\mathbb{R}$-tree is 0-hyperbolic. Another well-known example is the hyperbolic plane, which is $\log(2)$-hyperbolic, [NŠ16, Corollary 5.4]. Euclidean spaces of dimension greater than one are not Gromov hyperbolic. While Gromov hyperbolicity is a quasi-isometry invariant for intrinsic metric spaces [Väi05, Theorems 3.18 and 3.20], quasi-isometry invariance can fail for non-intrinsic spaces, see [Väi05, Remark 3.19] and also our examples in §3. In particular, a metric space that quasi-isometrically embeds into a Gromov hyperbolic space need not be Gromov hyperbolic.

A metric space $(X, d)$ is $\delta$-hyperbolic if and only if the four-point inequality holds, that is, for all $x, y, z, w \in X$,

$$xy + zw \leq \max\{xz + yw, yz + xw\} + 2\delta,$$

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We generalize the four-point inequality as follows. Let \((X, d)\) be a metric space. Let \(\mu, \delta \geq 0\). We say that a metric space \((X, d)\) satisfies the \((\mu, \delta)\)-four-point inequality if for all \(x, y, z, w \in X\),

\[
xy + zw \leq \mu \max\{xz + yw, xw + yz\} + 2\delta.
\]

In particular, \((X, d)\) is \(\delta\)-hyperbolic if and only if it satisfies the \((1, \delta)\)-four-point inequality.

We introduce the following numerical constants associated to a metric space.

**Definition (Quasi-hyperbolicity constants).** Let \((X, d)\) be a metric space.

(i) The **quasi-hyperbolicity constant** of \((X, d)\) is the number

\[
C(X, d) = \inf\{\mu \mid \text{there exists } \delta \geq 0 \text{ such that } (X, d) \text{ satisfies the } (\mu, \delta)\text{-four-point inequality}\}.
\]

(ii) The **restricted quasi-hyperbolicity constant** of \((X, d)\) is the number

\[
C_0(X, d) = \inf\{\mu \mid (X, d) \text{ satisfies the } (\mu, 0)\text{-four-point inequality}\}.
\]

Some basic properties of the quasi-hyperbolicity and restricted quasi-hyperbolicity constants of a metric space \((X, d)\) are readily derived, for example:

- \(C(X, d) \leq C_0(X, d) \leq 2\),
- if \((X, d)\) is bounded then \(C(X, d) = 0\), otherwise \(C(X, d) \geq 1\),
- if \((X, d)\) has at least two points then it is \(0\)-hyperbolic if and only if \(C_0(X, d) = 1\),
- if \((X, d)\) is Gromov hyperbolic and unbounded then \(C(X, d) = 1\).

Proofs of these and more properties are given in §2. In the absence of additional hypotheses, it is not true that \(C(X, d) = 1\) implies \((X, d)\) is Gromov hyperbolic. For example, given \(0 < \alpha < 1\), consider the graph, \(Y_\alpha\), of \(y = x^\alpha, \ x \geq 0\), as a subspace of the Euclidean plane, \((\mathbb{R}^2, d_E)\). We show \(C(Y_\alpha, d_E) = 1\), Proposition 3.4, however \(Y_\alpha\) is not Gromov hyperbolic if and only if \(1/2 < \alpha < 1\), Propositions 3.1 and 3.3. Nevertheless, if \((X, d)\) is a proper \(\text{CAT}(0)\)-space and \(C(X, d) = 1\) then \((X, d)\) is Gromov hyperbolic, see Proposition 3.8 and Question 3.7.

The appearance of a possibly positive \(\delta\) in a \((\mu, \delta)\)-four-point inequality suggests that \(C(X, d)\) can be insensitive to small scales. Indeed, \(C(X, d)\) is a rough isometry invariant of \((X, d)\), Corollary 2.15. Quasi-isometry is a less stringent condition than rough isometry and \(C(X, d)\) is not a quasi-isometry invariant of \((X, d)\). Examples of this phenomenon are given in §3.

While the restricted quasi-hyperbolicity constant, \(C_0(X, d)\), is obviously an isometry invariant it is not a rough isometry invariant; moreover, the constants \(C_0(X, d)\) and \(C(X, d)\) need not coincide. For example, if \((H^2, d_H)\) is the hyperbolic plane then \(C(H^2, d_H) = 1 < \sqrt{2} = C_0(H^2, d_H)\), see Example 2.11. The intuition supporting this example is that very small quadrilaterals in \(H^2\) are approximately Euclidean and contribute to \(C_0(H^2, d_H)\) but not to \(C(H^2, d_H)\). For spaces \((X, d)\) that are "four-point scalable in the large" (Definition 2.7) we show, Proposition 2.9, that \(C_0(X, d) = C(X, d)\). Examples of such spaces include Banach spaces and their metric snowflakes.
A CAT(0)-space is a geodesic metric space whose geodesic triangles are not fatter than corresponding comparison triangles in the Euclidean plane. Simply connected, complete Riemannian manifolds of non-positive sectional curvature are familiar examples of CAT(0)-spaces. We show, Theorem 4.2, that the restricted quasi-hyperbolicity constant of a metric space whose distance satisfies Ptolemy’s inequality and the quadrilateral inequality, in particular any CAT(0)-space, is bounded from above by $\sqrt{2}$. The quasi-hyperbolicity constant of any Euclidean space of dimension greater than one is equal to $\sqrt{2}$, Proposition 4.4.

Banach spaces are a particularly important class of metric spaces and their geometric properties have been extensively studied, [JL01]. For a Banach space $B$ with the metric determined by its norm, we write $C(B)$ for its quasi-hyperbolicity constant. We observe that $C(B) \geq J(B)$ where $J(B)$ is the James constant of $B$, see (5.7). Strong results for the James constant of a Banach space due to Gao and Lau, [GL90], and to Komuro, Saito and Tanaka, [KST16], lead to the following conclusion about $C(B)$.

**Theorem** (Theorem 5.8). If $B$ is a Banach space with $\dim B > 1$ then $C(B) \geq \sqrt{2}$. If $\dim B \geq 3$ and $C(B) = \sqrt{2}$ then $B$ is a Hilbert space.

Enflo [Enf69] introduced the notion of the *roundness* of a metric space, Definition 5.9, which is a real number greater than or equal to one. We show:

**Theorem** (Theorem 5.11). If $B$ is a Banach space with roundness $r(B)$ then $C(B) \leq 2^{1/r(B)}$.

This estimate allows us to calculate the quasi-hyperbolicity constant of a non-trivial $L_p$-space.

**Corollary** (Corollary 5.12). For a separable measure space $(\Omega, \Sigma, \mu)$ and $1 \leq p \leq \infty$, let $L_p(\Omega, \Sigma, \mu)$ be the corresponding $L_p$-space. If $\dim L_p(\Omega, \Sigma, \mu) \geq 2$ then $C(L_p(\Omega, \Sigma, \mu)) = \max\{2^{1/p}, 2^{1-1/p}\}$.

If $(X, d)$ is any metric space and $0 < \alpha < 1$ then $(X, d^\alpha)$ is also a metric space, called the $\alpha$-snowflake of $(X, d)$. We show, Theorem 6.2, that $C_0(X, d^\alpha) \leq 2^\alpha$. Applying this estimate, we calculate, Proposition 6.3, the quasi-hyperbolicity constant of the $\alpha$-snowflake of $(\mathbb{R}^n, d_\infty)$, where $d_\infty$ is the $L_\infty$-metric (“max metric”) on $\mathbb{R}^n$: For $n \geq 2$, $C(\mathbb{R}^n, d_\infty) = 2^\alpha$. The quasi-hyperbolicity constant of the $\alpha$-snowflake of of the Euclidean line $(\mathbb{R}^1, d_E)$ can be determined by solving an associated optimization problem, yielding the following calculation.

**Theorem** (Theorem 6.6). Let $0 < \alpha \leq 1$. Let $m \geq 1$ be the unique solution to the equation $(m - 1)^\alpha + (m + 1)^\alpha = 2$. Then $C(\mathbb{R}^1, d_E^\alpha) = m^\alpha$.

2. QUASI-HYPERBOLICITY AND RESTRICTED QUASI-HYPERBOLICITY CONSTANTS

We derive basic properties of the quasi-hyperbolicity constant and the restricted quasi-hyperbolicity constant of a metric space and examine their general behavior with regard to quasi-isometric embedding and, respectively, bilipschitz embedding.

Recall the following definition from the introduction.
Definition 2.1. Let $\mu, \delta \geq 0$. We say that a metric space $(X, d)$ satisfies the $(\mu, \delta)$-four-point inequality if for all $x, y, z, w \in X$,

$$xy + zw \leq \mu \max\{xz + yw, xw + yz\} + 2\delta.$$  

We make the following elementary observation concerning this definition.

Proposition 2.2. Let $(X, d)$ be a metric space.

(i) $(X, d)$ satisfies the $(2, 0)$-four-point inequality,

(ii) If $(X, d)$ is unbounded and satisfies the $(\mu, \delta)$-four-point inequality then $\mu \geq 1$.

(iii) If $(X, d)$ is bounded with diameter $D$ then it satisfies the $(0, D)$-four-point inequality.

Proof. (i). Let $x, y, z, w \in X$. Triangle inequality and symmetry of the metric yield:

$$xy \leq xz + yz, \quad xy \leq xw + yw, \quad zw \leq xz + xw, \quad zw \leq yz + yw.$$  

Adding these four inequalities and dividing by 2 gives $xy + zw \leq \max\{xz + yw, xw + yz\}$, that is, the $(2, 0)$-four-point inequality is satisfied.

(ii). Assume that $X$ is unbounded and satisfies the $(\mu, \delta)$-four-point inequality. Let $(x_n)$ and $(y_n)$ be sequences in $X$ such that $x_ny_n \to \infty$ as $n \to \infty$. By the $(\mu, \delta)$-four-point inequality, with $x = x_n$ and $y = z = w = y_n$, we have $x_ny_n \leq \mu x_ny_n + 2\delta$. Dividing by $x_ny_n$ and taking the limit as $n \to \infty$ yields $1 \leq \mu$.

Property (iii) is obvious. \(\square\)

Given points $x, y, z, w \in X$, not all identical, define

$$(2.3) \quad \Delta(x, y, z, w) = \frac{xy + zw}{\max\{xz + yw, xw + yz\}}.$$  

In the introduction, we defined the restricted quasi-hyperbolicity constant of $(X, d)$ by

$$C_0(X, d) = \inf\{\mu \mid (X, d) \text{ satisfies the (\mu, 0)-four-point inequality}\}.$$  

If $X$ has at least two points then

$$(2.4) \quad C_0(X, d) = \sup \Delta(x, y, z, w)$$  

where the supremum is taken over all $x, y, z, w \in X$, not all identical.

