THE QUASI-HYPERBOLICITY CONSTANT OF A METRIC SPACE

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ABSTRACT. We introduce the *quasi-hyperbolicity constant* of a metric space, a rough isometry invariant that measures how a metric space deviates from being Gromov hyperbolic. This number, for unbounded spaces, lies in the closed interval $[1, 2]$. The quasi-hyperbolicity constant of an unbounded Gromov hyperbolic space is equal to one. For a CAT(0)-space, it is bounded from above by $\sqrt{2}$. The quasi-hyperbolicity constant of a Banach space that is at least two dimensional is bounded from below by $\sqrt{2}$, and for a non-trivial L_p-space it is exactly $\max\{2^{1/p}, 2^{1-1/p}\}$. If $0 < \alpha < 1$ then the quasi-hyperbolicity constant of the α-snowflake of any metric space is bounded from above by 2^α. We give an exact calculation in the case of the α-snowflake of the Euclidean real line.

1. Introduction

Gromov hyperbolic spaces were introduced by Gromov in his seminal paper [Gro87] to study infinite groups as geometric objects. For a metric space (X, d), we use the abbreviated notation $xy = d(x, y)$ where convenient. Recall that for three points x, y, w in a metric space (X, d), the *Gromov product* of x and y with respect to w is defined as

$$(x \mid y)_w = \tfrac{1}{2}\,(xw + yw - xy)\,.$$

Given a non-negative constant δ, the metric space (X, d) is said to be δ*-hyperbolic* if

$$(x \mid y)_w \geq \min\{(x \mid z)_w, (y \mid z)_w\} - \delta$$

for all $x, y, z, w \in X$. A metric space (X, d) is said to be *Gromov hyperbolic* if it is δ-hyperbolic for some δ. Any $\mathbb{R}$-tree is 0-hyperbolic. Another well-known example is the hyperbolic plane, which is $\log(2)$-hyperbolic, [NŠ16, Corollary 5.4]. Euclidean spaces of dimension greater than one are not Gromov hyperbolic. While Gromov hyperbolicity is a quasi-isometry invariant for *intrinsic* metric spaces [Väi05, Theorems 3.18 and 3.20], quasi-isometry invariance can fail for non-intrinsic spaces, see [Väi05, Remark 3.19] and also our examples in §3. In particular, a metric space that quasi-isometrically embeds into a Gromov hyperbolic space need not be Gromov hyperbolic.

A metric space (X, d) is δ-hyperbolic if and only if the *the four-point inequality* holds, that is, for all $x, y, z, w \in X$,

$$xy + zw \leq \max\{xz + yw,\ yz + xw\} + 2\delta,$$

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see [Väi05, (2.12)].

We generalize the four-point inequality as follows. Let (X,d) be a metric space. Let $\mu, \delta \geq 0$. We say that a metric space (X,d) satisfies the (μ,δ)-*four-point inequality* if for all $x,y,z,w \in X$,

$$xy + zw \leq \mu \max\{xz + yw,\, xw + yz\} + 2\delta.$$

In particular, (X,d) is δ-hyperbolic if and only if it satisfies the $(1,\delta)$-four-point inequality.

We introduce the following numerical constants associated to a metric space.

Definition (Quasi-hyperbolicity constants)**.** Let (X,d) be a metric space.

(i) The *quasi-hyperbolicity constant* of (X,d) is the number

$$C(X,d) = \inf\{\mu \mid \text{there exists } \delta \geq 0 \text{ such that } (X,d) \text{ satisfies the } (\mu,\delta)\text{-four-point inequality}\}.$$

(ii) The *restricted quasi-hyperbolicity constant* of (X,d) is the number

$$C_0(X,d) = \inf\{\mu \mid (X,d) \text{ satisfies the } (\mu,0)\text{-four-point inequality}\}.$$

Some basic properties of the quasi-hyperbolicity and restricted quasi-hyperbolicity constants of a metric space (X,d) are readily derived, for example:

- $C(X,d) \leq C_0(X,d) \leq 2$,
- if (X,d) is bounded then $C(X,d) = 0$, otherwise $C(X,d) \geq 1$,
- if (X,d) has at least two points then it is 0-hyperbolic if and only if $C_0(X,d) = 1$,
- if (X,d) is Gromov hyperbolic and unbounded then $C(X,d) = 1$.

Proofs of these and more properties are given in §2. In the absence of additional hypotheses, it is not true that $C(X,d) = 1$ implies (X,d) is Gromov hyperbolic. For example, given $0 < \alpha < 1$, consider the graph, Y_α, of $y = x^\alpha$, $x \geq 0$, as a subspace of the Euclidean plane, $(\mathbb{R}^2, d_E)$. We show $C(Y_\alpha, d_E) = 1$, Proposition 3.4, however Y_α is *not* Gromov hyperbolic if and only if $1/2 < \alpha < 1$, Propositions 3.1 and 3.3. Nevertheless, if (X,d) is a proper CAT(0)-space and $C(X,d) = 1$ then (X,d) is Gromov hyperbolic, see Proposition 3.8 and Question 3.7.

The appearance of a possibly positive δ in a (μ,δ)-four-point inequality suggests that $C(X,d)$ can be insensitive to small scales. Indeed, $C(X,d)$ is a rough isometry invariant of (X,d), Corollary 2.15. Quasi-isometry is a less stringent condition than rough isometry and $C(X,d)$ is *not* a quasi-isometry invariant of (X,d). Examples of this phenomenon are given in §3.

While the restricted quasi-hyperbolicity constant, $C_0(X,d)$, is obviously an isometry invariant it is not a rough isometry invariant; moreover, the constants $C_0(X,d)$ and $C(X,d)$ need not coincide. For example, if (H^2, d_H) is the hyperbolic plane then $C(H^2, d_H) = 1 < \sqrt{2} = C_0(H^2, d_H)$, see Example 2.11. The intuition supporting this example is that very small quadrilaterals in H^2 are approximately Euclidean and contribute to $C_0(H^2, d_H)$ but not to $C(H^2, d_H)$. For spaces (X,d) that are "four-point scalable in the large" (Definition 2.7) we show, Proposition 2.9, that $C_0(X,d) = C(X,d)$. Examples of such spaces include Banach spaces and their metric snowflakes.

A CAT(0)-space is a geodesic metric space whose geodesic triangles are not fatter than corresponding comparison triangles in the Euclidean plane. Simply connected, complete Riemannian manifolds of non-positive sectional curvature are familiar examples of CAT(0)-spaces. We show, Theorem 4.2, that the restricted quasi-hyperbolicity constant of a metric space whose distance satisfies Ptolemy's inequality and the quadrilateral inequality, in particular any CAT(0)-space, is bounded from above by $\sqrt{2}$. The quasi-hyperbolicity constant of any Euclidean space of dimension greater than one is equal to $\sqrt{2}$, Proposition 4.4.

Banach spaces are a particularly important class of metric spaces and their geometric properties have been extensively studied, [JL01]. For a Banach space B with the metric determined by its norm, we write $C(B)$ for its quasi-hyperbolicity constant. We observe that $C(B) \geq J(B)$ where $J(B)$ is the *James constant* of B, see (5.7). Strong results for the James constant of a Banach space due to Gao and Lau, [GL90], and to Komuro, Saito and Tanaka, [KST16], lead to the following conclusion about $C(B)$.

Theorem (Theorem 5.8)**.** If B is a Banach space with $\dim B > 1$ then $C(B) \geq \sqrt{2}$. If $\dim B \geq 3$ and $C(B) = \sqrt{2}$ then B is a Hilbert space.

Enflo [Enf69] introduced the notion of the *roundness* of a metric space, Definition 5.9, which is a real number greater than or equal to one. We show:

Theorem (Theorem 5.11)**.** If B is a Banach space with roundness $r(B)$ then $C(B) \leq 2^{1/r(B)}$.

This estimate allows us to calculate the quasi-hyperbolicity constant of a non-trivial L_p-space.

Corollary (Corollary 5.12)**.** For a separable measure space (Ω, Σ, μ) and $1 \leq p \leq \infty$, let $L_p(\Omega, \Sigma, \mu)$ be the corresponding L_p-space. If $\dim L_p(\Omega, \Sigma, \mu) \geq 2$ then $C(L_p(\Omega, \Sigma, \mu)) = \max\{2^{1/p}, 2^{1-1/p}\}$.

If (X, d) is any metric space and $0 < \alpha < 1$ then (X, d^α) is also a metric space, called the *α-snowflake* of (X, d). We show, Theorem 6.2, that $C_0(X, d^\alpha) \leq 2^\alpha$. Applying this estimate, we calculate, Proposition 6.3, the quasi-hyperbolicity constant of the α-snowflake of $(\mathbb{R}^n, d_\infty)$, where d_∞ is the L_∞-metric ("max metric") on $\mathbb{R}^n$: For $n \geq 2$, $C(\mathbb{R}^n, d_\infty^\alpha) = 2^\alpha$. The quasi-hyperbolicity constant of the α-snowflake of of the Euclidean line $(\mathbb{R}^1, d_E)$ can be determined by solving an associated optimization problem, yielding the following calculation.

Theorem (Theorem 6.6)**.** Let $0 < \alpha \leq 1$. Let $m \geq 1$ be the unique solution to the equation $(m-1)^\alpha + (m+1)^\alpha = 2$. Then $C(\mathbb{R}^1, d_E^\alpha) = m^\alpha$.

2. Quasi-hyperbolicity and restricted quasi-hyperbolicity constants

We derive basic properties of the quasi-hyperbolicity constant and the restricted quasi-hyperbolicity constant of a metric space and examine their general behavior with regard to quasi-isometric embedding and, respectively, bilipschitz embedding.

Recall the following definition from the introduction.

Definition 2.1. Let $\mu, \delta \geq 0$. We say that a metric space (X,d) satisfies the (μ,δ)-*four-point inequality* if for all $x,y,z,w \in X$,

$$xy + zw \leq \mu \max\{xz + yw,\, xw + yz\} + 2\delta.$$

We make the following elementary observation concerning this definition.

