

LIMIT THEOREMS ASSOCIATED WITH THE PITMAN-YOR PROCESS

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Abstract

The Pitman-Yor process is a random discrete measure. The random weights or masses follow the two-parameter Poisson-Dirichlet distribution with parameters $0 < \alpha < 1, \theta > -\alpha$. The parameters α and θ correspond to the stable and gamma components, respectively. The distribution of atoms is given by a probability ν . In this article we consider the limit theorems for the Pitman-Yor process and the two-parameter Poisson-Dirichlet distribution. These include law of large numbers, fluctuations, and moderate or large deviation principles. The limiting procedures involve either α tends to zero or one. They arise naturally in genetics and physics such as the asymptotic coalescence time for explosive branching process and the approximation to generalized random energy model for disordered system.

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1. Introduction

For any $0 \leq \alpha < 1, \theta + \alpha > 0$, let $U_1(\alpha, \theta), U_2(\alpha, \theta), \dots$ be a sequence of independent random variables with $U_i(\alpha, \theta)$ having distribution $Beta(1 - \alpha, \theta + i\alpha)$ for $i \geq 1$. If we define

$$V_1(\alpha, \theta) = U_1(\alpha, \theta), V_n(\alpha, \theta) = (1 - U_1(\alpha, \theta)) \cdots (1 - U_{n-1}(\alpha, \theta))U_n(\alpha, \theta), \quad n \geq 2,$$

then the law of the decreasing order statistic

$$\mathbf{P}(\alpha, \theta) = (P_1(\alpha, \theta), P_2(\alpha, \theta), \dots)$$

of $(V_1(\alpha, \theta), V_2(\alpha, \theta), \dots)$ is the two-parameter Poisson-Dirichlet distribution $PD(\alpha, \theta)$. It is a probability on the infinite-dimensional simplex

$$\nabla_\infty = \{\mathbf{p} = (p_1, p_2, \dots) : p_1 \geq p_2 \geq \dots \geq 0, \sum_{i=1}^{\infty} p_i \leq 1\}.$$

Let S be a Polish space and ν a probability on S satisfying $\nu(\{x\}) = 0$ for all x in S . In this case we say ν is diffuse. The Pitman-Yor process with parameters α, θ and ν is the random measure

$$\Xi_{\alpha, \theta, \nu} = \sum_{i=1}^{\infty} P_i(\alpha, \theta) \delta_{\xi_i}.$$

where ξ_1, ξ_2, \dots are i.i.d. with common distribution ν and is independent of $\mathbf{P}(\alpha, \theta)$. The case $\alpha = 0$ corresponds to the Dirichlet process constructed in [20].

The distribution $PD(0, \theta)$ was introduced by Kingman in [27] as the law of relative jump sizes of a gamma subordinator over the interval $[0, \theta]$. It also arises in other context most notably population genetics. The distribution $PD(\alpha, 0)$ was introduced in Kingman [27] through the stable subordinator. In [29] and [33], $PD(\alpha, 0)$ was constructed from the ranked length of excursion intervals between zeros of a Brownian motion ($\alpha = 1/2$) or a recurrent Bessel process of order $2(1 - \alpha)$ for general α .

In this paper we focus on the case $\theta = 0$. This implies that the parameter α is in $(0, 1)$. Without loss of generality, we choose the space S to be $[0, 1]$ and the probability ν to be the uniform distribution on $[0, 1]$. Our main objective is to study the asymptotic behaviour of $PD(\alpha, 0)$ when α converges to zero, and the behaviour of both $PD(\alpha, 0)$ and $\Xi_{\alpha, 0, \nu}$ when α converges to one. There are many scenarios where the limiting

procedure of α approaching one or zero arises naturally. We consider two examples below.