We also defined the quasi-hyperbolicity constant of $(X, d)$ by

$$C(X, d) = \inf\{\mu \mid \text{there exists } \delta \geq 0 \text{ such that } (X, d) \text{ satisfies the (\mu, \delta)-four-point inequality}\}.$$  

The quasi-hyperbolicity constant and the restricted quasi-hyperbolicity constant have the following elementary properties.

Proposition 2.5. Let $(X, d)$ be a metric space.

(i) If $A \subset X$ and $d_A$ is the subspace metric then $C(A, d_A) \leq C(X, d)$ and $C_0(A, d_A) \leq C_0(X, d)$.

(ii) If $\lambda > 0$ then $C(X, \lambda d) = C(X, d)$ and $C_0(X, \lambda d) = C_0(X, d)$. 
(iii) $C(X, d) \leq C_0(X, d) \leq 2$.
(iv) If $(X, d)$ is unbounded then $1 \leq C(X, d)$.
(v) If $(X, d)$ is bounded then $C(X, d) = 0$.
(vi) If $(X, d)$ has at least two distinct points then $C_0(X, d) \geq 1$.
(vii) If $(X', d')$ is a metric completion of $(X, d)$ then $C(X, d) = C(X', d')$ and $C_0(X, d) = C_0(X', d')$.

Proof. Property (i) and the inequality $C(X, d) \leq C_0(X, d)$ are clear from the definitions of $C(X, d)$ and $C_0(X, d)$. Note that for $\lambda > 0$, $(X, d)$ satisfies the $(\mu, \delta)$-four-point inequality if and only if $(X, \lambda d)$ satisfies the $(\mu, \lambda \delta)$-four-point inequality. This implies (ii). The inequality $C_0(X, d) \leq 2$ in (iii) is a consequence of Proposition 2.2(i); (iv) follows from Proposition 2.2(ii); and (v) follows from Proposition 2.2(iii). If $x_0, y_0$ are distinct points in $X$ then $\Delta(x_0, y_0, y_0, y_0) = 1$, see (2.3), and so $C_0(X, d) \geq 1$ by (2.4). It is straightforward that a metric space $(X, d)$ satisfies the $(\mu, \delta)$-four-point inequality if and only if a metric completion of $(X, d)$ satisfies the $(\mu, \delta)$-four-point inequality. This implies (vii). \hfill \Box

Proposition 2.6. Let $(X, d)$ be a metric space.

(i) If $(X, d)$ is unbounded and Gromov hyperbolic then $C(X, d) = 1$.
(ii) If $(X, d)$ is a metric completion then it is 0-hyperbolic if and only if $C_0(X, d) = 1$.

Proof. (i). By Proposition 2.5(iv), $C(X, d) \geq 1$. Since, by definition, a Gromov hyperbolic space satisfies a $(1, \delta)$-four-point inequality for some $\delta \geq 0$ we have $C(X, d) \leq 1$. Hence $C(X, d) = 1$.
(ii). If $(X, d)$ is 0-hyperbolic then it satisfies the $(1,0)$-four-point inequality and so $C_0(X, d) \leq 1$. By Proposition 2.5(vi), $C_0(X, d) \geq 1$. Hence $C_0(X, d) = 1$. If $C_0(X, d) = 1$ then for every $x, y, z, w \in X$, not all identical, $\Delta(x, y, z, w) \leq 1$ and so $(X, d)$ satisfies the $(1,0)$-four-point inequality, that is, $(X, d)$ is 0-hyperbolic. \hfill \Box

Without additional hypotheses, the converse of Proposition 2.6(i) need not be true, in §3.2 we give examples of unbounded metric spaces with $C(X, d) = 1$ that are not Gromov hyperbolic (also see Question 3.7 and Proposition 3.8).

Definition 2.7. We say that a metric space $(X, d)$ is four-point scalable in the large if for every $x_1, x_2, x_3, x_4 \in X$ and for every $\lambda \geq 0$ there exists $x'_1, x'_2, x'_3, x'_4 \in X$ and $0 \leq \Lambda \leq \Lambda$ such that $d(x'_i, x'_j) = \Lambda d(x_i, x_j)$ for $1 \leq i, j \leq 4$.

Example 2.8. Let $V$ be a real vector space with a given norm $\| \cdot \|$. The norm determines a metric on $V$ given by $d(x, y) = \|x - y\|$. For any $0 < \alpha \leq 1$ the function $d^\alpha$ is also metric on $V$. The metric space $(V, d^\alpha)$ is called the (1-4)-snowflake of $(V, d)$. Note that $d^\alpha(\lambda x, \lambda y) = \lambda^\alpha d^\alpha(x, y)$ for any $\lambda > 0$ from which it easily follows that $(V, d^\alpha)$ is four-point scalable in the large. Let $S \subset V$ be a nonempty subset such that $\lambda x \in S$ for all $\lambda > 0$ and all $x \in S$. Then $S$, viewed as a metric subspace of $(V, d^\alpha)$, is also four-point scalable in the large.
Proposition 2.9. If \((X,d)\) is four-point scalable in the large then \(C(X,d) = C_0(X,d)\).

Proof. It suffices to show that if \((X,d)\) satisfies the \((\mu,\delta)\)-four-point inequality for a particular \((\mu,\delta)\) then it also satisfies the \((\mu,0)\)-four-point inequality. Assume that \((X,d)\) satisfies the \((\mu,\delta)\)-four-point inequality for some \(\mu \geq 1\) and \(\delta \geq 0\). Let \(x_1, x_2, x_3, x_4 \in X\). For each \(\lambda \geq 0\), let \(\Lambda \geq \lambda\) and \(x_i' \in X\) be such that \(x_i'x_j' = \Lambda x_i x_j\), \(1 \leq i,j \leq 4\). Note that the \((\mu,\delta)\)-four-point inequality for the points \(\{x_i'\}\) implies the \((\mu, \delta/\Lambda)\)-four-point inequality for \(\{x_i\}\). Since \(\Lambda\) can be chosen to be arbitrarily large, it follows that \(\{x_i\}\) satisfies the \((\mu,0)\)-four-point inequality. \(\square\)

Corollary 2.10. Let \(V\) be a real vector space with a given norm \(\|\cdot\|\) and corresponding metric, \(d(x,y) = \|x - y\|\). Let \(S \subset V\) be a nonempty subset such that \(\lambda x \in S\) for all \(\lambda > 0\) and all \(x \in S\). Then for all \(0 < \alpha \leq 1\), \(C(S, d^\alpha) = C_0(S, d^\alpha)\).

Proof. From Example 2.8, \((S, d^\alpha)\) is four-point scalar in the large and so the conclusion follows from Proposition 2.9. \(\square\)

Example 2.11 (Hyperbolic space). Let \(n > 1\) be an integer and let \((H^n, d_H)\) denote \(n\)-dimensional real hyperbolic space. For this space, \(C(H^n, d_H) = 1 < \sqrt{2} = C_0(H^n, d_H)\) and so Proposition 2.9 implies \((H^n, d_H)\) is not four-point scalable in the large. The space \((H^n, d_H)\) is Gromov hyperbolic and unbounded, hence \(C(H^n, d_H) = 1\) by Proposition 2.6(i). Since \(H^n\) has negative sectional curvature as a Riemannian manifold, \(C_0(H^n, d_H) = \sqrt{2}\) by Corollary 7.3.

Definition 2.12. Let \(C_1, C_2 > 0\) and \(L_1, L_2 \geq 0\). A map \(f: X \to Y\) between metric spaces \((X, d_X)\) and \((Y, d_Y)\) is a \(((C_1, L_1), (C_2, L_2))\)-quasi-isometric embedding if for all \(u, v \in X\),

\[C_1 d_X(u,v) - L_1 \leq d_Y(f(u), f(v)) \leq C_2 d_X(u,v) + L_2.\]

Some useful special cases of this definition include:

(i) A \(((C_1, 0), (C_2, 0))\)-quasi-isometric embedding \(f: X \to Y\) is also known as a \((C_1, C_2)\)-bilipschitz embedding.

(ii) A \(((1, k), (1, k))\)-quasi-isometric embedding \(f: X \to Y\) is also known as a \(k\)-rough isometric embedding. This condition is equivalent to: for all \(u, v \in X\), \(|d_Y(f(u), f(v)) - d_X(u,v)| \leq k\).

Lemma 2.13. If \(f: X \to Y\) is a \(((C_1, L_1), (C_2, L_2))\)-quasi-isometric embedding between metric spaces and \((Y, d_Y)\) satisfies the \((\mu, \delta)\)-four-point inequality for some \((\mu, \delta)\) then \((X, d_X)\) satisfies the \(\left(\frac{C_1}{C_2^2} \mu, \frac{1}{C_1}(\mu L_2 + L_1 + \delta)\right)\)-four-point inequality.
Proof. Let \( x, y, z, w \in X \) and let \( \bar{x}, \bar{y}, \bar{z}, \bar{w} \in Y \) be their respective images under \( f: X \to Y \). Then
\[
d_X(x, y) + d_X(z, w) \\
\leq \frac{1}{C_1} \left( d_Y(\bar{x}, \bar{y}) + d_Y(\bar{z}, \bar{w}) \right) + \frac{2L_1}{C_1} \\
\leq \frac{1}{C_1} \mu \max \{ d_Y(\bar{x}, \bar{z}) + d_Y(\bar{y}, \bar{w}), d_Y(\bar{x}, \bar{w}) + d_Y(\bar{y}, \bar{z}) \} + 2\delta \frac{d_Y(\bar{x}, \bar{y}) + d_Y(\bar{z}, \bar{w})}{C_1} \\
\leq \frac{1}{C_1} \mu \max \{ C_2(d_X(x, z) + d_X(y, w)) + 2L_2, C_2(d_X(x, w) + d_X(y, z)) + 2L_2 \} + \frac{2\delta}{C_1} + \frac{2L_1}{C_1} \\
= \frac{C_2}{C_1} \mu \max \{ d_X(x, z) + d_X(y, w), d_X(x, w) + d_X(y, z) \} + \frac{2\mu L_2}{C_1} + \frac{2\delta}{C_1} + \frac{2L_1}{C_1}
\]
which shows that \((X, d)\) satisfies the \( \left( \frac{C_2}{C_1} \mu, \frac{1}{C_1}(\mu L_2 + L_1 + \delta) \right) \)-four-point inequality. \( \square \)

Lemma 2.13 has the following immediate consequence.