Proposition 2.2. *Let (X,d) be a metric space.*

(i) *(X,d) satisfies the $(2,0)$-four-point inequality,*

(ii) *If (X,d) is unbounded and satisfies the (μ,δ)-four-point inequality then $\mu \geq 1$.*

(iii) *If (X,d) is bounded with diameter D then it satisfies the $(0,D)$-four-point inequality.*

Proof. (i). Let $x,y,z,w \in X$. Triangle inequality and symmetry of the metric yield:

$$xy \leq xz + yz, \quad xy \leq xw + yw, \quad zw \leq xz + xw, \quad zw \leq yz + yw.$$

Adding these four inequalities and dividing by 2 gives $xy + zw \leq (xz + yw) + (xw + zw)$. For real numbers a,b we have $a + b \leq 2\max\{a,b\}$ and so $xy + zw \leq 2\max\{xz + yw,\, xw + zw\}$, that is, the $(2,0)$-four-point inequality is satisfied.

(ii). Assume that X is unbounded and satisfies the (μ,δ)-four-point inequality. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $x_n y_n \to \infty$ as $n \to \infty$. By the (μ,δ)-four-point inequality, with $x = x_n$ and $y = z = w = y_n$, we have $x_n y_n \leq \mu\, x_n y_n + 2\delta$. Dividing by $x_n y_n$ and taking the limit as $n \to \infty$ yields $1 \leq \mu$.

Property (iii) is obvious. □

Given points $x,y,z,w \in X$, not all identical, define

$$\Delta(x,y,z,w) = \frac{xy + zw}{\max\{xz + yw, xw + yz\}}. \tag{2.3}$$

In the introduction, we defined the restricted quasi-hyperbolicity constant of (X,d) by

$$C_0(X,d) = \inf\{\mu \mid (X,d) \text{ satisfies the } (\mu,0)\text{-four-point inequality}\}.$$

If X has at least two points then

$$C_0(X,d) = \sup \Delta(x,y,z,w) \tag{2.4}$$

where the supremum is taken over all $x,y,z,w \in X$, not all identical.

We also defined the quasi-hyperbolicity constant of (X,d) by

$$C(X,d) = \inf\{\mu \mid \text{there exists } \delta \geq 0 \text{ such that } (X,d) \text{ satisfies the } (\mu,\delta)\text{-four-point inequality}\}.$$

The quasi-hyperbolicity constant and the restricted quasi-hyperbolicity constant have the following elementary properties.

Proposition 2.5. *Let (X,d) be a metric space.*

(i) *If $A \subset X$ and d_A is the subspace metric then $C(A,d_A) \leq C(X,d)$ and $C_0(A,d_A) \leq C_0(X,d)$.*

(ii) *If $\lambda > 0$ then $C(X,\lambda d) = C(X,d)$ and $C_0(X,\lambda d) = C_0(X,d)$.*

(iii) $C(X,d) \leq C_0(X,d) \leq 2$.

(iv) *If* (X,d) *is unbounded then* $1 \leq C(X,d)$.

(v) *If* (X,d) *is bounded then* $C(X,d) = 0$.

(vi) *If* (X,d) *has at least two distinct points then* $C_0(X,d) \geq 1$.

(vii) *If* $\big(X', d'\big)$ *is a metric completion of* (X,d) *then* $C(X,d) = C\big(X', d'\big)$ *and* $C_0(X,d) = C_0\big(X', d'\big)$.

Proof. Property (i) and the inequality $C(X,d) \leq C_0(X,d)$ are clear from the definitions of $C(X,d)$ and $C_0(X,d)$. Note that for $\lambda > 0$, (X,d) satisfies the (μ,δ)-four-point inequality if and only if $(X,\lambda d)$ satisfies the $(\mu,\lambda\delta)$-four-point inequality. This implies (ii). The inequality $C_0(X,d) \leq 2$ in (iii) is a consequence of Proposition 2.2(i); (iv) follows from Proposition 2.2(ii); and (v) follows from Proposition 2.2(iii). If x_0, y_0 are distinct points in X then $\Delta(x_0,y_0,y_0,y_0) = 1$, see (2.3), and so $C_0(X,d) \geq 1$ by (2.4). It is straightforward that a metric space (X,d) satisfies the (μ,δ)-four-point inequality if and only if a metric completion of (X,d) satisfies the (μ,δ)-four-point inequality. This implies (vii). □

Proposition 2.6. *Let* (X,d) *be a metric space.*

(i) *If* (X,d) *unbounded and Gromov hyperbolic then* $C(X,d) = 1$.

(ii) *If* (X,d) *has at least two points then it is* 0*-hyperbolic if and only if* $C_0(X,d) = 1$.

Proof. (i). By Proposition 2.5(iv), $C(X,d) \geq 1$. Since, by definition, a Gromov hyperbolic space satisfies a $(1,\delta)$-four-point inequality for some $\delta \geq 0$ we have $C(X,d) \leq 1$. Hence $C(X,d) = 1$.
(ii). If (X,d) is 0-hyperbolic then it satisfies the $(1,0)$-four-point inequality and so $C_0(X,d) \leq 1$. By Proposition 2.5(vi), $C_0(X,d) \geq 1$. Hence $C_0(X,d) = 1$. If $C_0(X,d) = 1$ then for every $x,y,z,w \in X$, not all identical, $\Delta(x,y,z,w) \leq 1$ and so (X,d) satisfies the $(1,0)$-four-point inequality, that is, (X,d) is 0-hyperbolic. □

Without additional hypotheses, the converse of Proposition 2.6(i) need not be true, in §3.2 we give examples of unbounded metric spaces with $C(X,d) = 1$ that are not Gromov hyperbolic (also see Question 3.7 and Proposition 3.8).

Definition 2.7. We say that a metric space (X,d) is *four-point scalable in the large* if for every $x_1,x_2,x_3,x_4 \in X$ and for every $\lambda \geq 0$ there exists $x_1', x_2', x_3', x_4' \in X$ and $\Lambda \geq \lambda$ such that $d(x_i', x_j') = \Lambda\, d(x_i,x_j)$ for $1 \leq i,j \leq 4$.

Example 2.8. Let V be a real vector space with a given norm $\|\cdot\|$. The norm determines a metric on V given by $d(x,y) = \|x-y\|$. For any $0 < \alpha \leq 1$ the function d^α is also metric on V. The metric space (V,d^α) is called the *α-snowflake* of (V,d). Note that $d^\alpha(\lambda x,\lambda y) = \lambda^\alpha d^\alpha(x,y)$ for any $\lambda > 0$ from which it easily follows that (V,d^α) is four-point scalable in the large. Let $S \subset V$ be a nonempty subset such that $\lambda x \in S$ for all $\lambda > 0$ and all $x \in S$. Then S, viewed as a metric subspace of (V,d^α), is also four-point scalable in the large.

Proposition 2.9. *If (X, d) is four-point scalable in the large then $C(X, d) = C_0(X, d)$.*

Proof. It suffices to show that if (X, d) satisfies the (μ, δ)-four-point inequality for a particular (μ, δ) then it also satisfies the $(\mu, 0)$-four-point inequality. Assume that (X, d) satisfies the (μ, δ)-four-point inequality for some $\mu \geq 1$ and $\delta \geq 0$. Let $x_1, x_2, x_3, x_4 \in X$. For each $\lambda \geq 0$, let $\Lambda \geq \lambda$ and $x_i' \in X$ be such that $x_i'x_j' = \Lambda\, x_i x_j$, $1 \leq i, j \leq 4$. Note that the (μ, δ)-four-point inequality for the points $\{x_i'\}$ implies the $(\mu, \delta/\Lambda)$-four-point inequality for $\{x_i\}$. Since Λ can be chosen to be arbitrarily large, it follows that $\{x_i\}$ satisfies the $(\mu, 0)$-four-point inequality. □

Corollary 2.10. *Let V be a real vector space with a given norm $\| \cdot \|$ and corresponding metric, $d(x, y) = \|x - y\|$. Let $S \subset V$ be a nonempty subset such that $\lambda x \in S$ for all $\lambda > 0$ and all $x \in S$. Then for all $0 < \alpha \leq 1$, $C(S, d^\alpha) = C_0(S, d^\alpha)$.*

Proof. From Example 2.8, (S, d^α) is four-point scalar in the large and so the conclusion follows from Proposition 2.9. □

Example 2.11 (Hyperbolic space)**.** Let $n > 1$ be an integer and let (H^n, d_H) denote n-dimensional real hyperbolic space. For this space, $C(H^n, d_H) = 1 < \sqrt{2} = C_0(H^n, d_H)$ and so Proposition 2.9 implies (H^n, d_H) is *not* four-point scalable in the large. The space (H^n, d_H) is Gromov hyperbolic and unbounded, hence $C(H^n, d_H) = 1$ by Proposition 2.6(i). Since H^n has negative sectional curvature as a Riemannian manifold, $C_0(H^n, d_H) = \sqrt{2}$ by Corollary 7.3.

Definition 2.12. Let $C_1, C_2 > 0$ and $L_1, L_2 \geq 0$. A map $f\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is a $((C_1, L_1), (C_2, L_2))$-*quasi-isometric embedding* if for all $u, v \in X$,

$$C_1 d_X(u, v) - L_1 \;\leq\; d_Y(f(u), f(v)) \;\leq\; C_2 d_X(u, v) + L_2.$$

Some useful special cases of this definition include:

(i) A $((C_1, 0), (C_2, 0))$-quasi-isometric embedding $f\colon X \to Y$ is also known as a (C_1, C_2)-*bilipschitz embedding*.

(ii) A $((1, k), (1, k))$-quasi-isometric embedding $f\colon X \to Y$ is also known as a k-*rough isometric embedding*. This condition is equivalent to: for all $u, v \in X$, $|d_Y(f(u), f(v)) - d_X(u, v)| \leq k$.