The first example is Derrida's random energy model (REM) introduced in [9] and [10]. This is a toy model for disordered system such as spin glasses. For any $N \geq 1$, let $S_N = \{-1, 1\}^N$ denote the configuration space. Then the REM is a family of i.i.d. random variables $\{H_N(\sigma) : \sigma \in S_N\}$ with common normal distribution of mean zero and variance N . Here $H_N(\sigma)$ is the Hamiltonian. Given the temperature T and $\beta = T^{-1}$, the Gibbs measure is a probability on S_N given by

$$Z_N^{-1} \exp\{-\beta H_N(\sigma)\}$$

where

$$Z_N = \sum_{\sigma \in S_N} \exp\{-\beta H_N(\sigma)\}$$

is the partition function. Let $T_c = \frac{1}{\sqrt{2 \ln 2}}$ and $\alpha = \frac{T}{T_c}$. Then for $T < T_c$ or equivalently $\beta > \sqrt{2 \ln 2}$, the decreasing order statistic of the Gibbs measure is known (cf. [37]) to converge to the Poisson-Dirichlet distribution $PD(\alpha, 0)$ as N tends to infinity. Thus α converging to zero corresponds to temperature going to zero while α converging to one corresponds to temperature rising to the critical value. To account for correlations, the generalized random energy model (GREM) involving hierarchical levels was introduced and studied in [11] and [12]. The generalization to continuum levels was done in [4] and the genealogy of the hierarchical systems is described by the Bolthausen-Sznitman coalescent. In deriving the infinitesimal rate of the coalescent (Proposition 4.11 in [3]), one needs to consider the limit of $PD(e^{-t}, 0)$ as t converges to zero or equivalently $\alpha = e^{-t}$ converging to one.

The second example is concerned with the coalescence time for an explosive branching process. Consider a Galton-Watson branching process with offspring distribution $\{p_j : j \geq 0\}$ in the domain of attraction of a stable law of order $0 < \gamma < 1$ and $p_0 = 0$. Let X_n denote the coalescence time of any two individuals chosen at random at generation n . Then it is shown in [1] that

$$\mathbb{P}\{n - X_n \leq k\} \rightarrow \pi(k) \text{ as } n \rightarrow \infty$$

where $\pi(k)$ can be calculated explicitly through $PD(\gamma^k, 0)$. In this case, $\alpha = \gamma^k$ converging to zero corresponds to k converging to infinity.

There have been intensive studies of the asymptotic behaviour for the Poisson-Dirichlet distribution and the Pitman-Yor process in recent years with motivations from probability theory, population genetics, and Bayesian statistics. The limiting procedures include the following three cases.

Case A: the parameter θ converges to infinity;

Case B: both θ and α converge to zero;

Case C: α converges to 1.

Results in Case A include central limit theorems in [22], [23], [24] and [25], large deviations in [6], [7], [14] and [28], and moderate deviations in [17]. It also includes the work in [13] and [21] where the large and moderate deviations for $\Xi_{0,\theta,\nu}$ were studied in a Bayesian context. In Case B, the large and moderate deviations were obtained in [15] and [18]. Recently the large deviations for $PD(\alpha, 0)$ in Case C is obtained in [19]. An extended account of asymptotic results for the Poisson-Dirichlet distribution and the Pitman-Yor process can be found in [16] and the references therein.

The main results of this paper include (i) the fluctuation and large deviations for $PD(\alpha, 0)$ in Case B when each coordinate is raised to a certain power (Theorem 2 and Theorem 4); (ii) the fluctuation theorem for $PD(\alpha, 0)$ in Case C (Theorem 3); (iii) the large deviations for $\Xi_{\alpha,0,\nu}$ in Case C (Theorem 8).

The existing asymptotic results for $PD(\alpha, \theta)$ in the literature in Case B involve only power 1. The scalings appear either as a shift or a multiplier. In (i) we studied the asymptotic behaviour of $PD(\alpha, 0)$ with various powers and discovered an interesting phase transition or critical behaviour in terms of the magnitude of the power. The scaling property of the stable subordinator plays the essential role here. The result in (ii) complements the result in [18] and reveals a non-Gaussian fluctuation.