**Proposition 2.14.** Let \( f: X \to Y \) be a map between metric spaces \((X, d_X)\) and \((Y, d_Y)\).

(i) If \( f \) is a \(((C_1, L_1), (C_2, L_2))\)-quasi-isometric embedding then \( C(X, d_X) \leq (C_2/C_1) C(Y, d_Y) \).

(ii) If \( f \) is a \((C_1, C_2)\)-bilipschitz embedding then \( C_0(X, d_X) \leq (C_2/C_1) C_0(Y, d_Y) \). \( \square \)

A map \( f: X \to Y \) between metric spaces \((X, d_X)\) and \((Y, d_Y)\) is a rough isometry if it is a \(k\)-rough isometric embedding for some \(k \geq 0\) and there exists \(R > 0\) such that \(f(X)\) is \(R\)-dense in \(Y\), that is, for every \(y \in Y\) there exists \(x \in X\) such that \(d_Y(f(x), y) < R\). Two metric spaces are roughly isometric if there exists a rough isometry between them. Note that rough isometry is a generally a stronger condition than quasi-isometry. Recall that \(f\) is a quasi-isometry if it is a \(((C_1, L_1), (C_2, L_2))\)-quasi-isometric embedding for some \((C_1, L_1), (C_2, L_2)\) and also \(f(X)\) is \(R\)-dense for some \(R\).

**Corollary 2.15.** If \((X, d_X)\) and \((Y, d_Y)\) are roughly isometric then \(C(X, d_X) = C(Y, d_Y)\).

**Proof.** Since \((X, d_X)\) and \((Y, d_Y)\) are assumed to be roughly isometric, there exists \(k \geq 0\) and \(R > 0\) and a \(k\)-rough isometric embedding \(f: X \to Y\) such that \(f(X)\) is \(R\)-dense in \(Y\). By Proposition 2.14(i), \(C(X, d_X) \leq C(Y, d_Y)\). Define \(g: Y \to X\) as follows. For each \(y \in Y\) we can choose \(x \in X\) such that \(d_Y(f(x), y) < R\) and declare \(g(y) = x\). Observe that for all \(y \in Y\), \(d_Y(f(g(y)), y) < R\). For all \(u, v \in Y\), \(|d_Y(f(g(u)), f(g(v))) - d_X(g(u), g(v))| \leq k\). Hence, for all \(u, v \in Y\), \(|d_Y(u, v) - d_X(g(u), g(v))| \leq k + 2R\) and so \(g\) is a \((k + 2R)\)-rough embedding. By Proposition 2.14(i), \(C(Y, d_Y) \leq C(X, d_X)\). It follows that \(C(X, d_X) = C(Y, d_Y)\). \( \square \)

### 3. Two families of examples

In §3.1, we exhibit spaces that are quasi-isometric to the Euclidean line yet with quasi-hyperbolicity constants that are greater than one and, consequently, are not Gromov hyperbolic. In §3.2, we give examples of metric spaces whose quasi-hyperbolicity constants are equal to one, yet are not Gromov hyperbolic. However, these are examples are not roughly geodesic. We show, using Bridson’s “Flat Plane Theorem”, that a proper CAT(0)-space whose quasi-hyperbolicity constant is equal to one is necessarily Gromov hyperbolic, see Proposition 3.8.
3.1. The graph of \( y = m|x| \) in the Euclidean plane.

Let \( m \geq 0 \). Consider the space \( X_m = \{(x, y) \in \mathbb{R}^2 \mid y = m|x|\} \) as a subspace of the Euclidean plane. The metric on \( X_m \) is given by
\[
d_E((u, m|u|),(v, m|v|)) = [(u-v)^2 + m^2(|u| - |v|)^2]^{1/2}.
\]
Let \((\mathbb{R}, d_E)\) be the Euclidean line, \( d_E(u, v) = |u - v|\). Let \( p : X_m \to \mathbb{R} \) be projection to the first coordinate, that is, \( p(x, y) = x \). For \( u, v \in \mathbb{R}, |u| - |v| \leq |u - v|\), and so, for \( u \neq v\),
\[
[(u-v)^2 + m^2(|u| - |v|)^2]^{1/2} = |u - v| \left[ 1 + m^2 \left( \frac{|u| - |v|}{u-v} \right)^2 \right]^{1/2} \leq (m^2+1)^{1/2} |u - v|,
\]
and thus for all \( u, v \)
\[
(m^2 + 1)^{-1/2}d_E((u, m|u|),(v, m|v|)) \leq |u - v| \leq d_E((u, m|u|),(v, m|v|)).
\]
Hence \( p \) is a \((m^2 + 1)^{-1/2},1\)-bilipschitz embedding of \( X_m \) into \( \mathbb{R} \). Since \( p \) is surjective, it is also a bilipschitz homeomorphism. In particular, \((X_m, d_E)\) and \((\mathbb{R}, d_E)\) are quasi-isometric.

Note that, since \((\mathbb{R}, d_E)\) is 0-hyperbolic, we have \( C(\mathbb{R}, d_E) = C_0(\mathbb{R}, d_E) = 1 \) by Proposition 2.6.

For \( m > 0 \), let \( \mu_m = \sqrt{\frac{m^2+1}{m^2+1-1}} \). A straightforward calculation yields
\[
\Delta(-\mu_m, \mu_m, (1, m), (-1, m), (\mu_m, \mu_m m)) = (2 - (m^2 + 1)^{-1})^{1/2}
\]
and so \( C_0(X_m, d_E) \geq (2 - (m^2 + 1)^{-1})^{1/2} \). Note that if \((x, y) \in X_m\) and \( \lambda > 0 \) then \( \lambda(x,y) \in X_m \) and so Corollary 2.10 gives \( C_0(X_m, d_E) = C(X_m, d_E) \). Hence \( C(X_m, d_E) \geq (2 - (m^2 + 1)^{-1})^{1/2} > 1 \) for \( m > 0 \). It follows from Proposition 2.6(i) that \((X_m, d_E)\) is not Gromov hyperbolic when \( m > 0 \). Combining Propositions 2.14 and 4.3 yields the non-sharp upper bound:
\[
C(X_m, d_E) \leq \min \left\{ \sqrt{7}, (m^2+1)^{1/2} \right\}.
\]
However, numerical calculations strongly suggest that the configuration \((-\mu_m, \mu_m m), (1, m), (-1, m), (\mu_m, \mu_m m)\) of four points in \( X_m \) is optimal, that is, \( C(X_m, d_E) = (2 - (m^2 + 1)^{-1})^{1/2} \) for all \( m > 0 \).

3.2. The graph of \( y = x^\alpha \), where \( 0 < \alpha < 1 \), in the Euclidean plane.

For \( 0 < \alpha < 1 \), let \( d_\alpha \) be the metric on the half-line, \([0, \infty)\), given by
\[
d_\alpha(x, y) = \left((x-y)^2 + (x^\alpha - y^\alpha)^2\right)^{1/2}.
\]
Let \( Y_\alpha = \{(x, y) \in \mathbb{R}^2 \mid y = x^\alpha, x \geq 0\} \) as a subspace of the Euclidean plane. Projection to the first coordinate, \((x, y) \mapsto x\), gives an isometry \((Y_\alpha, d_E) \to ([0, \infty), d_\alpha)\). The metric behavior of \(([0, \infty), d_\alpha)\) separates into two distinct cases, namely \( 0 < \alpha \leq 1/2 \) and \( 1/2 < \alpha < 1 \).

**Proposition 3.1.** If \( 0 < \alpha \leq 1/2 \) then for all \( x, y \geq 0 \), \( 0 \leq d_\alpha(x, y) - |x-y| \leq 1 \). Consequently, for \( 0 < \alpha \leq 1/2 \), \(([0, \infty), d_\alpha)\) is roughly isometric to the Euclidean half-line and is thus Gromov hyperbolic.
Proof. We first show that if $0 < \alpha \leq 1/2$ then for $u \geq 0$, $(u^2 + 2\alpha)^{1/2} - u \leq 1$. If $u \leq 1$ then
\[
(u^2 + 2\alpha)^{1/2} - u \leq (u + u^\alpha) - u = u^\alpha \leq 1.
\]
If $u \geq 1$ and $\alpha \leq 1/2$ then $u^{2\alpha} \leq u$ and
\[
(u^2 + u^{2\alpha})^{1/2} - u \leq \frac{u^{2\alpha}}{(u^2 + u^{2\alpha})^{1/2} + u} \leq \frac{u}{u^2 + u^{2\alpha})^{1/2} + u} \leq 1.
\]
For $0 < \alpha < 1$ and $x, y \geq 0$, $|x^\alpha - y^\alpha| \leq |x - y|^\alpha$. Hence for $0 < \alpha \leq 1/2$ and $x, y \geq 0$, and using the inequality $(u^2 + u^{2\alpha})^{1/2} - u \leq 1$ with $u = |x - y|$, we have
\[
0 \leq ((x - y)^2 + (x^\alpha - y^\alpha)^2)^{1/2} - |x - y| \leq (|x - y|^2 + |x - y|^2|u|) - |x - y| \leq 1,
\]
establishing the conclusion of the Proposition. \hfill \Box

In [BH99, 1.23 Exercise, p.412] it is asserted that $(0, \infty), d_{1/2}$ is not Gromov hyperbolic. This is not accurate as demonstrated by Proposition 3.1, however, we show in Proposition 3.3 that $(0, \infty), d_{\alpha}$ is not Gromov hyperbolic if $1/2 < \alpha < 1$.