Lemma 2.13. *If $f\colon X \to Y$ is a $((C_1, L_1), (C_2, L_2))$-quasi-isometric embedding between metric spaces and (Y, d_Y) satisfies the (μ, δ)-four-point inequality for some (μ, δ) then (X, d_X) satisfies the $\left(\frac{C_2}{C_1}\mu,\ \frac{1}{C_1}(\mu L_2 + L_1 + \delta)\right)$-four-point inequality.*

Proof. Let $x, y, z, w \in X$ and let $\bar{x}, \bar{y}, \bar{z}, \bar{w} \in Y$ be their respective images under $f\colon X \to Y$. Then

$$\begin{aligned}
&d_X(x, y) + d_X(z, w)\\
&\quad\le \tfrac{1}{C_1}\big(d_Y(\bar{x}, \bar{y}) + d_Y(\bar{z}, \bar{w})\big) + \tfrac{2L_1}{C_1}\\
&\quad\le \tfrac{1}{C_1}\big(\mu \max\{d_Y(\bar{x}, \bar{z}) + d_Y(\bar{y}, \bar{w}),\ d_Y(\bar{x}, \bar{w}) + d_Y(\bar{y}, \bar{z})\} + 2\delta\big) + \tfrac{2L_1}{C1}\\
&\quad\le \tfrac{1}{C_1}\mu \max\big\{C_2\big(d_X(x, z) + d_X(y, w)\big) + 2L_2,\ C_2\big(d_X(x, w) + d_X(y, z)\big) + 2L_2\big\} + \tfrac{2\delta}{C_1} + \tfrac{2L_1}{C_1}\\
&\quad= \tfrac{C_2}{C_1}\mu \max\{d_X(x, z) + d_X(y, w),\ d_X(x, w) + d_X(y, z)\} + \tfrac{2\mu L_2}{C_1} + \tfrac{2\delta}{C_1} + \tfrac{2L_1}{C_1}
\end{aligned}$$

which shows that (X, d) satisfies the $\Big(\frac{C_2}{C_1}\mu,\ \frac{1}{C_1}(\mu L_2 + L_1 + \delta)\Big)$-four-point inequality. □

Lemma 2.13 has the following immediate consequence.

Proposition 2.14. *Let $f\colon X \to Y$ be a map between metric spaces (X, d_X) and (Y, d_Y).*

(i) *If f is a $((C_1, L_1), (C_2, L_2))$-quasi-isometric embedding then $C(X, d_X) \le (C_2/C_1)\, C(Y, d_Y)$.*
(ii) *If f is a (C_1, C_2)-bilipschitz embedding then $C_0(X, d_X) \le (C_2/C_1)\, C_0(Y, d_Y)$.* □

A map $f\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is a *rough isometry* if it is a k-rough isometric embedding for some $k \ge 0$ and there exists $R > 0$ such that $f(X)$ is R-dense in Y, that is, for every $y \in Y$ there exists $x \in X$ such that $d_Y(f(x), y) < R$. Two metric spaces are *roughly isometric* if there exists a rough isometry between them. Note that rough isometry is a generally a stronger condition than *quasi-isometry*. Recall that f is a quasi-isometry if it is a $((C_1, L_1), (C_2, L_2))$-quasi-isometric embedding for some $(C_1, L_1), (C_2, L_2)$ and also $f(X)$ is R-dense for some R.

Corollary 2.15. *If (X, d_X) and (Y, d_Y) are roughly isometric then $C(X, d_X) = C(Y, d_Y)$.*

Proof. Since (X, d_X) and (Y, d_Y) are assumed to be roughly isometric, there exists $k \ge 0$ and $R > 0$ and a k-rough isometric embedding $f\colon X \to Y$ such that $f(X)$ is R-dense in Y. By Proposition 2.14(i), $C(X, d_X) \le C(Y, d_Y)$. Define $g\colon Y \to X$ as follows. For each $y \in Y$ we can choose $x \in X$ such that $d_Y(f(x), y) < R$ and declare $g(y) = x$. Observe that for all $y \in Y$, $d_Y(f(g(y)), y) < R$. For all $u, v \in Y$, $|d_Y(f(g(u)), f(g(v))) - d_X(g(u), g(v))| \le k$. Hence, for all $u, v \in Y$, $|d_Y(u, v) - d_X(g(u), g(v))| \le k + 2R$ and so g is a $(k + 2R)$-rough embedding. By Proposition 2.14(i), $C(Y, d_Y) \le C(X, d_X)$. It follows that $C(X, d_X) = C(Y, d_Y)$. □

3. Two families of examples

In §3.1, we exhibit spaces that are quasi-isometric to the Euclidean line yet with quasi-hyperbolicity constants that are greater than one and, consequently, are not Gromov hyperbolic. In §3.2, we give examples of metric spaces whose quasi-hyperbolicity constants are equal to one, yet are not Gromov hyperbolic. However, these are examples are not roughly geodesic. We show, using Bridson's "Flat Plane Theorem", that a proper CAT(0)-space whose quasi-hyperbolicity constant is equal to one is necessarily Gromov hyperbolic, see Proposition 3.8.

3.1. **The graph of $y = m|x|$ in the Euclidean plane.**
Let $m \geq 0$. Consider the space $X_m = \{(x, y) \in \mathbb{R}^2 \mid y = m\,|x|\}$ as a subspace of the Euclidean plane. The metric on X_m is given by

$$d_E\left((u, m\,|u|), (v, m\,|v|)\right) = \left[(u-v)^2 + m^2(|u| - |v|)^2\right]^{1/2}.$$

Let $(\mathbb{R}, d_E)$ be the Euclidean line, $d_E(u, v) = |u - v|$. Let $p\colon X_m \to \mathbb{R}$ be projection to the first coordinate, that is, $p(x, y) = x$. For $u, v \in \mathbb{R}$, $|\,|u| - |v|\,| \leq |u - v|$, and so, for $u \neq v$,

$$\left[(u-v)^2 + m^2(|u| - |v|)^2\right]^{1/2} = |u - v| \left[1 + m^2\left(\frac{|u| - |v|}{u - v}\right)^2\right]^{1/2} \leq (m^2+1)^{1/2}\,|u - v|,$$

and thus for all u, v

$$(m^2+1)^{-1/2} d_E\left((u, m\,|u|), (v, m\,|v|)\right) \leq |u - v| \leq d_E\left((u, m\,|u|), (v, m\,|v|)\right).$$

Hence p is a $((m^2+1)^{-1/2}, 1)$-bilipschitz embedding of X_m into $\mathbb{R}$. Since p is surjective, it is also a bilipschitz homeomorphism. In particular, (X_m, d_E) and $(\mathbb{R}, d_E)$ are quasi-isometric.

Note that, since $(\mathbb{R}, d_E)$ is 0-hyperbolic, we have $C(\mathbb{R}, d_E) = C_0(\mathbb{R}, d_E) = 1$ by Proposition 2.6.

For $m > 0$, let $\mu_m = \frac{\sqrt{m^2+1}+1}{\sqrt{m^2+1}-1}$. A straightforward calculation yields

$$\Delta\left((-\mu_m, \mu_m\, m),\, (1, m),\, (-1, m),\, (\mu_m, \mu_m\, m)\right) = \left(2 - (m^2+1)^{-1}\right)^{1/2}$$

and so $C_0(X_m, d_E) \geq \left(2 - (m^2+1)^{-1}\right)^{1/2}$. Note that if $(x, y) \in X_m$ and $\lambda > 0$ then $\lambda(x, y) \in X_m$ and so Corollary 2.10 gives $C_0(X_m, d_E) = C(X_m, d_E)$. Hence $C(X_m, d_E) \geq \left(2 - (m^2+1)^{-1}\right)^{1/2} > 1$ for $m > 0$. It follows from Proposition 2.6(i) that (X_m, d_E) is not Gromov hyperbolic when $m > 0$. Combining Propositions 2.14 and 4.3 yields the non-sharp upper bound:

$$C(X_m, d_E) \leq \min\left\{\sqrt{2},\, (m^2+1)^{1/2}\right\}.$$

However, numerical calculations strongly suggest that the configuration $(-\mu_m, \mu_m\, m)$, $(1, m)$, $(-1, m)$, $(\mu_m, \mu_m\, m)$ of four points in X_m is optimal, that is, $C(X_m, d_E) = \left(2 - (m^2+1)^{-1}\right)^{1/2}$ for all $m > 0$.

3.2. **The graph of $y = x^\alpha$, where $0 < \alpha < 1$, in the Euclidean plane.**
For $0 < \alpha < 1$, let d_α be the metric on the half-line, $[0, \infty)$, given by

$$d_\alpha(x, y) = \left((x-y)^2 + (x^\alpha - y^\alpha)^2\right)^{1/2}.$$

Let $Y_\alpha = \{(x, y) \in \mathbb{R}^2 \mid y = x^\alpha,\, x \geq 0\}$ as a subspace of the Euclidean plane. Projection to the first coordinate, $(x, y) \mapsto x$, gives an isometry $(Y_\alpha, d_E) \to ([0, \infty), d_\alpha)$. The metric behavior of $([0, \infty), d_\alpha)$ separates into two distinct cases, namely $0 < \alpha \leq 1/2$ and $1/2 < \alpha < 1$.

Proposition 3.1. *If $0 < \alpha \leq 1/2$ then for all $x, y \geq 0$, $0 \leq d_\alpha(x, y) - |x - y| \leq 1$. Consequently, for $0 < \alpha \leq 1/2$, $([0, \infty), d_\alpha)$ is roughly isometric to the Euclidean half-line and is thus Gromov hyperbolic.*

Proof. We first show that if $0 < \alpha \le 1/2$ then for $u \ge 0$, $\left(u^2 + u^{2\alpha}\right)^{1/2} - u \le 1$. If $u \le 1$ then

$$\left(u^2 + u^{2\alpha}\right)^{1/2} - u \;\le\; (u + u^\alpha) - u \;=\; u^\alpha \;\le\; 1.$$

If $u \ge 1$ and $\alpha \le 1/2$ then $u^{2\alpha} \le u$ and

$$\left(u^2 + u^{2\alpha}\right)^{1/2} - u = \frac{u^{2\alpha}}{\left(u^2 + u^{2\alpha}\right)^{1/2} + u} \;\le\; \frac{u}{\left(u^2 + u^{2\alpha}\right)^{1/2} + u} \;\le\; 1.$$

For $0 < \alpha < 1$ and $x, y \ge 0$, $|x^\alpha - y^\alpha| \le |x - y|^\alpha$. Hence for $0 < \alpha \le 1/2$ and $x, y \ge 0$, and using the inequality $\left(u^2 + u^{2\alpha}\right)^{1/2} - u \le 1$ with $u = |x - y|$, we have

$$0 \le \left((x-y)^2 + (x^\alpha - y^\alpha)^2\right)^{1/2} - |x - y| \le \left(|x-y|^2 + |x-y|^{2\alpha}\right)^{1/2} - |x - y| \le 1,$$

establishing the conclusion of the Proposition. □

In [BH99, 1.23 Exercise, p.412] it is asserted that $([0,\infty), d_{1/2})$ is not Gromov hyperbolic. This is not accurate as demonstrated by Proposition 3.1, however, we show in Proposition 3.3 that $([0,\infty), d_\alpha)$ is not Gromov hyperbolic if $1/2 < \alpha < 1$.