The large deviations in (iii) provide more information on the microscopic transition structure at the critical temperature for the REM. At the instant when the temperature starts to move below the critical value T_c , a portion of mass of the uniform measure ν may be lost and is replaced by an atomic portion with finite atoms. This represents the emerging of finite number of energy valleys and the energy landscape of the system becomes a mixture of valleys and “flat” regions. The emerging of energy valleys follow the order where the small number of energy valleys is more likely to occur than a large number of valleys. Our result will validate this picture and show that large deviations

may lead to the disappearance of all “flat” regions. In comparison with the large deviation result in [19], the result in (iii) reveals the new information that no singular components arise through large deviations.

The behaviour of $\Xi_{\alpha,0,\nu}$ in Case C resembles the behaviour of $\Xi_{\alpha,\theta,\nu}$ in Case A in terms of the law of large numbers. But our result in (iii) will show that the two cases are very different in terms of large deviations.

The paper is organized as follows. In Section 2, we introduce the necessary terminologies and review the subordinator representation for $PD(\alpha, 0)$. Section 3 contains the law of large numbers, fluctuation, and large deviations associated with $PD(\alpha, 0)$ as α converges to zero or one. In Section 4, we establish the large deviation principle for $\Xi_{\alpha,0,\nu}$ under the limit of α converging to one. Large deviations for $PD(\alpha, 0)$ are closely related to the large deviations for $\Xi_{\alpha,0,\nu}$. Result for one usually gives some hint for the other. But a direct derivation from one to the other is difficult due to the topologies involved and the lack of continuous map. Thus our proof here will be completely new in comparison to the result in [19]. We close the paper in Section 5 with some concluding remarks on the differences between Case A and Case C .

2. Preliminaries

In this section, we introduce the necessary terminologies of large deviations and the subordinator representation of $PD(\alpha, 0)$. All results are stated in the form that is sufficient for our purposes. We refer to Dembo and Zeitouni [8] for a comprehensive introduction to the general theory of large deviations.

2.1. Terminologies

Let E be a complete separable metric space equipped with metric d . Consider a family of E -valued random variables $\{Y_\lambda : \lambda > 0\}$ with corresponding distributions $\{Q_\lambda : \lambda > 0\}$. Assume that Y_λ converges in probability to a constant as λ tends to a number λ_0 in $[0, +\infty]$.

Definition 1. (1). The family of probability measures $\{Q_\lambda : \lambda > 0\}$ (or the family $\{Y_\lambda : \lambda > 0\}$) is said to satisfy a *large deviation principle* with *speed* $a(\lambda)$ and *rate*

function $I(\cdot)$ if for any closed set F and open set G in E

$$\begin{aligned} \text{(upper bound)} \quad & \limsup_{\lambda \rightarrow \lambda_0} a^{-1}(\lambda) \log Q_\lambda\{F\} \leq - \inf_{x \in F} I(x), \\ \text{(lower bound)} \quad & \liminf_{\lambda \rightarrow \lambda_0} a^{-1}(\lambda) \log Q_\lambda\{G\} \geq - \inf_{x \in G} I(x), \end{aligned}$$

and $I(\cdot)$ is lower semicontinuous. The rate function $I(\cdot)$ is *good* if for any $c > 0$

the level set $\{x : I(x) \leq c\}$ is compact.

The set $\{x : I(x) < \infty\}$ is called the *effective domain* of the rate function.

(2). The family $\{Q_\lambda : \lambda > 0\}$ is said to satisfy a *local large deviation principle* with speed $a(\lambda)$ and rate function $I(\cdot)$ if for every x in E

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \limsup_{\lambda \rightarrow \lambda_0} a^{-1}(\lambda) \log P\{d(Y_\lambda, x) \leq \delta\} \\ & = \lim_{\delta \rightarrow 0} \liminf_{\lambda \rightarrow \lambda_0} a^{-1}(\lambda) \log P\{d(Y_\lambda, x) < \delta\} \\ & = -I(x). \end{aligned}$$

(3). The family $\{Q_\lambda : \lambda > 0\}$ is *exponentially tight* with speed $a(\lambda)$ if for every $L > 0$, there is a compact subset K_L of E such that

$$\limsup_{\lambda \rightarrow \lambda_0} a^{-1}(\lambda) \log P\{Y_\lambda \notin K_L\} \leq -L.$$

Remark. It is known that a local large deviation principle combined with exponential tightness implies the large deviation principle with a good rate function (cf. [36]).