Lemma 3.2. Let $f(\alpha) = 2^{2\alpha - 2} + \frac{1}{\alpha}(1 - 2\alpha)^2 - \frac{1}{2}(1 - 2\alpha)^2 - 2^{4\alpha - 3}$. If $0 < \alpha < 1$ then $f(\alpha) > 0$ and
\[
\lim_{t \to \infty} \left( d_\alpha(t, 4t) + d_\alpha(0, 2t) - d_\alpha(t, 2t) - d_\alpha(0, 4t) \right) / t^{2\alpha - 1} = f(\alpha).
\]
Proof. Consider the polynomial $g(x) = \frac{1}{12}x^4 - \frac{7}{24}x^2 + x - \frac{1}{5} = \frac{1}{12}(x - 2)^2(x^2 + 4x - 2)$. Using the factored expression for $g(x)$, we see that $g(x) > 0$ for $1 < x < 2$. Note that $f(\alpha) = g(2\alpha)$. Hence $f(\alpha) > 0$ for $0 < \alpha < 1$. For $s \geq 0$, let
\[
h(s) = \left(3^2 + (1 - 4^\alpha)^2s\right)^{1/2} + \left(2^2 + 2\alpha s\right)^{1/2} - \left(1 + (1 - 2\alpha)^2s\right)^{1/2} - \left(4^2 + 2\alpha^2s\right)^{1/2}.
\]
A straightforward calculation reveals that, for $t > 0$,
\[
\theta(t) = (d_\alpha(t, 4t) + d_\alpha(0, 2t) - d_\alpha(t, 2t) - d_\alpha(0, 4t)) / t^{2\alpha - 1} = h(t^{2\alpha - 2}) / t^{2\alpha - 2}.
\]
Since $2\alpha - 2 < 0$, $\lim_{t \to \infty} t^{2\alpha - 2} = 0$ and so
\[
\lim_{t \to \infty} \theta(t) = \lim_{s \to 0} \frac{h(s)}{s} = h'(0) = f(\alpha)
\]
yielding the conclusion of the Lemma. \hfill \Box

Proposition 3.3. If $1/2 < \alpha < 1$ then $(0, \infty), d_\alpha$ is not Gromov hyperbolic.

Proof. For $x, y, z, w \in [0, \infty)$ and $0 < \alpha < 1$, let
\[
\text{Gr}_\alpha(x, y, z, w) = d_\alpha(x, y) + d_\alpha(z, w) - \max \{d_\alpha(x, z) + d_\alpha(y, w), d_\alpha(x, w) + d_\alpha(y, z)\}.
\]
Note that $(0, \infty), d_\alpha$ is not Gromov hyperbolic if and only if $\sup_{x,y,z,w} \text{Gr}_\alpha(x, y, z, w) = \infty$.

For $t > 0$, let
\[
h(t) = \frac{d_\alpha(t, 2t) + d_\alpha(0, 4t)}{d_\alpha(0, t) + d_\alpha(2t, 4t)} = \frac{(1 + (1 - 2\alpha)^2t^{2\alpha - 2})^{1/2} + (4^2 + 4\alpha^2t^{2\alpha - 2})^{1/2}}{(1 + t^{2\alpha - 2})^{1/2} + (2^2 + (2\alpha^2 - 4\alpha^2)t^{2\alpha - 2})^{1/2}}.
\]
Since $2\alpha - 2 < 0$, $\lim_{t \to \infty} t^{2\alpha - 2} = 0$ and so the above expression for $h(t)$ yields $\lim_{t \to \infty} h(t) = 5/3$. Hence $d_\alpha(t, 2t) + d_\alpha(0, 4t) > d_\alpha(0, t) + d_\alpha(2t, 4t)$ for sufficiently large $t$ which implies that $\text{Gr}_\alpha(t, 4t, 0, 2t) = d_\alpha(t, 4t) + d_\alpha(0, 2t) - d_\alpha(t, 2t) - d_\alpha(0, 4t)$ for sufficiently large $t$. If $1/2 < \alpha < 1$ then $2\alpha - 1 > 0$ and so Lemma 3.2 implies that $\lim_{t \to \infty} \text{Gr}_\alpha(t, 4t, 0, 2t) = \infty$. □

**Proposition 3.4.** If $0 < \alpha < 1$ then $C([0, \infty), d_\alpha) = 1$.

**Proof.** Let $L > 0$. If $x, y \geq 0$ and $|x - y| \geq L$ then

$$\frac{|x^\alpha - y^\alpha|}{|x - y|} \leq \frac{|x - y|^\alpha}{|x - y|} = |x - y|^{\alpha - 1} \leq L^{\alpha - 1},$$

and so for $|x - y| \geq L$,

$$d_\alpha(x, y) \leq \left(|x - y|^2 + (L^{\alpha - 1}|x - y|)^2\right)^{1/2} \leq \left(1 + L^{2\alpha - 2}\right)^{1/2} |x - y|.$$

If $x, y \geq 0$ and $|x - y| \leq L$ then

$$d_\alpha(x, y) \leq \left(|x - y|^2 + |x - y|^{2\alpha}\right)^{1/2} \leq \left(L^2 + L^{2\alpha}\right)^{1/2} = L \left(1 + L^{2\alpha - 2}\right)^{1/2}.$$

It follows that for all $x, y \geq 0$

$$|x - y| \leq d_\alpha(x, y) \leq \left(1 + L^{2\alpha - 2}\right)^{1/2} |x - y| + L \left(1 + L^{2\alpha - 2}\right)^{1/2}. \quad (3.5)$$

Let $d_E(x, y) = |x - y|$, the Euclidean metric on $[0, \infty)$. By Proposition 2.6(i), $C([0, \infty), d_E) = 1$. Proposition 2.14(i) and (3.5) imply that $C([0, \infty), d_\alpha) \leq \left(1 + L^{2\alpha - 2}\right)^{1/2}$. Since $2\alpha - 2 < 0$, we have that $\lim_{t \to \infty} \left(1 + L^{2\alpha - 2}\right)^{1/2} = 1$. Hence $C([0, \infty), d_\alpha) \leq 1$. Furthermore, by Proposition 2.5(iv), $C([0, \infty), d_\alpha) \geq 1$ and so $C([0, \infty), d_\alpha) = 1$. □

**Remark 3.6.** It follows from the inequality (3.5) that the identity map $([0, \infty), d_E) \to ([0, \infty), d_\alpha)$ is a quasi-isometry. In this inequality, there is a trade-off between the “distortion”, $(1 + L^{2\alpha - 2})^{1/2}$, and the “roughness”, $L \left(1 + L^{2\alpha - 2}\right)^{1/2}$, that is, an attempt to adjust the parameter $L$ to make the distortion small (close to 1) makes the roughness large and vice versa.

We showed that for $1/2 < \alpha < 1$ the space $([0, \infty), d_\alpha)$ is not Gromov hyperbolic but, nevertheless, $C([0, \infty), d_\alpha) = 1$.

**Question 3.7.** Assume that $(X, d)$ is a geodesic metric space or, more generally, roughly geodesic. Does $C(X, d) = 1$ imply that $(X, d)$ is Gromov hyperbolic?

For $1/2 < \alpha < 1$, the space $([0, \infty), d_\alpha)$ is not roughly geodesic and so does not provide a negative answer to this question. Some evidence in favor of an affirmative answer to Question 3.7 is given by the following result (see §4 for a discussion of CAT(0)-spaces).

**Proposition 3.8.** Let $(X, d)$ be a proper CAT(0)-space. If $C(X, d) = 1$ then $(X, d)$ is Gromov hyperbolic.
Proof. Assume the proper CAT(0)-space \((X, d)\) is not Gromov hyperbolic. Bridson’s Flat Plane Theorem, [Bri95, Theorem A], asserts that there exists an isometric embedding of a Euclidean plane, \((V, d_E)\), into \(X\). Hence \(C(V, d_E) \leq C(X, d)\). By Proposition 4.4, \(C(V, d_E) = \sqrt{2}\) and so \(C(X, d) \geq \sqrt{2}\). In particular, \(C(X, d) \neq 1\). \(\square\)

4. The Ptolemy and quadrilateral inequalities, CAT(0)-spaces

The notion of a CAT(0)-space generalizes the concept of a simply connected, complete Riemannian manifold of non-positive sectional curvature to geodesic metric spaces. We show that the restricted quasi-hyperbolicity constant of a CAT(0)-space is bounded from above by \(\sqrt{2}\). Indeed, the restricted quasi-hyperbolicity constant of any metric space whose distance satisfies Ptolemy’s inequality and the quadrilateral inequality, in particular any CAT(0)-space, is bounded from above by \(\sqrt{2}\), Theorem 4.2. The quasi-hyperbolicity constant of any Euclidean space of dimension greater than one is equal to \(\sqrt{2}\), Proposition 4.4.

Definition 4.1. Let \((X, d)\) be a metric space.

(i) The metric \(d\) satisfies Ptolemy’s inequality if for all \(x, y, z, w \in X\),
\[(xy)(zw) \leq (xz)(yw) + (xw)(yz).\]
In this case we say \((X, d)\) is Ptolemaic.

(ii) The metric \(d\) satisfies the quadrilateral inequality if for all \(x, y, z, w \in X\),
\[(xy)^2 + (zw)^2 \leq (xz)^2 + (yw)^2 + (xw)^2 + (yz)^2.\]
In this case we say \((X, d)\) is 2-round (see Definition 5.9).

Recall that a Euclidean space is a real vector space \(V\) together with a positive definite inner product, \((u, v) \mapsto \langle u, v \rangle\). The inner product yields a Euclidean norm, \(\|x\| = (x, x)^{1/2}\), and a corresponding Euclidean metric, \(d(u, v) = \|x - y\|\). It is classical mathematics that a Euclidean space with its Euclidean metric is Ptolemaic and 2-round.

Theorem 4.2. If the metric space \((X, d)\) is Ptolemaic and 2-round then \(C_0(X, d) \leq \sqrt{2}\).

Proof. Assume \((X, d)\) is Ptolemaic and 2-round. Then for \(x, y, z, w \in X\),
\[(xy)(zw) \leq (xz)(yw) + (xw)(yz)\]
\[(xy)^2 + (zw)^2 \leq (xz)^2 + (yw)^2 + (xw)^2 + (yz)^2.\]

Multiplying the first inequality by 2 and adding it to the second one yields:
\[(xy + zw)^2 \leq (xz + yw)^2 + (xw + yz)^2.\]
For non-negative real numbers \(a, b\) we have \(\sqrt{a^2 + b^2} \leq \sqrt{2} \max\{a, b\}\) and so the above inequality implies
\[xy + zw \leq \sqrt{2} \max\{xz + yw, xw + yz\}\]
from which it follows that $C_0(X, d) \leq \sqrt{2}$.

Informally, a CAT(0)-space is a geodesic metric space whose geodesic triangles are not fatter than corresponding comparison triangles in the Euclidean plane, see [BH99, II.1.1, page 158] for the precise definition. Since any configuration of four points in a CAT(0)-space has a “subembedding” into Euclidean space, [BH99, page 164], a CAT(0)-space is Ptolemaic and 2-round.