Lemma 3.2. *Let* $f(\alpha) = 2^{2\alpha-2} + \frac{1}{6}(1 - 2^{2\alpha})^2 - \frac{1}{2}(1 - 2^\alpha)^2 - 2^{4\alpha-3}$. *If* $0 < \alpha < 1$ *then* $f(\alpha) > 0$ *and*

$$\lim_{t\to\infty} \left(d_\alpha(t, 4t) + d_\alpha(0, 2t) - d_\alpha(t, 2t) - d_\alpha(0, 4t)\right) / \, t^{2\alpha-1} = f(\alpha).$$

Proof. Consider the polynomial $g(x) = \frac{1}{24}x^4 - \frac{7}{12}x^2 + x - \frac{1}{3} = \frac{1}{24}(x-2)^2(x^2 + 4x - 2)$. Using the factored expression for $g(x)$, we see that $g(x) > 0$ for $1 < x < 2$. Note that $f(\alpha) = g(2^\alpha)$. Hence $f(\alpha) > 0$ for $0 < \alpha < 1$. For $s \ge 0$, let

$$h(s) = \left(3^2 + (1 - 4^\alpha)^2 s\right)^{1/2} + \left(2^2 + 2^{2\alpha} s\right)^{1/2} - \left(1 + (1 - 2^\alpha)^2 s\right)^{1/2} - \left(4^2 + 4^{2\alpha} s\right)^{1/2}.$$

A straightforward calculation reveals that, for $t > 0$,

$$\theta(t) \;=\; \left(d_\alpha(t, 4t) + d_\alpha(0, 2t) - d_\alpha(t, 2t) - d_\alpha(0, 4t)\right) / \, t^{2\alpha-1} = h(t^{2\alpha-2}) \, / \, t^{2\alpha-2}.$$

Since $2\alpha - 2 < 0$, $\lim_{t\to\infty} t^{2\alpha-2} = 0$ and so

$$\lim_{t\to\infty} \theta(t) = \lim_{s\to 0} \frac{h(s)}{s} = h'(0) = f(\alpha)$$

yielding the conclusion of the Lemma. □

Proposition 3.3. *If* $1/2 < \alpha < 1$ *then* $([0,\infty), d_\alpha)$ *is not Gromov hyperbolic.*

Proof. For $x, y, z, w \in [0, \infty)$ and $0 < \alpha < 1$, let

$$\mathrm{Gr}_\alpha(x, y, z, w) \;=\; d_\alpha(x, y) + d_\alpha(z, w) - \max\left\{d_\alpha(x, z) + d_\alpha(y, w), d_\alpha(x, w) + d_\alpha(y, z)\right\}.$$

Note that $([0,\infty), d_\alpha)$ is not Gromov hyperbolic if and only if $\sup_{x,y,z,w} \mathrm{Gr}_\alpha(x, y, z, w) = \infty$.

For $t > 0$, let

$$h(t) = \frac{d_\alpha(t, 2t) + d_\alpha(0, 4t)}{d_\alpha(0, t) + d_\alpha(2t, 4t)} = \frac{\left(1 + (1 - 2^\alpha)^2 \, t^{2\alpha-2}\right)^{1/2} + \left(4^2 + 4^{2\alpha} \, t^{2\alpha-2}\right)^{1/2}}{(1 + t^{2\alpha-2})^{1/2} + (2^2 + (2^\alpha - 4^\alpha)^2 \, t^{2\alpha-2})^{1/2}}.$$

Since $2\alpha - 2 < 0$, $\lim_{t\to\infty} t^{2\alpha-2} = 0$ and so the above expression for $h(t)$ yields $\lim_{t\to\infty} h(t) = 5/3$. Hence $d_\alpha(t,2t) + d_\alpha(0,4t) > d_\alpha(0,t) + d_\alpha(2t,4t)$ for sufficiently large t which implies that $\mathrm{Gr}_\alpha(t,4t,0,2t) = d_\alpha(t,4t) + d_\alpha(0,2t) - d_\alpha(t,2t) - d_\alpha(0,4t)$ for sufficiently large t. If $1/2 < \alpha < 1$ then $2\alpha - 1 > 0$ and so Lemma 3.2 implies that $\lim_{t\to\infty} \mathrm{Gr}_\alpha(t,4t,0,2t) = \infty$. □

Proposition 3.4. *If $0 < \alpha < 1$ then $C([0,\infty), d_\alpha) = 1$.*

Proof. Let $L > 0$. If $x, y \geq 0$ and $|x-y| \geq L$ then

$$\frac{|x^\alpha - y^\alpha|}{|x-y|} \leq \frac{|x-y|^\alpha}{|x-y|} = |x-y|^{\alpha-1} \leq L^{\alpha-1},$$

and so for $|x-y| \geq L$,

$$d_\alpha(x,y) \leq \left(|x-y|^2 + \left(L^{\alpha-1}|x-y|\right)^2\right)^{1/2} \leq \left(1 + L^{2\alpha-2}\right)^{1/2}|x-y|.$$

If $x, y \geq 0$ and $|x-y| \leq L$ then

$$d_\alpha(x,y) \leq \left(|x-y|^2 + |x-y|^{2\alpha}\right)^{1/2} \leq \left(L^2 + L^{2\alpha}\right)^{1/2} = L\left(1 + L^{2\alpha-2}\right)^{1/2}.$$

It follows that for all $x, y \geq 0$

$$|x-y| \leq d_\alpha(x,y) \leq \left(1 + L^{2\alpha-2}\right)^{1/2}|x-y| + L\left(1 + L^{2\alpha-2}\right)^{1/2}. \tag{3.5}$$

Let $d_E(x,y) = |x-y|$, the Euclidean metric on $[0,\infty)$. By Proposition 2.6(i), $C([0,\infty), d_E) = 1$. Proposition 2.14(i) and (3.5) imply that $C([0,\infty), d_\alpha) \leq \left(1 + L^{2\alpha-2}\right)^{1/2}$. Since $2\alpha - 2 < 0$, we have that $\lim_{L\to\infty}\left(1 + L^{2\alpha-2}\right)^{1/2} = 1$. Hence $C([0,\infty), d_\alpha) \leq 1$. Furthermore, by Proposition 2.5(iv), $C([0,\infty), d_\alpha) \geq 1$ and so $C([0,\infty), d_\alpha) = 1$. □

Remark 3.6. It follows from the inequality (3.5) that the identity map $([0,\infty), d_E) \to ([0,\infty), d_\alpha)$ is a quasi-isometry. In this inequality, there is a trade-off between the "distortion", $(1 + L^{2\alpha-2})^{1/2}$, and the "roughness", $L\left(1 + L^{2\alpha-2}\right)^{1/2}$, that is, an attempt to adjust the parameter L to make the distortion small (close to 1) makes the roughness large and vice versa.

We showed that for $1/2 < \alpha < 1$ the space $([0,\infty), d_\alpha)$ is not Gromov hyperbolic but, nevertheless, $C([0,\infty), d_\alpha) = 1$.

Question 3.7. Assume that (X,d) is a geodesic metric space or, more generally, roughly geodesic. Does $C(X,d) = 1$ imply that (X,d) is Gromov hyperbolic?

For $1/2 < \alpha < 1$, the space $([0,\infty), d_\alpha)$ is not roughly geodesic and so does not provide a negative answer to this question. Some evidence in favor of an affirmative answer to Quesition 3.7 is given by the following result (see §4 for a discussion of CAT(0)-spaces).

Proposition 3.8. *Let (X,d) be a proper* CAT(0)*-space. If $C(X,d) = 1$ then (X,d) is Gromov hyperbolic.*

Proof. Assume the proper CAT(0)-space (X, d) is not Gromov hyperbolic. Bridson's *Flat Plane Theorem*, [Bri95, Theorem A], asserts that there exists an isometric embedding of a Euclidean plane, (V, d_E), into X. Hence $C(V, d_E) \leq C(X, d)$. By Proposition 4.4, $C(V, d_E) = \sqrt{2}$ and so $C(X, d) \geq \sqrt{2}$. In particular, $C(X, d) \neq 1$. □

4. The Ptolemy and quadrilateral inequalities, CAT(0)-spaces

The notion of a CAT(0)-space generalizes the concept of a simply connected, complete Riemannian manifold of non-positive sectional curvature to geodesic metric spaces. We show that the restricted quasi-hyperbolicity constant of a CAT(0)-space is bounded from above by $\sqrt{2}$. Indeed, the restricted quasi-hyperbolicity constant of any metric space whose distance satisfies Ptolemy's inequality and the quadrilateral inequality, in particular any CAT(0)-space, is bounded from above by $\sqrt{2}$, Theorem 4.2. The quasi-hyperbolicity constant of any Euclidean space of dimension greater than one is equal to $\sqrt{2}$, Proposition 4.4.

Definition 4.1. Let (X, d) be a metric space.

(i) The metric d satisfies *Ptolemy's inequality* if for all $x, y, z, w \in X$,

$$(xy)(zw) \leq (xz)(yw) + (xw)(yz).$$

In this case we say (X, d) is *Ptolemaic*.

(ii) The metric d satisfies the *quadrilateral inequality* if for all $x, y, z, w \in X$,

$$(xy)^2 + (zw)^2 \leq (xz)^2 + (yw)^2 + (xw)^2 + (yz)^2.$$

In this case we say (X, d) is 2*-round* (see Definition 5.9).

Recall that a *Euclidean space* is a real vector space V together with a positive definite inner product, $(u, v) \mapsto \langle u, v\rangle$. The inner product yields a *Euclidean norm*, $\|x\| = \langle x, x\rangle^{1/2}$, and a corresponding *Euclidean metric*, $d(u, v) = \|x - y\|$. It is classical mathematics that a Euclidean space with its Euclidean metric is Ptolemaic and 2-round.