Definition 2. (1). The family $\{Y_\lambda : \lambda > 0\}$ is said to satisfy a *fluctuation theorem* if there exists functions $b(\lambda), c(\lambda)$, and a finite non-deterministic random variable W such that

$$\lim_{\lambda \rightarrow \lambda_0} b(\lambda) = \infty, b(\lambda)[Y_\lambda - c(\lambda)] \Rightarrow W, \lambda \rightarrow \lambda_0$$

where \Rightarrow denotes convergence in distribution.

(2). Assume that the family $\{Y_\lambda : \lambda > 0\}$ satisfy the fluctuation theorem above. Let $e(\lambda)$ satisfy

$$\lim_{\lambda \rightarrow \lambda_0} e(\lambda) = \infty, \lim_{\lambda \rightarrow \lambda_0} \frac{e(\lambda)}{b(\lambda)} = 0. \quad (1)$$

Then the family $\{Q_\lambda : \lambda > 0\}$ or equivalently the family $\{Y_\lambda : \lambda > 0\}$ is said to satisfy a *moderate deviation principle* with speed $a(\lambda)$ (depending on $e(\lambda)$) and (good) rate

function $I(\cdot)$ if the family $\{e(\lambda)[Y_\lambda - c(\lambda)] : \lambda > 0\}$ satisfies a large deviation principle with speed $a(\lambda)$ and (good) rate function $I(\cdot)$. Thus the moderate deviation principle for $\{Y_\lambda : \lambda > 0\}$ is the large deviation principle for $\{e(\lambda)[Y_\lambda - c(\lambda)] : \lambda > 0\}$.

2.2. Subordinator Representation

For any $0 < \alpha < 1$, let ρ_t be the stable subordinator with index α and Lévy measure

$$\Lambda_\alpha(dx) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-(1+\alpha)} dx, \quad x > 0.$$

The boundary case $\alpha = 1$ corresponds to the straight line $\rho_t = t$. When α converges to zero, ρ_t becomes a killed subordinator with killing rate one ([2]).

For any $t > 0$, let $J^1(\rho_t) \geq J^2(\rho_t) \geq \dots$ denote the jump sizes of ρ_t over the interval $[0, t]$. Then the following representation holds.

Theorem 1. (Perman, Pitman, and Yor [29]) *For any $t > 0$, the law of*

$$\left(\frac{J^1(\rho_t)}{\rho_t}, \frac{J^2(\rho_t)}{\rho_t}, \dots \right) \quad (2)$$

is $PD(\alpha, 0)$.

For any $n \geq 1$, let $Z_n = \Lambda_\alpha(J^n(\rho_1), \infty)$. Then $Z_1 < Z_2 < \dots$ and $Z_1, Z_2 - Z_1, Z_3 - Z_2, \dots$ are i.i.d. exponential random variables with parameter 1 (cf. [29]). Noting that $\Lambda_\alpha(x, \infty) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}$, it follows that

$$\frac{J^n(\rho_1)}{\rho_1} = \frac{Z_n^{-1/\alpha}}{\sum_{i=1}^{\infty} Z_i^{-1/\alpha}} \quad (3)$$

and

$$\rho_1 = \Gamma(1-\alpha)^{-1/\alpha} \sum_{i=1}^{\infty} Z_i^{-1/\alpha}. \quad (4)$$

Thus by Theorem 1 the law of

$$\left(\frac{Z_1^{-1/\alpha}}{\sum_{i=1}^{\infty} Z_i^{-1/\alpha}}, \frac{Z_2^{-1/\alpha}}{\sum_{i=1}^{\infty} Z_i^{-1/\alpha}}, \dots \right) \quad (5)$$

is $PD(\alpha, 0)$. In other words we have the representation

$$P_n(\alpha, 0) = \frac{Z_n^{-1/\alpha}}{\sum_{i=1}^{\infty} Z_i^{-1/\alpha}}, \quad n = 1, 2, \dots$$