**Corollary 4.3.** If $(X, d)$ is a subspace of a CAT(0)-space then $C_0(X, d) \leq \sqrt{2}$.

**Proof.** Since a CAT(0)-space is Ptolemaic and 2-round, so is any subspace. The conclusion follows from Theorem 4.2. □

**Proposition 4.4.** Let $V$ be a Euclidean space and $d$ its Euclidean metric. If $\dim V \geq 2$ then $C(V, d) = C_0(V, d) = \sqrt{2}$.

**Proof.** By Theorem 4.2, $C_0(V, d) \leq \sqrt{2}$. Since $\dim V \geq 2$, there are orthogonal unit vectors $u, v \in V$. A calculation using the inner product of $V$ yields $\Delta(u, v, 0, u+v) = \sqrt{2}$ and thus $C_0(V, d) \geq \sqrt{2}$. Hence $C_0(V, d) = \sqrt{2}$. Also, by Corollary 2.10, $C(V, d) = C_0(V, d)$. □

Remarkably, a geodesic metric space that is 2-round is necessarily a CAT(0)-space, [BN08, Sat09] and so Corollary 4.3 yields the following proposition.

**Proposition 4.5.** Let $(X, d)$ be a geodesic metric space. If $(X, d)$ is 2-round then $C_0(X, d) \leq \sqrt{2}$. □

**Remark 4.6.** Let $(X, d)$ be any metric space. Blumenthal [Blu70, Theorem 52.1] showed that if $0 < \alpha \leq 1/2$ then the $\alpha$-snowflake $(X, d^\alpha)$ has the property that any four points in it can be isometrically embedded into Euclidean space. Hence, in the case $0 < \alpha \leq 1/2$, $(X, d^\alpha)$ is Ptolemaic and 2-round and so Theorem 4.2 implies that $C_0(X, d^\alpha) \leq \sqrt{2}$. An improvement and extension of this estimate is given by Theorem 6.2.

5. Banach spaces

In contrast to a CAT(0)-space, whose quasi-hyperbolicity constant is bounded from above by $\sqrt{2}$, the quasi-hyperbolicity constant of a Banach space $B$ of dimension greater than one is bounded from below by $\sqrt{2}$ with equality holding, assuming that the dimension of $B$ is at least three, only when $B$ is a Hilbert space, see Theorem 5.8. This is a consequence of strong results for the James constant of $B$ due to Gao and Lau, [GL90], and to Komuro, Saito and Tanaka, [KST16]. Enflo [Enf69] introduced the notion of the roundness of a metric space. We show, Theorem 5.11, that if $B$ is a Banach space with roundness $r(B)$ then its quasi-hyperbolicity constant is bounded from above by $2^{1/r(B)}$ and use this to show that the quasi-hyperbolicity constant of a non-trivial $L_p$-space, where $1 \leq p \leq \infty$, is $\max\{2^{1/p}, 2^{1-1/p}\}$, see Corollary 5.12.
Let $B = (V, \| \cdot \|)$ be a real Banach space. The norm of $B$, $\| \cdot \|$, yields a metric $d(u, v) = \| u - v \|$ on the real vector space $V$ and we use notation $C(B)$ for $C(V, d)$. Note that by Corollary 2.10 we have $C_0(V, d) = C(V, d) = C(B)$.

Let $1 \leq p \leq \infty$. Recall the $p$-norm on $\mathbb{R}^n$, denoted by $\| x \|_p$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, is given by

$$\| x \|_p = \left\{ \begin{array}{ll}
(\sum |x_1|^p + \cdots + |x_n|^p)^{1/p} & \text{if } 1 \leq p < \infty, \\
\max \{|x_1|, \ldots, |x_n|\} & \text{if } p = \infty.
\end{array} \right.$$  

We write $\ell^n_p = (\mathbb{R}^n, \| \cdot \|_p)$ and $d_p(u, v) = \| u - v \|_p$. The $p$-norms on $\mathbb{R}^n$ are related by the following well-known inequality. If $1 \leq p \leq q \leq \infty$ then for all $x \in \mathbb{R}^n$

$$(5.1) \quad \| x \|_q \leq \| x \|_p \leq n^{1/p-1/q} \| x \|_q$$

where, by convention, $1/\infty = 0$.

Note that $\ell^n_2$ is a Euclidean space and so by Proposition 4.4, $C(\ell^n_2) = \sqrt{2}$ for $n \geq 2$.

**Proposition 5.2.** For $1 \leq p \leq \infty$, $C(\ell^n_p) = \max\{2^{1/p}, 2^{1-1/p}\}$.

**Proof.** If $1 \leq p \leq 2$ then by (5.1), $\| x \|_2 \leq \| x \|_p \leq 2^{1/p-1/2} \| x \|_2$. By Proposition 2.14,

$$C(\ell^n_p) \leq 2^{1/p-1/2} C(\ell^n_2) = 2^{1/p}.$$  

Observe $\Delta((-1, 1), (1, -1), (-1, -1), (1, 1)) = 2^{1/p}$ and so $C(\ell^n_p) \geq 2^{1/p}$. Thus $C(\ell^n_p) = 2^{1/p}$.

If $2 \leq p \leq \infty$ then by (5.1), $2^{1/p-1/2} \| x \|_2 \leq \| x \|_p \leq \| x \|_2$. By Proposition 2.14,

$$C(\ell^n_p) \leq 2^{1/2-1/p} C(\ell^n_2) = 2^{1-1/p}.$$  

Observe $\Delta((0, 1), (0, -1), (-1, 0), (1, 0)) = 2^{1-1/p}$ and so $C(\ell^n_p) \geq 2^{1-1/p}$. Thus $C(\ell^n_p) = 2^{1-1/p}$. \(\Box\)

Proposition 5.2 generalizes to non-trivial $L_p$-spaces, see Corollary 5.12.

The *Banach-Mazur distance* between two isomorphic Banach spaces $E$ and $F$ is defined by

$$d_{BM}(E, F) = \inf\{ \| T \| \| T^{-1} \| \mid T: E \to F \text{ is an isomorphism} \}.$$  

For example, if $1 \leq p \leq q \leq 2$ or $2 \leq p \leq q \leq \infty$ then $d_{BM}(\ell^n_p, \ell^n_q) = n^{1/p-1/q}$, [TJ89, Proposition 37.6]. Proposition 2.14 yields the following comparison.

**Proposition 5.3.** If $E$ and $F$ are isomorphic Banach spaces then $C(E) \leq d_{BM}(E, F) C(F)$. \(\Box\)

Because of Theorem 5.8 below, the inequality of Proposition 5.3 can only give useful information when $d_{BM}(E, F) < \sqrt{2}$.

Since, up to a translation, any four points of a Banach space lie in some subspace of dimension at most three,

$$(5.4) \quad C(B) = \sup\{C(V) \mid V \text{ is a subspace of } B \text{ with } \dim V \leq 3\}.$$  

A Banach space $B$ is *finitely representable* in another Banach space $B'$ if for every finite dimensional subspace $F$ of $B$ and every $\varepsilon > 0$ there is a subspace $F'$ of $B'$ and an isomorphism $T: F \to F'$ such that $\| T \| \| T^{-1} \| \leq 1 + \varepsilon$. 

Proposition 5.5. If $B$ is finitely representable in $B'$ then $C(B) \leq C(B')$.

Proof. Let $\varepsilon > 0$. Let $V$ be a subspace of $B$ with $\dim V \leq 3$. Since $B$ is finitely representable in $B'$, there exists a subspace $V'$ of $B'$ and an isomorphism $T: V \rightarrow V'$ such that $\|T\| \|T^{-1}\| \leq 1 + \varepsilon$. By Proposition 2.14, $C(V) \leq (1 + \varepsilon) C(V')$ and so $C(V) \leq (1 + \varepsilon) C(B')$ because $C(V') \leq C(B')$. It follows from (5.4) that $C(B) \leq (1 + \varepsilon) C(B')$. Since $\varepsilon$ is arbitrary, we conclude $C(B) \leq C(B')$. □

Corollary 5.6. Let $B$ be a Banach space and $B^{**}$ its second dual. Then $C(B) = C(B^{**})$.

Proof. The canonical map $B \rightarrow B^{**}$ is an isometric embedding and hence $C(B) \leq C(B^{**})$. In any Banach space $B$, the second dual $B^{**}$ is finitely representable in in $B$, [JL01, §9], and so by Proposition 5.5, $C(B^{**}) \leq C(B)$. It follows that $C(B) = C(B^{**})$. □

The James constant of a Banach space $B$ is defined by:

$$J(B) = \sup \{ \min(\|x - y\|, \|x + y\|) \mid \|x\| = \|y\| = 1 \}$$

If $\|x\| = \|y\| = 1$ then $\Delta(x, y, 0, x + y) = \frac{1}{2}(\|x - y\| + \|x + y\|)$ and thus

$$C(B) \geq \sup \left\{ \frac{1}{2}(\|x - y\| + \|x + y\|) \mid \|x\| = \|y\| = 1 \right\} \geq J(B) \tag{5.7}$$

A Banach space $B$ is said to be non-trivial if $\dim(B) \geq 2$.

Theorem 5.8. If $B$ is any non-trivial Banach space then $C(B) \geq \sqrt{2}$. If $\dim B \geq 3$ and $C(B) = \sqrt{2}$ then $B$ is a Hilbert space.

Proof. Gao and Lau, [GL90, Theorem 2.5], show $J(B) \geq \sqrt{2}$ for any non-trivial Banach space $B$. Furthermore, Komuro, Saito and Tanaka, [KST16], show that $\dim B \geq 3$ and $J(B) = \sqrt{2}$ implies $B$ is a Hilbert space. The conclusion of the theorem follows from (5.7). □

Definition 5.9 ([Enf69]). Let $(X, d)$ be a metric space and $p \geq 1$. The space $(X, d)$ is said to be $p$-round if for all $x, y, z, w \in X$, $(xy)^p + (zw)^p \leq (xz)^p + (yw)^p + (xw)^p + (yz)^p$. The roundness of $(X, d)$ is $r(X, d) = \sup \{ p \mid (X, d)$ is $p$-round $\}$.

Note that if $r(X, d) < \infty$ then the supremum is attained. Enflo, [Enf69], observed that $r(X, d) \geq 1$ and that if $(X, d)$ has the midpoint property\footnote{A metric space $(X, d)$ has the midpoint property if for every $x, y \in X$ there exists $z \in X$ such that $d(x, z) = d(z, y) = \frac{1}{2}d(x, y)$.} then $r(X, d) \leq 2$. In particular, if $B$ is a Banach space then $1 \leq r(B) \leq 2$, where $r(B)$ is the roundness of $B$ as a metric space.