Theorem 4.2. *If the metric space (X, d) is Ptolemaic and 2-round then $C_0(X, d) \leq \sqrt{2}$.*

Proof. Assume (X, d) is Ptolemaic and 2-round. Then for $x, y, z, w \in X$,

$$\begin{aligned} (xy)(zw) &\leq (xz)(yw) + (xw)(yz) \quad \text{and} \\ (xy)^2 + (zw)^2 &\leq (xz)^2 + (yw)^2 + (xw)^2 + (yz)^2. \end{aligned}$$

Multiplying the first inequality by 2 and adding it to the second one yields:

$$(xy + zw)^2 \leq (xz + yw)^2 + (xw + yz)^2.$$

For non-negative real numbers a, b we have $\sqrt{a^2 + b^2} \leq \sqrt{2}\max\{a, b\}$ and so the above inequality implies

$$xy + zw \leq \sqrt{2}\max\{xz + yw,\ xw + yz\}$$

from which it follows that $C_0(X,d) \leq \sqrt{2}$. □

Informally, a CAT(0)-space is a geodesic metric space whose geodesic triangles are are not fatter than corresponding comparison triangles in the Euclidean plane, see [BH99, II.1.1, page 158] for the precise definition. Since any configuration of four points in a CAT(0)-space has a "subembedding" into Euclidean space, [BH99, page 164], a CAT(0)-space is Ptolemaic and 2-round.

Corollary 4.3. *If (X,d) is a subspace of a* CAT(0)*-space then $C_0(X,d) \leq \sqrt{2}$.*

Proof. Since a CAT(0)-space is Ptolemaic and 2-round, so is any subspace. The conclusion follows from Theorem 4.2. □

Proposition 4.4. *Let V be a Euclidean space and d its Euclidean metric. If $\dim V \geq 2$ then $C(V,d) = C_0(V,d) = \sqrt{2}$.*

Proof. By Theorem 4.2, $C_0(V,d) \leq \sqrt{2}$. Since $\dim V \geq 2$, there are orthogonal unit vectors $u, v \in V$. A calculation using the inner product of V yields $\Delta(u,v,0,u+v) = \sqrt{2}$ and thus $C_0(V,d) \geq \sqrt{2}$. Hence $C_0(V,d) = \sqrt{2}$. Also, by Corollary 2.10, $C(V,d) = C_0(V,d)$. □

Remarkably, a geodesic metric space that is 2-round is necessarily a CAT(0)-space, [BN08, Sat09] and so Corollary 4.3 yields the following proposition.

Proposition 4.5. *Let (X,d) be a geodesic metric space. If (X,d) is 2-round then $C_0(X,d) \leq \sqrt{2}$.* □

Remark 4.6. Let (X,d) be any metric space. Blumenthal [Blu70, Theorem 52.1] showed that if $0 < \alpha \leq 1/2$ then the α-snowflake (X,d^α) has the property that any four points in it can be isometrically embedded into Euclidean space. Hence, in the case $0 < \alpha \leq 1/2$, (X,d^α) is Ptolemaic and 2-round and so Theorem 4.2 implies that $C_0(X,d^\alpha) \leq \sqrt{2}$. An improvement and extension of this estimate is given by Theorem 6.2.

5. Banach spaces

In contrast to a CAT(0)-space, whose quasi-hyperbolicity constant is bounded from above by $\sqrt{2}$, the quasi-hyperbolicity constant of a Banach space B of dimension greater that one is bounded from *below* by $\sqrt{2}$ with equality holding, assuming that the dimension of B is at least three, only when B is a Hilbert space, see Theorem 5.8. This is a consequence of strong results for the James constant of B due to Gao and Lau, [GL90], and to Komuro, Saito and Tanaka, [KST16]. Enflo [Enf69] introduced the notion of the *roundness* of a metric space. We show, Theorem 5.11, that if B is a Banach space with roundness $r(B)$ then its quasi-hyperbolicity constant is bounded from above by $2^{1/r(B)}$ and use this to show that the quasi-hyperbolicity constant of a non-trivial L^p-space, where $1 \leq p \leq \infty$, is $\max\{2^{1/p}, 2^{1-1/p}\}$, see Corollary 5.12.

Let $B = (V, \|\cdot\|)$ be a real Banach space. The norm of B, $\|\cdot\|$, yields a metric $d(u, v) = \|u - v\|$ on the real vector space V and we use notation $C(B)$ for $C(V, d)$. Note that by Corollary 2.10 we have $C_0(V, d) = C(V, d) = C(B)$.

Let $1 \le p \le \infty$. Recall the *p-norm* on $\mathbb{R}^n$, denoted by $\|x\|_p$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, is given by

$$\|x\|_p = \begin{cases} (|x_1|^p + \cdots + |x_n|^p)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{|x_1|, \ldots, |x_n|\} & \text{if } p = \infty. \end{cases}$$

We write $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$ and $d_p(u, v) = \|u - v\|_p$. The p-norms on $\mathbb{R}^n$ are related by the following well-known inequality. If $1 \le p \le q \le \infty$ then for all $x \in \mathbb{R}^n$

$$\|x\|_q \;\le\; \|x\|_p \;\le\; n^{1/p-1/q}\|x\|_q \tag{5.1}$$

where, by convention, $1/\infty = 0$.

Note that ℓ_2^n is a Euclidean space and so by Proposition 4.4, $C(\ell_2^n) = \sqrt{2}$ for $n \ge 2$.

Proposition 5.2. *For $1 \le p \le \infty$, $C(\ell_p^2) = \max\{2^{1/p}, 2^{1-1/p}\}$.*

Proof. If $1 \le p \le 2$ then by (5.1), $\|x\|_2 \;\le\; \|x\|_p \;\le\; 2^{1/p-1/2}\|x\|_2$. By Proposition 2.14,

$$C(\ell_p^2) \;\le\; 2^{1/p-1/2}\, C(\ell_2^2) \;=\; 2^{1/p}.$$

Observe $\Delta((-1, 1), (1, -1), (-1, -1), (1, 1)) = 2^{1/p}$ and so $C(\ell_p^2) \ge 2^{1/p}$. Thus $C(\ell_p^2) = 2^{1/p}$.

If $2 \le p \le \infty$ then by (5.1), $2^{1/p-1/2}\|x\|_2 \;\le\; \|x\|_p \;\le\; \|x\|_2$. By Proposition 2.14,

$$C(\ell_p^2) \;\le\; 2^{1/2-1/p}\, C(\ell_2^2) \;=\; 2^{1-1/p}.$$

Observe $\Delta((0, 1), (0, -1), (-1, 0), (1, 0)) = 2^{1-1/p}$ and so $C(\ell_p^2) \ge 2^{1-1/p}$. Thus $C(\ell_p^2) = 2^{1-1/p}$. □

Proposition 5.2 generalizes to non-trivial L_p-spaces, see Corollary 5.12.

The *Banach-Mazur distance* between two isomorphic Banach spaces E and F is defined by

$$d_{\mathrm{BM}}(E, F) = \inf\{\|T\|\,\|T^{-1}\| \mid T\colon E \to F \text{ is an isomorphism}\}.$$

For example, if $1 \le p \le q \le 2$ or $2 \le p \le q \le \infty$ then $d_{\mathrm{BM}}(\ell_p^n, \ell_q^n) = n^{1/p-1/q}$, [TJ89, Proposition 37.6]. Proposition 2.14 yields the following comparison.

Proposition 5.3. *If E and F are isomorphic Banach spaces then $C(E) \le d_{\mathrm{BM}}(E, F)\, C(F)$.* □

Because of Theorem 5.8 below, the inequality of Proposition 5.3 can only give useful information when $d_{\mathrm{BM}}(E, F) < \sqrt{2}$.

Since, up to a translation, any four points of a Banach space lie in some subspace of dimension at most three,

$$C(B) = \sup\{C(V) \mid V \text{ is a subspace of } B \text{ with } \dim V \le 3\}. \tag{5.4}$$

A Banach space B is *finitely representable* in another Banach space B' if for every finite dimensional subspace F of B and every $\varepsilon > 0$ there is a subspace F' of B' and an isomorphism $T\colon F \to F'$ such that $\|T\|\,\|T^{-1}\| \le 1 + \varepsilon$.

Proposition 5.5. *If B is finitely representable in B' then $C(B) \leq C(B')$.*

Proof. Let $\varepsilon > 0$. Let V be a subspace of B with $\dim V \leq 3$. Since B is finitely representable in B', there exists a subspace V' of B' and an isomorphism $T\colon V \to V'$ such that $\|T\|\,\|T^{-1}\| \leq 1+\varepsilon$. By Proposition 2.14, $C(V) \leq (1+\varepsilon)\,C(V')$ and so $C(V) \leq (1+\varepsilon)\,C(B')$ because $C(V') \leq C(B')$. It follows from (5.4) that $C(B) \leq (1+\varepsilon)\,C(B')$. Since ε is arbitrary, we conclude $C(B) \leq C(B')$. □

Corollary 5.6. *Let B be a Banach space and B^{**} its second dual. Then $C(B) = C(B^{**})$.*

Proof. The canonical map $B \to B^{**}$ is an isometric embedding and hence $C(B) \leq C(B^{**})$. In any Banach space B, the second dual B^{**} is finitely representable in in B, [JL01, §9], and so by Proposition 5.5, $C(B^{**}) \leq C(B)$. It follows that $C(B) = C(B^{**})$. □

The *James constant* of a Banach space B is defined by:

$$J(B) = \sup\{\min(\|x-y\|, \|x+y\|) \mid \|x\| = \|y\| = 1\}$$

If $\|x\| = \|y\| = 1$ then $\Delta(x, y, 0, x+y) = \frac{1}{2}(\|x-y\| + \|x+y\|)$ and thus

$$C(B) \geq \sup\{\tfrac{1}{2}(\|x-y\| + \|x+y\|) \mid \|x\| = \|y\| = 1\} \geq J(B) \tag{5.7}$$

A Banach space B is said to be *non-trivial* if $\dim(B) \geq 2$.

Theorem 5.8. *If B is any non-trivial Banach space then $C(B) \geq \sqrt{2}$. If $\dim B \geq 3$ and $C(B) = \sqrt{2}$ then B is a Hilbert space.*

Proof. Gao and Lau, [GL90, Theorem 2.5], show $J(B) \geq \sqrt{2}$ for any non-trivial Banach space B. Furthermore, Komuro, Saito and Tanaka, [KST16], show that $\dim B \geq 3$ and $J(B) = \sqrt{2}$ implies B is a Hilbert space. The conclusion of the theorem follows from (5.7). □

Definition 5.9 ([Enf69])**.** Let (X, d) be a metric space and $p \geq 1$. The space (X, d) is said to be *p-round* if for all $x, y, z, w \in X$, $(xy)^p + (zw)^p \leq (xz)^p + (yw)^p + (xw)^p + (yz)^p$. The *roundness* of (X, d) is $r(X, d) = \sup\{p \mid (X, d) \text{ is } p\text{-round}\}$.