3. Limit Theorems for $PD(\alpha, 0)$

Take a sample of size two from a population with frequencies following the distribution $PD(\alpha, 0)$. Then the probability that the two samples are of the same type is given by

$$\varphi_2(\mathbf{P}(\alpha, 0)) := \sum_{i=1}^{\infty} P_i^2(\alpha, 0).$$

This function is called the homozygosity in population genetics. It is closely associated with the Shannon entropy in communication, the Herfindahl-Hirschman index in economics, and the Gini-Simpson index in ecology. It is a measure of concentration of the population in terms of types with small values corresponding to lower concentration.

A direct application of Pitman's sampling formula ([30], [32]) leads to

$$\mathbb{E}_{\alpha, 0}[\varphi_2(\mathbf{P}(\alpha, 0))] = 1 - \alpha.$$

This implies that $\mathbf{P}(\alpha, 0)$ converges in probability to $(1, 0, \dots)$ and $(0, 0, \dots)$ as α converges to 0 and 1, respectively. The objective of this section is to obtain more detailed information associated with these limits including fluctuation and large deviations.

3.1. Convergence and Limit

Let $0 < \gamma(\alpha) \leq 1$ and $\iota(\alpha) > 0$ be such that

$$\lim_{\alpha \rightarrow 0} \frac{\gamma(\alpha)}{\alpha} = c_1 \in [0, +\infty] \quad (6)$$

and

$$\lim_{\alpha \rightarrow 1} \frac{\iota(\alpha)}{\Gamma(1 - \alpha)} = c_2 \in [0, \infty). \quad (7)$$

Theorem 2. *Let*

$$\mathbf{P}^{\gamma(\alpha)}(\alpha, 0) = (P_1^{\gamma(\alpha)}(\alpha, 0), P_2^{\gamma(\alpha)}(\alpha, 0), \dots).$$

If c_1 is finite, then $\mathbf{P}^{\gamma(\alpha)}(\alpha, 0)$ converges almost surely to $(1, (\frac{Z_1}{Z_2})^{c_1}, (\frac{Z_1}{Z_3})^{c_1}, \dots)$ as α converges to 0. If $c_1 = \infty$, then $\mathbf{P}^{\gamma(\alpha)}(\alpha, 0)$ converges to $(1, 0, \dots)$ in probability as α converges to 0.

Proof: Set

$$\tilde{\mathbf{Z}} = (Z_1^{-1}, Z_2^{-1}, \dots).$$

Then we have

$$\left(\sum_{i=1}^{\infty} Z_i^{-1/\alpha}\right)^\alpha = \|\tilde{\mathbf{Z}}\|_{1/\alpha}.$$

When α approaches to zero, $\|\tilde{\mathbf{Z}}\|_{1/\alpha}$ converges almost surely to $\|\tilde{\mathbf{Z}}\|_\infty = Z_1^{-1}$. This implies that

$$\mathbf{P}^\alpha(\alpha, 0) = \left(\frac{Z_1^{-1}}{\|\tilde{\mathbf{Z}}\|_{1/\alpha}}, \frac{Z_2^{-1}}{\|\tilde{\mathbf{Z}}\|_{1/\alpha}}, \dots\right)$$

converges almost surely to $(1, \frac{Z_1}{Z_2}, \frac{Z_1}{Z_3}, \dots)$ as α converges to zero. Writing $\mathbf{P}^{\gamma(\alpha)}(\alpha, 0)$ as

$$\left(\left(\frac{Z_1^{-1}}{\|\tilde{\mathbf{Z}}\|_{1/\alpha}}\right)^{\gamma(\alpha)/\alpha}, \left(\frac{Z_2^{-1}}{\|\tilde{\mathbf{Z}}\|_{1/\alpha}}\right)^{\gamma(\alpha)/\alpha}, \dots\right).$$