Lemma 5.10. Let $B$ be a Banach space that is $p$-round. Then for any vectors $e, f \in B$

$$\quad (\|e\| + \|f\|)^p \leq \|e - f\|^p + \|e + f\|^p.$$
Proof. In the “p-round inequality” of Definition 5.9, letting \( x = e + f, \ y = e - f, \ w = 2e, \) and \( z = 0 \) gives
\[
\|2e\|^p + \|2f\|^p \leq 2\|e - f\|^p + 2\|e + f\|^p
\]
and so
\[
2^{p-1} (\|e\|^p + \|f\|^p) \leq \|e - f\|^p + \|e + f\|^p.
\]
By (5.1), with \( n = 2, (\|e\| + \|f\|)^p \leq 2^{p-1} (\|e\|^p + \|f\|^p) \) from which the conclusion follows. \( \square \)

**Theorem 5.11.** If \( B \) is a Banach space then \( C(B) \leq 2^{1/r(B)} \).

*Proof.* Let \( p = r(B) \). Then \( B \) is \( p \)-round. Let \( x, y, z, w \in B \). Let \( a = x - z, \ b = w - y, \ c = w - x, \ d = y - z, \ e = y - x, \) and \( f = w - z \). Note that \( f = a + c = b + d \) and \( e = d - a = c - b \). Hence \( e + f = c + d \) and \( f - e = a + b \). By Lemma 5.10,
\[
(\|e\| + \|f\|)^p \leq \|e - f\|^p + \|e + f\|^p
\]
\[
= \|a + b\|^p + \|c + d\|^p
\]
\[
\leq (\|a\| + \|b\|)^p + (\|c\| + \|d\|)^p \quad \text{(by triangle inequality)}.
\]
It follows that
\[
\|e\| + \|f\| \leq ((\|a\| + \|b\|)^p + (\|c\| + \|d\|)^p)^{1/p}
\]
\[
\leq 2^{1/p} \max (\|a\| + \|b\|, \|c\| + \|d\|) \quad \text{(by (5.1))}.
\]
Thus the \((2^{1/p}, 0)\)-four-point inequality holds and so \( C(B) \leq 2^{1/p} \). \( \square \)

**Corollary 5.12.** Let \((\Omega, \Sigma, \mu)\) be a separable measure space, that is, the \( \sigma \)-algebra \( \Sigma \) is generated by a countable collection of subsets of \( \Omega \). Let \( 1 \leq p \leq \infty \) and let \( L_p(\Omega, \Sigma, \mu) \) be the corresponding \( L_p \)-space. If \( \dim L_p(\Omega, \Sigma, \mu) \geq 2 \) then \( C(L_p(\Omega, \Sigma, \mu)) = \max\{2^{1/p}, 2^{1-1/p}\} \).

*Proof.* Denote \( B = L_p(\Omega, \Sigma, \mu) \). Assume \( \dim B \geq 2 \). In the case \( 1 \leq p \leq 2 \), Enflo, [Enf69], showed that \( r(B) = p \) and so \( C(B) \leq 2^{1/p} \) by Theorem 5.11. In the case \( 2 \leq p \leq \infty \), by [LTW97, Proposition 1.4 and Remark 1.5], \( r(B) = 1/(1 - 1/p) \) and so \( C(B) \leq 2^{1-1/p} \) by Theorem 5.11. Hence for \( 1 \leq p \leq \infty \), \( C(B) \leq \max\{2^{1/p}, 2^{1-1/p}\} \).

The classification theory of \( L_p \) spaces (see [JL01, §4]) gives that, for \( 1 \leq p < \infty \), the space \( B = L_p(\Omega, \Sigma, \mu) \) is isometric to one of the Banach spaces in the list
\[
\ell_p^n, \ell_p, L_p(0, 1), \ell_p \oplus_p L_p(0, 1), \ell_p^n \oplus_p L_p(0, 1) \quad n = 1, 2, \ldots
\]
Here, \( \ell_p \) denotes the space of sequences \((x_n)_{n=1}^{\infty}\) with \( \sum_{n=1}^{\infty} |x_n|^p < \infty \) and \( L_p(0, 1) \) denotes the space of measurable functions (modulo null sets) on the unit interval such that \( \int_0^1 |f(x)|^p dx < \infty \), and \( \oplus \) denotes the \( \ell_p \) direct sum, that is, \( \|a \oplus b\| = (\|a\|^p + \|b\|^p)^{1/p} \). Each of the spaces in the list \((5.13)\) (in the case of \( \ell_p^n \), assume \( n \geq 2 \)) contains a subspace isometric to \( \ell_p^2 \) and so \( C(B) \geq C(\ell_p^2) = \max\{2^{1/p}, 2^{1-1/p}\} \) by Proposition 5.2. Hence \( C(B) = \max\{2^{1/p}, 2^{1-1/p}\} \). In the case \( p = \infty \) note that \( B \) contains a subspace isometric to \( \ell_\infty^2 \) which implies that \( C(B) = 2 \). \( \square \)
Question 5.14. Let \((X, d)\) be a geodesic metric space. Is \(C_0(X, d) \leq 2^{1/r(X,d)}\)?

By Proposition 4.5, this is true in the case \(r(X, d) = 2\).

6. Snowflaked metric spaces

Recall that if \(0 < \alpha \leq 1\) and \((X, d)\) is any metric space then \((X, d^\alpha)\) is also a metric space, called the \(\alpha\)-snowflake of \((X, d)\). We show that \(C_0(X, d^\alpha) \leq 2^\alpha\), Theorem 6.2, and give some applications of this estimate. We determine the quasi-hyperbolicity constant of the \(\alpha\)-snowflake of the Euclidean real line, Theorem 6.6.

Lemma 6.1. Let \(a_{ij} \in \mathbb{R}, i, j \in \{1, 2, 3, 4\}\), be such that \(a_{ij} = a_{ji}\). Let \(\lambda \geq 1\). If \(a_{ij} \leq \lambda \max\{a_{ik}, a_{kj}\}\) for all \(i, j, k\), then \(a_{ij} + a_{kl} \leq \lambda \max\{a_{ik} + a_{j\ell}, a_{i\ell} + a_{jk}\}\) for all \(i, j, k, \ell\).

Note that if \(L, M\) and \(S\) denote the largest, medium and smallest of the three sums \(a_{ij} + a_{kl}, a_{ik} + a_{j\ell}\) and \(a_{i\ell} + a_{jk}\) for some choice of \(i, j, k, \ell \in \{1, 2, 3, 4\}\), then the conclusion of the lemma is equivalent to \(L \leq \lambda M\).

Proof. Fix \(i, j, k, \ell \in \{1, 2, 3, 4\}\). Without loss of generality, assume that \(L = a_{ij} + a_{kl}\) is the largest sum and assume that \(a_{kl} \leq a_{ij}\). Since \(a_{ij} \leq \lambda \max\{a_{ik}, a_{kj}\}\) and \(a_{ij} \leq \lambda \max\{a_{i\ell}, a_{k\ell}\}\), we have
\[
a_{ij} + a_{kl} \leq a_{ij} + a_{ij} \leq \lambda \max\{a_{ik} + a_{i\ell}, a_{ik} + a_{j\ell}, a_{kj} + a_{i\ell}, a_{kj} + a_{k\ell}\}.
\]
If \(a_{ik} \geq a_{kj}\) and \(a_{i\ell} \geq a_{k\ell}\) then
\[
M = a_{ik} + a_{i\ell} = \max\{a_{ik} + a_{i\ell}, a_{ik} + a_{j\ell}, a_{kj} + a_{i\ell}, a_{kj} + a_{k\ell}\}
\]
and if \(a_{ik} \leq a_{kj}\) and \(a_{i\ell} \leq a_{k\ell}\) then
\[
M = a_{kj} + a_{k\ell} = \max\{a_{ik} + a_{i\ell}, a_{ik} + a_{j\ell}, a_{kj} + a_{i\ell}, a_{kj} + a_{k\ell}\}.
\]
In both cases, \(L \leq \lambda M\). Furthermore, if \(a_{ik} \geq a_{kj}\) and \(a_{i\ell} \leq a_{k\ell}\) then \(a_{ij} \leq \lambda \max\{a_{ik}, a_{kj}\} = \lambda a_{ik}\) and \(a_{ij} \leq \lambda \max\{a_{i\ell}, a_{k\ell}\} = \lambda a_{i\ell}\), and since \(a_{kl} \leq \lambda \max\{a_{kj}, a_{k\ell}\}\),
\[
a_{ij} + a_{kl} \leq a_{ij} + \lambda \max\{a_{kj}, a_{k\ell}\} = \max\{a_{ij} + \lambda a_{kj}, a_{ij} + \lambda a_{k\ell}\}
\]
\[
\leq \lambda \max\{a_{i\ell} + \lambda a_{kj}, a_{kj} + \lambda a_{k\ell}\} = \lambda \max\{a_{i\ell} + a_{kj}, a_{ik} + a_{k\ell}\}.
\]
Finally, if \(a_{ik} \leq a_{kj}\) and \(a_{i\ell} \geq a_{k\ell}\) then
\[
a_{ij} \leq \lambda \max\{a_{ik}, a_{kj}\} = \lambda a_{kj}\;
\]
and since \(a_{kl} \leq \lambda \max\{a_{ki}, a_{i\ell}\}\), we have
\[
a_{ij} + a_{kl} \leq a_{ij} + \lambda \max\{a_{ki}, a_{i\ell}\} \leq \lambda \max\{a_{ij} + a_{ki}, a_{kj} + a_{i\ell}\},
\]
that is, \(L \leq \lambda M\). \(\square\)

Theorem 6.2. Let \(0 < \alpha \leq 1\). For any metric space \((X, d)\), \(C_0(X, d^\alpha) \leq 2^\alpha\).
The conclusion follows from Lemma 6.1 with \( a_{ij} = (x_i,x_j)^\alpha \) and \( \lambda = 2^\alpha \).

As in §5, \(d_p\), where \(1 \leq p \leq \infty\), denotes the metric on \(\mathbb{R}^n\) determined by the standard \(p\)-norm.