Note that if $r(X, d) < \infty$ then the supremum is attained. Enflo, [Enf69], observed that $r(X, d) \geq 1$ and that if (X, d) has the *midpoint property*[1] then $r(X, d) \leq 2$. In particular, if B is a Banach space then $1 \leq r(B) \leq 2$, where $r(B)$ is the roundness of B as a metric space.

Lemma 5.10. *Let B be a Banach space that is p-round. Then for any vectors $e, f \in B$*

$$(\|e\| + \|f\|)^p \leq \|e-f\|^p + \|e+f\|^p.$$

[1]A metric space (X, d) has the *midpoint property* if for every $x, y \in X$ there exists $z \in X$ such that $d(x, z) = d(z, y) = \frac{1}{2}d(x, y)$.

Proof. In the "p-round inequality" of Definition 5.9, letting $x = e + f$, $y = e - f$, $w = 2e$, and $z = 0$ gives

$$\|2e\|^p + \|2f\|^p \le 2\|e - f\|^p + 2\|e + f\|^p$$

and so

$$2^{p-1}\left(\|e\|^p + \|f\|^p\right) \le \|e - f\|^p + \|e + f\|^p.$$

By (5.1), with $n = 2$, $(\|e\| + \|f\|)^p \le 2^{p-1}\left(\|e\|^p + \|f\|^p\right)$ from which the conclusion follows. □

Theorem 5.11. *If B is a Banach space then $C(B) \le 2^{1/r(B)}$.*

Proof. Let $p = r(B)$. Then B is p-round. Let $x, y, z, w \in B$. Let $a = x - z$, $b = w - y$, $c = w - x$, $d = y - z$, $e = y - x$, and $f = w - z$. Note that $f = a + c = b + d$ and $e = d - a = c - b$. Hence $e + f = c + d$ and $f - e = a + b$. By Lemma 5.10,

$$\begin{aligned}(\|e\| + \|f\|)^p &\le \|e - f\|^p + \|e + f\|^p \\ &= \|a + b\|^p + \|c + d\|^p \\ &\le (\|a\| + \|b\|)^p + (\|c\| + \|d\|)^p \quad \text{(by triangle inequality).}\end{aligned}$$

It follows that

$$\begin{aligned}\|e\| + \|f\| &\le \left((\|a\| + \|b\|)^p + (\|c\| + \|d\|)^p\right)^{1/p} \\ &\le 2^{1/p}\max\left(\|a\| + \|b\|, \|c\| + \|d\|\right) \quad \text{(by (5.1)).}\end{aligned}$$

Thus the $(2^{1/p}, 0)$-four-point inequality holds and so $C(B) \le 2^{1/p}$. □

Corollary 5.12. *Let (Ω, Σ, μ) be a separable measure space, that is, the σ-algebra Σ is generated by a countable collection of subsets of Ω. Let $1 \le p \le \infty$ and let $L_p(\Omega, \Sigma, \mu)$ be the corresponding L_p-space. If $\dim L_p(\Omega, \Sigma, \mu) \ge 2$ then $C(L_p(\Omega, \Sigma, \mu)) = \max\{2^{1/p}, 2^{1-1/p}\}$.*

Proof. Denote $B = L_p(\Omega, \Sigma, \mu)$. Assume $\dim B \ge 2$. In the case $1 \le p \le 2$, Enflo, [Enf69], showed that $r(B) = p$ and so $C(B) \le 2^{1/p}$ by Theorem 5.11. In the case $2 \le p \le \infty$, by [LTW97, Proposition1.4 and Remark 1.5], $r(B) = 1/(1 - 1/p)$ and so $C(B) \le 2^{1-1/p}$ by Theorem 5.11. Hence for $1 \le p \le \infty$, $C(B) \le \max\{2^{1/p}, 2^{1-1/p}\}$.

The classification theory of L_p spaces (see [JL01, §4]) gives that, for $1 \le p < \infty$, the space $B = L_p(\Omega, \Sigma, \mu)$ is isometric to one of the Banach spaces in the list

$$\ell_p^n,\ \ell_p,\ L_p(0,1),\ \ell_p \oplus_p L_p(0,1),\ \ell_p^n \oplus_p L_p(0,1) \qquad n = 1, 2, \ldots \tag{5.13}$$

Here, ℓ_p denotes the space of sequences $(x_n)_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} |x_n|^p < \infty$ and $L_p(0,1)$ denotes the space of measurable functions (modulo null sets) on the unit interval such that $\int_0^1 |f(x)|^p dx < \infty$, and $\oplus_p$ denotes the ℓ_p direct sum, that is, $\|a \oplus b\| = (\|a\|^p + \|b\|^p)^{1/p}$. Each of the spaces in the list (5.13) (in the case of ℓ_p^n, assume $n \ge 2$) contains a subspace isometric to ℓ_p^2 and so $C(B) \ge C(\ell_p^2) = \max\{2^{1/p}, 2^{1-1/p}\}$ by Proposition 5.2. Hence $C(B) = \max\{2^{1/p}, 2^{1-1/p}\}$. In the case $p = \infty$ note that B contains a subspace isometric to ℓ_∞^2 which implies that $C(B) = 2$. □

Question 5.14. Let (X,d) be a geodesic metric space. Is $C_0(X,d) \le 2^{1/r(X,d)}$?

By Proposition 4.5, this is true in the case $r(X,d) = 2$.

6. Snowflaked metric spaces

Recall that if $0 < \alpha \le 1$ and (X,d) is any metric space then (X,d^α) is also a metric space, called the α-*snowflake* of (X,d). We show that $C_0(X,d^\alpha) \le 2^\alpha$, Theorem 6.2, and give some applications of this estimate. We determine the quasi-hyperbolicity constant of the α-snowflake of the Euclidean real line, Theorem 6.6.

Lemma 6.1. *Let $a_{ij} \in \mathbb{R}$, $i,j \in \{1,2,3,4\}$, be such that $a_{ij} = a_{ji}$. Let $\lambda \ge 1$. If $a_{ij} \le \lambda \max\{a_{ik}, a_{kj}\}$ for all i,j,k, then $a_{ij} + a_{k\ell} \le \lambda \max\{a_{ik} + a_{j\ell},\, a_{i\ell} + a_{jk}\}$ for all i,j,k,ℓ.*

Note that if L, M and S denote the largest, medium and smallest of the three sums $a_{ij} + a_{k\ell}$, $a_{ik} + a_{j\ell}$ and $a_{i\ell} + a_{jk}$ for some choice of $i,j,k,\ell \in \{1,2,3,4\}$, then the conclusion of the lemma is equivalent to $L \le \lambda M$.

Proof. Fix $i,j,k,\ell \in \{1,2,3,4\}$. Without loss of generality, assume that $L = a_{ij} + a_{k\ell}$ is the largest sum and assume that $a_{k\ell} \le a_{ij}$. Since $a_{ij} \le \lambda \max\{a_{ik}, a_{kj}\}$ and $a_{ij} \le \lambda \max\{a_{i\ell},\, a_{\ell j}\}$, we have

$$a_{ij} + a_{k\ell} \le a_{ij} + a_{ij} \le \lambda \max\{a_{ik} + a_{i\ell},\, a_{ik} + a_{\ell j},\, a_{kj} + a_{i\ell},\, a_{kj} + a_{\ell j}\}.$$

If $a_{ik} \ge a_{kj}$ and $a_{\ell j} \ge a_{i\ell}$ then

$$M = a_{ik} + a_{\ell j} = \max\{a_{ik} + a_{i\ell},\, a_{ik} + a_{\ell j},\, a_{kj} + a_{i\ell},\, a_{kj} + a_{\ell j}\}$$

and if $a_{ik} \le a_{kj}$ and $a_{\ell j} \le a_{i\ell}$ then

$$M = a_{kj} + a_{i\ell} = \max\{a_{ik} + a_{i\ell},\, a_{ik} + a_{\ell j},\, a_{kj} + a_{i\ell},\, a_{kj} + a_{\ell j}\}.$$

In both cases, $L \le \lambda M$. Furthermore, if $a_{ik} \ge a_{kj}$ and $a_{\ell j} \le a_{i\ell}$ then $a_{ij} \le \lambda \max\{a_{ik},\, a_{kj}\} = \lambda a_{ik}$ and $a_{ij} \le \lambda \max\{a_{i\ell},\, a_{\ell j}\} = \lambda a_{i\ell}$, and since $a_{k\ell} \le \lambda \max\{a_{kj},\, a_{\ell j}\}$,

$$\begin{aligned} a_{ij} + a_{k\ell} &\le a_{ij} + \lambda \max\{a_{kj},\, a_{\ell j}\} = \max\{a_{ij} + \lambda a_{kj},\, a_{ij} + \lambda a_{\ell j}\} \\ &\le \max\{\lambda a_{i\ell} + \lambda a_{kj},\, \lambda a_{ik} + \lambda a_{\ell j}\} = \lambda \max\{a_{i\ell} + a_{kj},\, a_{ik} + a_{\ell j}\}. \end{aligned}$$

Finally, if $a_{ik} \le a_{kj}$ and $a_{\ell j} \ge a_{i\ell}$ then

$$a_{ij} \le \lambda \max\{a_{ik},\, a_{kj}\} = \lambda a_{kj} \quad \text{and} \quad a_{ij} \le \lambda \max\{a_{i\ell},\, a_{\ell j}\} = \lambda a_{\ell j},$$

and since $a_{k\ell} \le \lambda \max\{a_{ki},\, a_{i\ell}\}$, we have

$$a_{ij} + a_{k\ell} \le a_{ij} + \lambda \max\{a_{ki},\, a_{i\ell}\} \le \lambda \max\{a_{\ell j} + a_{ki},\, a_{kj} + a_{i\ell}\},$$

that is, $L \le \lambda M$. □

Theorem 6.2. *Let $0 < \alpha \le 1$. For any metric space (X,d), $C_0(X,d^\alpha) \le 2^\alpha$.*

Proof. Let $x_i \in X$, $i = 1, 2, 3, 4$. It suffices to show that if $i, j, k, l \in \{1, 2, 3, 4\}$ then

$$(x_i x_j)^\alpha + (x_k x_l)^\alpha \le 2^\alpha \max\{(x_i x_k)^\alpha + (x_j x_l)^\alpha,\ (x_i x_l)^\alpha + (x_j x_k)^\alpha\}.$$

Observe that for all i, j, k triangle inequality implies $x_i x_j \le x_i x_k + x_j x_k \le 2 \max\{x_i x_k, x_j x_k\}$. Hence

$$(x_i x_j)^\alpha \le 2^\alpha \max\{(x_i x_k)^\alpha, (x_j x_k)^\alpha\}.$$

The conclusion follows from Lemma 6.1 with $a_{ij} = (x_i x_j)^\alpha$ and $\lambda = 2^\alpha$. □

As in §5, d_p, where $1 \le p \le \infty$, denotes the metric on $\mathbb{R}^n$ determined by the standard p-norm.

Proposition 6.3. *If $0 < \alpha \le 1$ and $n \ge 2$ then $C(\mathbb{R}^n, d_\infty^\alpha) = 2^\alpha$*

Proof. By Proposition 6.2, $C_0(\mathbb{R}^n, d_\infty^\alpha) \le 2^\alpha$. Consider following the four points in $\mathbb{R}^n$:

$$x = (0, 1, 0, \dots, 0),\ y = (0, -1, 0, \dots, 0),\ z = (-1, 0, \dots, 0),\ w = (1, 0, \dots, 0).$$

A calculation using the metric d_∞^α yields $\Delta(x, y, z, w) = 2^\alpha$ and thus $C_0(\mathbb{R}^n, d_\infty^\alpha) \ge 2^\alpha$. Hence $C_0(\mathbb{R}^n, d_\infty^\alpha) = 2^\alpha$. By Corollary 2.10, $C(\mathbb{R}^n, d_\infty^\alpha) = C_0(\mathbb{R}^n, d_\infty^\alpha)$. □

The same technique gives a non-sharp estimate for $C(\mathbb{R}^n, d_2^\alpha)$, where $n \ge 2$, as follows.

Proposition 6.4. *If $0 < \alpha \le 1$ and $n \ge 2$ then $2^{\alpha/2} \le C(\mathbb{R}^n, d_2^\alpha) \le 2^{\min\{\alpha, 1/2\}}$.*

Proof. By Proposition 6.2, $C(\mathbb{R}^n, d_2^\alpha) \le 2^\alpha$. Schoenberg showed, [Sch37, Theorem 1], that $(\mathbb{R}^n, d_2^\alpha)$ isometrically embeds into (infinite dimensional) Hilbert space and hence $C(\mathbb{R}^n, d_2^\alpha) \le 2^{1/2}$. Consequently, $C(\mathbb{R}^n, d_2^\alpha) \le 2^{\min\{\alpha, 1/2\}}$. For the four points $x, y, z, w \in \mathbb{R}^n$ specified in the proof of Proposition 6.3, we have $\Delta(x, y, z, w) = 2^{\alpha/2}$, yielding the lower bound for $C(\mathbb{R}^n, d_2^\alpha)$. □

Numerical calculations suggest the following exact value for $C(\mathbb{R}^n, d_2^\alpha)$ when $n \ge 2$.

Conjecture 6.5. *Let $0 < \alpha < 1$. If $n \ge 2$ then $C(\mathbb{R}^n, d_2^\alpha) = 2^{\alpha/2}$.*

The α-snowflakes of the Euclidean line turns out to be of a different nature than the spaces $(\mathbb{R}^n, d_2^\alpha)$ with $n \ge 2$, as revealed in the following theorem.

Theorem 6.6. *Let $0 < \alpha \le 1$ and $d_E^\alpha(x, y) = |x - y|^\alpha, x, y \in \mathbb{R}$. Let $m \ge 1$ be the unique solution to the equation $(m - 1)^\alpha + (m + 1)^\alpha = 2$. Then $C(\mathbb{R}^1, d_E^\alpha) = m^\alpha$.*

Observe that by Corollary 2.10,

$$C(\mathbb{R}^1, d_E^\alpha) = C_0(\mathbb{R}^1, d_E^\alpha) = \sup \Delta(x, y, z, w), \tag{6.7}$$

where

$$\Delta(x, y, z, w) = \frac{|x - y|^\alpha + |z - w|^\alpha}{\max\{|x - z|^\alpha + |y - w|^\alpha,\ |x - w|^\alpha + |y - z|^\alpha\}}$$

and the supremum in (6.7) is taken over all $x, y, z, w \in \mathbb{R}$, not all identical. Since the map $(x, y, z, w) \mapsto \Delta(x, y, z, w)$ is translation and scale invariant, we may assume that $x = 0$, $y = 1 + s$, $z = 1 - t$, and $w = 2$, with $(t, s) \in D = \{(t, s) \in [-1, 1] \times [-1, 1] \mid t + s \geq 0\}$. Then

$$(t, s) \mapsto \Delta(0, 1+s, 1-t, 2) = \frac{(1+s)^\alpha + (1+t)^\alpha}{\max\limits_{(t,s)\in D}\{(1-t)^\alpha + (1-s)^\alpha, (t+s)^\alpha + 2^\alpha\}}$$

is continuous on the compact set D and

$$C(\mathbb{R}^1, d_E^\alpha) = \max_{(t,s)\in D} \Delta(0, 1+s, 1-t, 2).$$

Furthermore, if $F, G \colon D \to \mathbb{R}$ are given by

$$F(t, s) = \frac{(1+t)^\alpha + (1+s)^\alpha}{(1-t)^\alpha + (1-s)^\alpha} \text{ and } G(t, s) = \frac{(1+t)^\alpha + (1+s)^\alpha}{(t+s)^\alpha + 2^\alpha}, \tag{6.8}$$

and $D_1 = \{(t, s) \in D \mid F(t, s) \leq G(t, s)\}$ and $D_2 = \{(t, s) \in D \mid F(t, s) \geq G(t, s)\}$, then

$$\Delta(0, 1-t, 1+s, 2) = \min_{(t,s)\in D}\{F(t, s), G(t, s)\} = \begin{cases} F(t, s), & (t, s) \in D_1 \\ G(t, s), & (t, s) \in D_2, \end{cases}$$

and

$$C(\mathbb{R}^1, d_E^\alpha) = \max\left\{\max_{(t,s)\in D_1} F(t, s), \max_{(t,s)\in D_2} G(t, s)\right\}. \tag{6.9}$$

The following lemma shows that the maximum in (6.9) is attained on $D_0 = D_1 \cap D_2$.

Lemma 6.10. *Let $0 < \alpha < 1$. Let $F, G \colon D \to \mathbb{R}$ be given by (6.8) and let $D_0 = \{(t, s) \in D \mid F(t, s) = G(t, s)\}$. Then*

$$C(\mathbb{R}^1, d_E^\alpha) = \max_{(t,s)\in D_0} F(t, s).$$

Proof. We show that F and G attain their maximum on the boundary of D_1 and D_2, respectively. Indeed, the partial derivatives of F,

$$F_t(t, s) = \frac{\alpha(1+t)^{\alpha-1}}{(1-t)^\alpha + (1-s)^\alpha} + \frac{\alpha((1+t)^\alpha + (1+s)^\alpha)(1-t)^{\alpha-1}}{((1-t)^\alpha + (1-s)^\alpha)^2}$$

$$F_s(t, s) = \frac{\alpha(1+s)^{\alpha-1}}{(1-t)^\alpha + (1-s)^\alpha} + \frac{\alpha((1+t)^\alpha + (1+s)^\alpha)(1-s)^{\alpha-1}}{((1-t)^\alpha + (1-s)^\alpha)^2}$$

are defined for all $(t, s) \in (-1, 1)^2, t + s > 0$ and $F_t > 0$ and $F_s > 0$. Thus $\max_{(t,s)\in D_1} F(t, s)$ is attained on the boundary $\partial D_1 = D_0 \cup \{(t, s) \in D \mid t + s = 1\}$. Note that $F(t, s) \geq 1$ for $(t, s) \in D$ and $F(t, s) = 1$ if and only if $t + s = 1$. Hence

$$\max_{(t,s)\in D_1} F(t, s) = \max_{(t,s)\in D_0} F(t, s). \tag{6.11}$$

The partial derivatives of G

$$G_t(t, s) = \frac{\alpha(1+t)^{\alpha-1}}{(t+s)^\alpha + 2^\alpha} - \frac{\alpha((1+t)^\alpha + (1+s)^\alpha)(t+s)^{\alpha-1}}{((t+s)^\alpha + 2^\alpha)^2}$$

$$G_s(t, s) = \frac{\alpha(1+s)^{\alpha-1}}{(t+s)^\alpha + 2^\alpha} - \frac{\alpha((1+t)^\alpha + (1+s)^\alpha)(t+s)^{\alpha-1}}{((t+s)^\alpha + 2^\alpha)^2}$$

are defined for all $(t,s) \in (-1,1)^2, t+s > 0$ and $G_t = G_s = 0$ if and only if $t = s = 1$. Thus $\max_{(t,s)\in D_2} G(t,s)$ is attained on the boundary $\partial D_2 = D_0 \cup \{(t,s) \in D \mid t = 1\} \cup \{(t,s) \in D \mid s = 1\}$. Note also that $G(t,s) \geq 1$ for $(t,s) \in D$ and $G(t,s) = 1$ if and only if $t = 1$ or $s = 1$. Hence

$$\max_{(t,s)\in D_2} G(t,s) = \max_{(t,s)\in D_0} G(t,s). \tag{6.12}$$

The conclusion follows from (6.9) together with (6.11) and (6.12). □

The following result shows that $\max_{(t,s)\in D_0} F(t,s)$ is attained when $t = s$.

Lemma 6.13. *Let* $0 < \alpha < 1$. *Let* $F, G \colon D \to \mathbb{R}$ *be given by (6.8) and let* $D_0 = \{(t,s) \in D \mid F(t,s) = G(t,s)\}$. *Then*

$$\max_{(t,s)\in D_0} F(t,s) = \left(\frac{1+a}{1-a}\right)^{\alpha},$$

where $0 < a < 1$ *is the unique solution of* $F(a,a) = G(a,a)$.