**Proposition 6.3.** If \(0 < \alpha \leq 1\) and \(n \geq 2\) then \(C(\mathbb{R}^n,d_\infty^\alpha) = 2^\alpha\).

**Proof.** By Proposition 6.2, \(C(\mathbb{R}^n,d_\infty^\alpha) \leq 2^\alpha\). Consider following the four points in \(\mathbb{R}^n\):

\[
x = (0,1,0,\ldots,0),\ y = (0,-1,0,\ldots,0),\ z = (-1,0,\ldots,0),\ w = (1,0,\ldots,0).
\]

A calculation using the metric \(d_\infty^\alpha\) yields \(\Delta(x,y,z,w) = 2^\alpha\) and thus \(C(\mathbb{R}^n,d_\infty^\alpha) \geq 2^\alpha\). Hence \(C(\mathbb{R}^n,d_\infty^\alpha) = 2^\alpha\). By Corollary 2.10, \(C(\mathbb{R}^n,d_\infty^\alpha) = C_0(\mathbb{R}^n,d_\infty^\alpha)\). \(\square\)

The same technique gives a non-sharp estimate for \(C(\mathbb{R}^n,d_2^\alpha)\), where \(n \geq 2\), as follows.

**Proposition 6.4.** If \(0 < \alpha \leq 1\) and \(n \geq 2\) then \(2^{\alpha/2} \leq C(\mathbb{R}^n,d_2^\alpha) \leq 2^{\min\{\alpha,1/2\}}\).

**Proof.** By Proposition 6.2, \(C(\mathbb{R}^n,d_2^\alpha) \leq 2^\alpha\). Schoenberg showed, [Sch37, Theorem 1], that \((\mathbb{R}^n,d_2^\alpha)\) isometrically embeds into (infinite dimensional) Hilbert space and hence \(C(\mathbb{R}^n,d_2^\alpha) \leq 2^{1/2}\). Consequently, \(C(\mathbb{R}^n,d_2^\alpha) \leq 2^{\min\{\alpha,1/2\}}\). For the four points \(x,y,z,w \in \mathbb{R}^n\) specified in the proof of Proposition 6.3, we have \(\Delta(x,y,z,w) = 2^{\alpha/2}\), yielding the lower bound for \(C(\mathbb{R}^n,d_2^\alpha)\). \(\square\)

Numerical calculations suggest the following exact value for \(C(\mathbb{R}^n,d_2^\alpha)\) when \(n \geq 2\).

**Conjecture 6.5.** Let \(0 < \alpha < 1\). If \(n \geq 2\) then \(C(\mathbb{R}^n,d_2^\alpha) = 2^{\alpha/2}\).

The \(\alpha\)-snowflakes of the Euclidean line turns out to be of a different nature than the spaces \((\mathbb{R}^n,d_2^\alpha)\) with \(n \geq 2\), as revealed in the following theorem.

**Theorem 6.6.** Let \(0 < \alpha \leq 1\) and \(d_E^\alpha(x,y) = |x-y|^\alpha, x,y \in \mathbb{R}\). Let \(m \geq 1\) be the unique solution to the equation \((m-1)^\alpha + (m+1)^\alpha = 2\). Then \(C(\mathbb{R}^1,d_E^\alpha) = m^\alpha\).

Observe that by Corollary 2.10,

\[
(6.7)\quad C(\mathbb{R}^1,d_E^\alpha) = C_0(\mathbb{R}^1,d_E^\alpha) = \sup \Delta(x,y,z,w),
\]

where

\[
\Delta(x,y,z,w) = \frac{|x-y|^\alpha + |z-w|^\alpha}{\max\{|x-z|^\alpha + |y-w|^\alpha, |x-w|^\alpha + |y-z|^\alpha\}}
\]
and the supremum in (6.7) is taken over all \( x, y, z, w \in \mathbb{R} \), not all identical. Since the map \((x, y, z, w) \mapsto \Delta(x, y, z, w)\) is translation and scale invariant, we may assume that \( x = 0, y = 1 + s, z = 1 - t, \) and \( w = 2 \), with \((t, s) \in D = \{(t, s) \in [-1, 1] \times [-1, 1] \mid t + s \geq 0\}\). Then

\[
(t, s) \mapsto \Delta(0, 1 + s, 1 - t, 2) = \frac{(1 + s)^\alpha + (1 + t)^\alpha}{\max_{(t, s) \in D} \{(1 - t)^\alpha + (1 - s)^\alpha, (t + s)^\alpha + 2^\alpha\}}
\]

is continuous on the compact set \( D \) and

\[
C(\mathbb{R}^4, d_E^\alpha) = \max_{(t, s) \in D} \Delta(0, 1 + s, 1 - t, 2).
\]

Furthermore, if \( F, G: D \to \mathbb{R} \) are given by

\[
F(t, s) = \frac{(1 + t)^\alpha + (1 + s)^\alpha}{(1 - t)^\alpha + (1 - s)^\alpha} \quad \text{and} \quad G(t, s) = \frac{(1 + t)^\alpha + (1 + s)^\alpha}{(t + s)^\alpha + 2^\alpha},
\]

and \( D_1 = \{(t, s) \in D \mid F(t, s) \leq G(t, s)\} \) and \( D_2 = \{(t, s) \in D \mid F(t, s) \geq G(t, s)\} \), then

\[
\Delta(0, 1 - t, 1 + s, 2) = \min_{(t, s) \in D} \{ F(t, s), G(t, s) \} = \begin{cases} F(t, s), & (t, s) \in D_1 \\ G(t, s), & (t, s) \in D_2, \end{cases}
\]

and

\[
C(\mathbb{R}^4, d_E^\alpha) = \max \left\{ \max_{(t, s) \in D_1} F(t, s), \max_{(t, s) \in D_2} G(t, s) \right\}.
\]

The following lemma shows that the maximum in (6.9) is attained on \( D_0 = D_1 \cap D_2 \).

**Lemma 6.10.** Let \( 0 < \alpha < 1 \). Let \( F, G: D \to \mathbb{R} \) be given by (6.8) and let \( D_0 = \{(t, s) \in D \mid F(t, s) = G(t, s)\} \). Then

\[
C(\mathbb{R}^4, d_E^\alpha) = \max_{(t, s) \in D_0} F(t, s).
\]

**Proof.** We show that \( F \) and \( G \) attain their maximum on the boundary of \( D_1 \) and \( D_2 \), respectively. Indeed, the partial derivatives of \( F \),

\[
F_t(t, s) = \frac{\alpha(1 + t)^{\alpha - 1}}{(1 - t)^\alpha + (1 - s)^\alpha} + \frac{\alpha((1 + t)^\alpha + (1 + s)^\alpha)(1 - t)^{\alpha - 1}}{((1 - t)^\alpha + (1 - s)^\alpha)^2}
\]

\[
F_s(t, s) = \frac{\alpha(1 + s)^{\alpha - 1}}{(1 - t)^\alpha + (1 - s)^\alpha} + \frac{\alpha((1 + t)^\alpha + (1 + s)^\alpha)(1 - s)^{\alpha - 1}}{((1 - t)^\alpha + (1 - s)^\alpha)^2}
\]

are defined for all \((t, s) \in (-1, 1)^2, t + s > 0 \) and \( F_t > 0 \) and \( F_s > 0 \). Thus \( \max_{(t, s) \in D_1} F(t, s) \) is attained on the boundary \( \partial D_1 = D_0 \cup \{(t, s) \in D \mid t + s = 1\} \). Note that \( F(t, s) \geq 1 \) for \((t, s) \in D \) and \( F(t, s) = 1 \) if and only if \( t + s = 1 \). Hence

\[
\max_{(t, s) \in D_1} F(t, s) = \max_{(t, s) \in D_0} F(t, s).
\]

The partial derivatives of \( G \),

\[
G_t(t, s) = \frac{\alpha(1 + t)^{\alpha - 1}}{(t + s)^\alpha + 2^\alpha} - \frac{\alpha((1 + t)^\alpha + (1 + s)^\alpha)(t + s)^{\alpha - 1}}{((t + s)^\alpha + 2^\alpha)^2}
\]

\[
G_s(t, s) = \frac{\alpha(1 + s)^{\alpha - 1}}{(t + s)^\alpha + 2^\alpha} - \frac{\alpha((1 + t)^\alpha + (1 + s)^\alpha)(t + s)^{\alpha - 1}}{((t + s)^\alpha + 2^\alpha)^2}
\]

are defined for all \((t, s) \in (-1, 1)^2, t + s > 0 \) and \( F_t > 0 \) and \( F_s > 0 \). Thus \( \max_{(t, s) \in D_1} F(t, s) \) is attained on the boundary \( \partial D_1 = D_0 \cup \{(t, s) \in D \mid t + s = 1\} \). Note that \( F(t, s) \geq 1 \) for \((t, s) \in D \) and \( F(t, s) = 1 \) if and only if \( t + s = 1 \). Hence

\[
\max_{(t, s) \in D_1} F(t, s) = \max_{(t, s) \in D_0} F(t, s).
\]

The partial derivatives of \( G \),
are defined for all \((t, s) \in (-1, 1)^2, t + s > 0\) and \(G_t = G_s = 0\) if and only if \(t = s = 1\). Thus \(\max_{(t,s) \in D_2} G(t, s)\) is attained on the boundary \(\partial D_2 = D_0 \cup \{(t, s) \in D \mid t = 1\} \cup \{(t, s) \in D \mid s = 1\}\). Note also that \(G(t, s) \geq 1\) for \((t, s) \in D\) and \(G(t, s) = 1\) if and only if \(t = 1\) or \(s = 1\). Hence

\[
(6.12) \quad \max_{(t,s) \in D_2} G(t, s) = \max_{(t,s) \in D_0} G(t, s).
\]

The conclusion follows from (6.9) together with (6.11) and (6.12). \(\square\)

The following result shows that \(\max_{(t,s) \in D_0} F(t, s)\) is attained when \(t = s\).

**Lemma 6.13.** Let \(0 < \alpha < 1\). Let \(F, G : D \to \mathbb{R}\) be given by (6.8) and let \(D_0 = \{(t, s) \in D \mid F(t, s) = G(t, s)\}\). Then

\[
\max_{(t,s) \in D_0} F(t, s) = \left(\frac{1 + a}{1 - a}\right)^\alpha,
\]

where \(0 < a < 1\) is the unique solution of \(F(a, a) = G(a, a)\).

**Proof.** Notice that if \((t, s) \in D_0\) then \(t = -1\) if and only if \(s = 1\) and \(F(-1, 1) = 1\). By symmetry, \(F(1, -1) = 1\). Since \(F(t, s) \geq 1\) on \(D_0\), the maximum of \(F\) on \(D_0\), is not attained at \((-1, 1)\) or \((1, -1)\). Let \((a, b) \in D_0\), with \(a \neq \pm 1\). If \(F\) attains a local extremum at \((a, b)\) subject to the constrain \(F(t, s) = G(t, s)\), then the level curves \(\{(t, s) \in D \mid F(t, s) = F(a, b)\}\) and \(\{(t, s) \in D \mid F(t, s) - G(t, s) = 0\}\) are both tangent at \((a, b)\). Since \(F_s(a, b) - G_s(a, b) \neq 0\), by the Implicit Function Theorem, there exists an open neighbourhood \(U \subseteq (-1, 1)\) of \(a\) and a function \(\omega = \omega(t)\) such that \(F(t, \omega(t)) - G(t, \omega(t)) = 0\) for \(t \in U\). Furthermore,

\[
\omega'(t) = -\frac{(t + \omega)^\alpha - (1 - t)^\alpha}{(t + \omega)^\alpha + (1 - \omega)^\alpha}
\]

for all \(t \in U\). Similarly, since \(F_s(a, b) \neq 0\), there exists an open neighbourhood \(V \subseteq (-1, 1)\) of \(a\) and a function \(\nu = \nu(t)\) on \(V\) such that \(F(t, \nu(t)) = F(a, b)\) on \(V\). Also, for all \(t \in V\),

\[
\nu'(t) = -\frac{(1 + t)^\alpha - F(a, b)(1 - t)^\alpha}{(1 + \nu)^\alpha + F(a, b)(1 - \nu)^\alpha}.
\]

Hence, a necessary condition for \((a, b)\) to be a point of local extremum for \(F|_{\partial D_0}\) is that \(\omega'(a) = \nu'(a)\). Using that \(\omega(a) = \nu(a) = b\), that is,

\[
\frac{(a + b)^\alpha - (1 - a)^\alpha}{(a + b)^\alpha + (1 - b)^\alpha} = \frac{(1 + a)^\alpha - F(a, b)(1 - a)^\alpha}{(1 + b)^\alpha + F(a, b)(1 - b)^\alpha}.
\]

Equivalently,

\[
(a + b)^\alpha \left[ (1 + b)^\alpha F(a, b)(1 - b)^\alpha - (1 + a)^\alpha F(a, b)(1 - a)^\alpha - (1 + b)^\alpha - (1 - b)^\alpha (1 + a)^\alpha \right] = 0.
\]
Using that \( F(a, b) = \frac{(1+a)^\alpha + (1+b)^\alpha}{(1-a)^\alpha + (1-b)^\alpha} = \frac{(1+a)^\alpha + (1+b)^\alpha}{(a+b)^\alpha + 2^\alpha} \), the above equality holds if and only if

\[
(a + b)^{-1} \{(1 + b)^{-1} - (1 + a)^{-1}\} \{(1 - a)^{\alpha} + (1 - b)^{\alpha}\} \\
\quad + (1 - b)^{-1} - (1 - a)^{\alpha}\} \{(1 + a) + (1 + b)^{\alpha}\} \\
\quad + (1 - a)^{-1} (1 + b)^{-1} - (1 + a)^{\alpha - 1} (1 - b)^{-1} \{(a + b)^{\alpha} + 2^\alpha\} = 0,
\]
equivalently,

\[
2(a + b)^{\alpha - 1} \{(1 - b^2)^{\alpha - 1} - (1 - a^2)^{\alpha - 1}\} + 2^\alpha \{(1 - a)^{\alpha - 1}(1 + b)^{\alpha - 1} - (1 + a)^{\alpha - 1}(1 - b)^{\alpha - 1}\} = 0
\]

Factoring out \(2(a + b)^{\alpha - 1}(1 - b^2)^{\alpha - 1} \neq 0\) yields

\[
(6.14) \quad 1 - \left(\frac{1-b}{1-a}\right)^{1-\alpha} \left(\frac{1+b}{1+a}\right)^{1-\alpha} - \left(\frac{a+b}{2}\right)^{1-\alpha} \left[\left(\frac{1+b}{1+a}\right)^{1-\alpha} - \left(\frac{1-b}{1-a}\right)^{1-\alpha}\right] = 0
\]

Assume \(a < b\). Since \(a + b > 0\), this implies \(b > 0\) and \(-b < a < b\). In particular, \(a^2 < b^2\). Let \(x = \frac{1-b}{1-a}\) and \(y = \frac{1+b}{1+a}\). Then \(0 < x < 1 < y\), and \(0 < xy < 1\). Note that \(\frac{a+b}{x} = \frac{1-x}{y}\). We claim that the expression on the left hand side of (6.14) is negative. That is, we claim,

\[
1 - (xy)^{1-\alpha} - \left(\frac{1-xy}{y-x}\right)^{1-\alpha} \left(y^{1-\alpha} - x^{1-\alpha}\right) < 0.
\]

Indeed, multiplying the above inequality by \((1-xy)^{\alpha-1} > 0\) yields

\[
\frac{1 - (xy)^{1-\alpha}}{(1-xy)^{1-\alpha}} - \frac{y^{1-\alpha} - x^{1-\alpha}}{(y-x)^{1-\alpha}} < 0,
\]
equivalently,

\[
\frac{1 - (xy)^{1-\alpha}}{(1-xy)^{1-\alpha}} - \frac{1 - (x/y)^{1-\alpha}}{(1-x/y)^{1-\alpha}} < 0
\]

which is valid since the function \(t \mapsto \frac{1-t^{1-\alpha}}{(1-t)^{1-\alpha}}, 0 < t < 1\), is decreasing and \(0 < x/y < xy < 1\).

Note that the expression on the left hand side of (6.14) is positive if \(a > b\). Thus, (6.14) holds if and only if \(a = b\). Finally, notice that \(F(a,a) = G(a,a)\) has unique solution \(0 < a < 1\). Since \(F(a,a) = [(1+a)/(1-a)]^\alpha > 1\), the conclusion follows. \(\square\)

**Proof of Theorem 6.6.** If \(\alpha = 1\), the conclusion holds with \(m = 1\) by Proposition 2.6, since the space \((\mathbb{R}^1, d_E)\) is 0-hyperbolic. Let \(0 < \alpha < 1\). By Lemmas 6.10 and 6.13

\[
C(\mathbb{R}^1, d_E^\alpha) = \max_{(t,s) \in D_0} F(t,s) = F(a,a) = \left(\frac{1+a}{1-a}\right)^\alpha = m^\alpha
\]

where \(m = \frac{1+a}{1-a} > 1\) is the unique solution of

\[
2 = \left(\frac{1+a-(1-a)}{1-a}\right)^\alpha + \left(\frac{1+a+(1-a)}{1-a}\right)^\alpha = (m-1)^\alpha + (m+1)^\alpha.
\]

**Remark 6.15.** Let \(0 < \alpha \leq 1\). It is **not** true in general that for any metric space \((X,d)\) the inequality \(C(X,d^\alpha) \leq (C(X,d))^\alpha\) holds. For example, if \(\alpha = 1/2\) then \(m = 5/4\) as in Theorem 6.6 and so

\[
C(\mathbb{R}^1, d_E^{1/2}) = \sqrt{5}/2 > (C(\mathbb{R}^1, d_E))^{1/2} = \sqrt{1} = 1.
\]
7. Distances on Riemannian manifolds

We show that the restricted quasi-hyperbolicity constant of the metric space associated to a Riemannian manifold of dimension greater than one is bounded from below by $\sqrt{2}$.

**Proposition 7.1.** If $M$ is a Riemannian manifold of dimension greater than one and $d_M$ is the distance on $M$ induced by the given Riemannian metric then $C_0(M, d_M) \geq \sqrt{2}$.

**Proof.** Let $p \in M$ and let $\exp_p : T_p M \to M$ denote the Riemannian exponential map. The Riemannian metric on $M$ endows the tangent space, $T_p M$, with an inner product and we write $d_E$ for the corresponding Euclidean distance on $T_p M$. For a vector $X \in T_p M$ and a scalar $t$, let $X_t = \exp_p(tX) \in M$. If $X, Y \in T_p M$ then

\[(7.2) \lim_{t \to 0} \frac{d_M(X_t, Y_t)}{t} = d_E(X, Y).\]

This is a consequence of the fact that in normal coordinates $\{x^i\}$ the components $g_{ij}(x)$ of the Riemannian metric satisfy the estimate $|g_{ij}(x) - \delta_{ij}| \leq C\|x\|^2$ for some $C$.

For $X, Y, Z, W \in T_p M$, not all identical,

\[\Delta(X_t, Y_t, Z_t, W_t) = \frac{d_M(X_t, Y_t) + d_M(Z_t, W_t)}{\max\{d_M(X_t, Z_t) + d_M(Y_t, W_t), d_M(X_t, W_t) + d_M(Y_t, Z_t)\}} \leq \frac{d_M(X_t, Y_t) + d_M(Z_t, W_t)}{\max\{d_M(X_t, Z_t)/t + d_M(Y_t, W_t)/t, d_M(X_t, W_t)/t + d_M(Y_t, Z_t)/t}\}

By (7.2), $\lim_{t \to 0} \Delta(X_t, Y_t, Z_t, W_t) = \Delta(X, Y, Z, W)$, where the second $\Delta$ is with respect to $d_E$. Since $\dim M > 1$, there are orthogonal unit vectors $U, V \in T_p M$. Since

\[C_0(M, d_M) \geq \Delta(U_t, V_t, 0_t, (U + V)_t),\]

it follows that

\[C_0(M, d_M) \geq \lim_{t \to 0} \Delta(U_t, V_t, 0_t, (U + V)_t) = \Delta(U, V, 0, U + V) = \sqrt{2},\]

establishing the conclusion of the proposition. \qed

**Corollary 7.3.** Let $M$ be a simply connected, complete Riemannian manifold of non-positive sectional curvature with associated distance $d_M$. Then $C_0(M, d_M) = \sqrt{2}$.

**Proof.** By [BH99], Chapter II.1, Theorem 1A.6], the metric space $(M, d_M)$ is a CAT(0)-space and so $C_0(M, d_M) \leq \sqrt{2}$ by Corollary 4.3. By Proposition 7.1, $C_0(M, d_M) \geq \sqrt{2}$. Thus $C_0(M, d_M) = \sqrt{2}$. \qed

**References**


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