Proof. Notice that if $(t,s) \in D_0$ then $t = -1$ if and only if $s = 1$ and $F(-1,1) = 1$. By symmetry, $F(1,-1) = 1$. Since $F(t,s) \geq 1$ on D_0, the maximum of $F|_{D_0}$, the restriction of F to D_0, is not attained at $(-1,1)$ or $(1,-1)$. Let $(a,b) \in D_0$, with $a \neq \pm 1$. If F attains a local extremum at (a,b) subject to the constrain $F(t,s) = G(t,s)$, then the level curves $\{(t,s) \in D \mid F(t,s) = F(a,b)\}$ and $\{(t,s) \in D \mid F(t,s) - G(t,s) = 0\}$ are both tangent at (a,b). Since $F_s(a,b) - G_s(a,b) \neq 0$, by the Implicit Function Theorem, there exists an open neighbourhood $U \subseteq (-1,1)$ of a and a function $\omega = \omega(t)$ such that $F(t,\omega(t)) - G(t,\omega(t)) = 0$ for $t \in U$. Furthermore,

$$\omega'(t) = -\frac{(t+\omega)^{\alpha-1} + (1-t)^{\alpha-1}}{(t+\omega)^{\alpha-1} + (1-\omega)^{\alpha-1}}$$

for all $t \in U$. Similarly, since $F_s(a,b) \neq 0$, there exists an open neighbourhood $V \subseteq (-1,1)$ of a and a function $\nu = \nu(t)$ on V such that $F(t,\nu(t)) = F(a,b)$ on V. Also, for all $t \in V$,

$$\nu'(t) = -\frac{(1+t)^{\alpha-1} + F(a,b)(1-t)^{\alpha-1}}{(1+\nu)^{\alpha-1} + F(a,b)(1-\nu)^{\alpha-1}}.$$

Hence, a necessary condition for (a,b) to be a point of local extremum for $F|_{D_0}$ is that $\omega'(a) = \nu'(a)$. Using that $\omega(a) = \nu(a) = b$, that is,

$$\frac{(a+b)^{\alpha-1} + (1-a)^{\alpha-1}}{(a+b)^{\alpha-1} + (1-b)^{\alpha-1}} = \frac{(1+a)^{\alpha-1} + F(a,b)(1-a)^{\alpha-1}}{(1+b)^{\alpha-1} + F(a,b)(1-b)^{\alpha-1}},$$

equivalently,

$$\begin{aligned}(a+b)^{\alpha-1}\big[(1+b)^{\alpha-1} &+ F(a,b)(1-b)^{\alpha-1} - (1+a)^{\alpha-1} - F(a,b)(1-a)^{\alpha-1}\big] \\ &+ (1-a)^{\alpha-1}(1+b)^{\alpha-1} - (1-b)^{\alpha-1}(1+a)^{\alpha-1} = 0.\end{aligned}$$

Using that $F(a,b) = \frac{(1+a)^\alpha+(1+b)^\alpha}{(1-a)^\alpha+(1-b)^\alpha} = \frac{(1+a)^\alpha+(1+b)^\alpha}{(a+b)^\alpha+2^\alpha}$, the above equality holds if and only if

$$\begin{aligned}(a+b)^{\alpha-1}\big\{&[(1+b)^{\alpha-1}-(1+a)^{\alpha-1}][(1-a)^\alpha+(1-b)^\alpha]\\&+[(1-b)^{\alpha-1}-(1-a)^{\alpha-1}][(1+a)^\alpha+(1+b)^\alpha]\big\}\\&+[(1-a)^{\alpha-1}(1+b)^{\alpha-1}-(1+a)^{\alpha-1}(1-b)^{\alpha-1}][(a+b)^\alpha+2^\alpha]=0,\end{aligned}$$

equivalently,

$$2(a+b)^{\alpha-1}\left[(1-b^2)^{\alpha-1}-(1-a^2)^{\alpha-1}\right]+2^\alpha[(1-a)^{\alpha-1}(1+b)^{\alpha-1}-(1+a)^{\alpha-1}(1-b)^{\alpha-1}]=0$$

Factoring out $2(a+b)^{\alpha-1}(1-b^2)^{\alpha-1}\neq 0$ yields

$$1-\left(\tfrac{1-b}{1-a}\right)^{1-\alpha}\left(\tfrac{1+b}{1+a}\right)^{1-\alpha}-\left(\tfrac{a+b}{2}\right)^{1-\alpha}\left[\left(\tfrac{1+b}{1+a}\right)^{1-\alpha}-\left(\tfrac{1-b}{1-a}\right)^{1-\alpha}\right]=0 \tag{6.14}$$

Assume $a<b$. Since $a+b>0$, this implies $b>0$ and $-b<a<b$. In particular, $a^2<b^2$. Let $x=\frac{1-b}{1-a}$ and $y=\frac{1+b}{1+a}$. Then $0<x<1<y$, and $0<xy<1$. Note that $\frac{a+b}{2}=\frac{1-xy}{y-x}$. We claim that the expression on the left hand side of (6.14) is negative. That is, we claim,

$$1-(xy)^{1-\alpha}-\left(\frac{1-xy}{y-x}\right)^{1-\alpha}\left(y^{1-\alpha}-x^{1-\alpha}\right)<0.$$

Indeed, multiplying the above inequality by $(1-xy)^{\alpha-1}>0$ yields

$$\frac{1-(xy)^{1-\alpha}}{(1-xy)^{1-\alpha}}-\frac{y^{1-\alpha}-x^{1-\alpha}}{(y-x)^{1-\alpha}}<0,$$

equivalently,

$$\frac{1-(xy)^{1-\alpha}}{(1-xy)^{1-\alpha}}-\frac{1-(x/y)^{1-\alpha}}{(1-x/y)^{1-\alpha}}<0$$

which is valid since the function $t\mapsto\frac{1-t^{1-\alpha}}{(1-t)^{1-\alpha}}$, $0<t<1$, is decreasing and $0<x/y<xy<1$.

Note that the expression on the left hand side of (6.14) is positive if $a>b$. Thus, (6.14) holds if and only if $a=b$. Finally, notice that $F(a,a)=G(a,a)$ has unique solution $0<a<1$. Since $F(a,a)=[(1+a)/(1-a)]^\alpha>1$, the conclusion follows. □

Proof of Theorem 6.6. If $\alpha=1$, the conclusion holds with $m=1$ by Proposition 2.6, since the space $(\mathbb{R}^1,d_E)$ is 0-hyperbolic. Let $0<\alpha<1$. By Lemmas 6.10 and 6.13

$$C(\mathbb{R}^1,d_E^\alpha)=\max_{(t,s)\in D_0}F(t,s)=F(a,a)=\left(\tfrac{1+a}{1-a}\right)^\alpha=m^\alpha$$

where $m=\frac{1+a}{1-a}>1$ is the unique solution of

$$2=\left(\frac{1+a-(1-a)}{1-a}\right)^\alpha+\left(\frac{1+a+(1-a)}{1-a}\right)^\alpha=(m-1)^\alpha+(m+1)^\alpha.$$ □

Remark 6.15. Let $0<\alpha\le 1$. It is *not* true in general that for any metric space (X,d) the inequality $C(X,d^\alpha)\le(C(X,d))^\alpha$ holds. For example, if $\alpha=1/2$ then $m=5/4$ as in Theorem 6.6 and so

$$C(\mathbb{R}^1,d_E^{1/2})=\sqrt{5}/2>(C(\mathbb{R}^1,d_E))^{1/2}=\sqrt{1}=1.$$

7. Distances on Riemannian manifolds

We show that the restricted quasi-hyperbolicity constant of the metric space associated to a Riemannian manifold of dimension greater than one is bounded from below by $\sqrt{2}$.

Proposition 7.1. *If M is a Riemannian manifold of dimension greater than one and d_M is the distance on M induced by the given Riemannian metric then $C_0(M, d_M) \geq \sqrt{2}$.*

Proof. Let $p \in M$ and let $\exp_p : T_pM \to M$ denote the Riemannian exponential map. The Riemannian metric on M endows the tangent space, T_pM, with an inner product and we write d_E for the corresponding Euclidean distance on T_pM. For a vector $X \in T_pM$ and a scalar t, let $X_t = \exp_p(tX) \in M$. If $X, Y \in T_pM$ then

$$\lim_{t\to 0} \frac{d_M(X_t, Y_t)}{t} = d_E(X, Y). \tag{7.2}$$

This is a consequence of the fact that in normal coordinates $\{x^i\}$ the components $g_{ij}(x)$ of the Riemannian metric satisfy the estimate $|g_{ij}(x) - \delta_{ij}| \leq C\|x\|^2$ for some C.

For $X, Y, Z, W \in T_pM$, not all identical,

$$\begin{aligned}\Delta(X_t,\, Y_t,\, Z_t\,, W_t) &= \frac{d_M(X_t, Y_t) + d_M(Z_t, W_t)}{\max\{d_M(X_t, Z_t) + d_M(Y_t, W_t),\, d_M(X_t, W_t) + d_M(Y_t, Z_t)\}} \\ &= \frac{d_M(X_t, Y_t)/t + d_M(Z_t, W_t)/t}{\max\{d_M(X_t, Z_t)/t + d_M(Y_t, W_t)/t,\, d_M(X_t, W_t)/t + d_M(Y_t, Z_t)/t\}}\end{aligned}$$

By (7.2), $\lim_{t\to 0} \Delta(X_t,\, Y_t,\, Z_t\,, W_t) = \Delta(X,\, Y,\, Z,\, W)$, where the second Δ is with respect to d_E. Since $\dim M > 1$, there are orthogonal unit vectors $U, V \in T_pM$. Since

$$C_0(M, d_M) \geq \Delta(U_t,\, V_t,\, 0_t,\, (U + V)_t),$$

it follows that

$$C_0(M, d_M) \geq \lim_{t\to 0} \Delta(U_t,\, V_t,\, 0_t,\, (U + V)_t) = \Delta(U,\, V,\, 0,\, U + V) = \sqrt{2},$$

establishing the conclusion of the proposition. □

Corollary 7.3. *Let M be a simply connected, complete Riemannian manifold of non-positive sectional curvature with associated distance d_M. Then $C_0(M, d_M) = \sqrt{2}$.*

Proof. By [BH99, Chapter II.1, Theorem 1A.6], the metric space (M, d_M) is a CAT(0)-space and so $C_0(M, d_M) \leq \sqrt{2}$ by Corollary 4.3. By Proposition 7.1, $C_0(M, d_M) \geq \sqrt{2}$. Thus $C_0(M, d_M) = \sqrt{2}$. □

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