Then by continuity we obtain that $\mathbf{P}^{\gamma(\alpha)}(\alpha, 0)$ converges almost surely to

$$\left(1, \left(\frac{Z_1}{Z_2}\right)^{c_1}, \left(\frac{Z_1}{Z_3}\right)^{c_1}, \dots\right)$$

as α converges to zero. If $c_1 = \infty$, then for any $M \geq 1$ one has $\frac{\gamma(\alpha)}{\alpha} > M$ for small enough α . Thus for any $n > 1$

$$\lim_{\alpha \rightarrow 0} P_n^{\gamma(\alpha)}(\alpha, 0) \leq \lim_{\alpha \rightarrow 0} P_n^M(\alpha, 0) = \left(\frac{Z_1}{Z_n}\right)^M.$$

Since M is arbitrary, we obtain

$$\lim_{\alpha \rightarrow 0} P_n^{\gamma(\alpha)}(\alpha, 0) = 0, \text{ a.s., } n > 1.$$

Finally for $n = 1$, we have

$$P_1(\alpha, 0) \leq P_1^{\gamma(\alpha)}(\alpha, 0) \leq 1.$$

Noting that

$$\mathbb{E}[P_1(\alpha, 0)] \leq \mathbb{E}[\varphi_2(\mathbf{P}(\alpha, 0))] = 1 - \alpha.$$

It follows that $P_1(\alpha, 0)$ converges to 1 in probability which implies that $P_1^{\gamma(\alpha)}(\alpha, 0)$ converges to one in probability. \square

Theorem 3. *As α converges to 1, $\iota(\alpha)\mathbf{P}(\alpha, 0)$ converges to $c_2(Z_1^{-1}, Z_2^{-1}, \dots)$ in probability.*

Proof: Let $S_\alpha = \rho_1^{-\alpha}$. Then the law of S_α is the Mittag-Leffler distribution with density function

$$g_\alpha(s) = \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} \Gamma(\alpha k + \alpha + 1) \frac{\sin(\alpha k \pi)}{\alpha k \pi}$$

and

$$\sum_{i=1}^{\infty} Z_i^{-1/\alpha} = \left(\frac{S_\alpha}{\Gamma(1-\alpha)}\right)^{-1/\alpha} \quad (8)$$

$$\mathbb{E}[S_\alpha^r] = \frac{\Gamma(r+1)}{\Gamma(\alpha r+1)}, \quad r > -1. \quad (9)$$

This implies that

$$\begin{aligned} \mathbb{E}[(S_\alpha - 1)^2] &= \frac{2}{\Gamma(2\alpha+1)} - \frac{2}{\Gamma(\alpha+1)} + 1 \\ &\rightarrow 0, \quad \alpha \rightarrow 1. \end{aligned}$$

Hence S_α converges to 1 in probability as α converges to 1.

By (4), one has

$$\begin{aligned} \iota(\alpha)\mathbf{P}(\alpha, 0) &= \frac{\iota(\alpha)}{\Gamma(1-\alpha)} \Gamma(1-\alpha)^{1-\frac{1}{\alpha}} \left(\left(\frac{Z_1}{S_\alpha}\right)^{-1/\alpha}, \left(\frac{Z_2}{S_\alpha}\right)^{-1/\alpha}, \dots\right) \\ &= \frac{\iota(\alpha)}{\Gamma(1-\alpha)} \Gamma(1-\alpha)^{1-\frac{1}{\alpha}} \exp\left\{\frac{1}{\alpha} \log S_\alpha\right\} (Z_1^{-1/\alpha}, Z_2^{-1/\alpha}, \dots). \end{aligned}$$

Since S_α converges to one in probability and $(Z_1^{-1/\alpha}, Z_2^{-1/\alpha}, \dots)$ converges to

$$(Z_1^{-1}, Z_2^{-1}, \dots)$$

almost surely as α converges to one, we conclude that $\iota(\alpha)\mathbf{P}(\alpha, 0)$ converges in probability to

$$c_2(Z_1^{-1}, Z_2^{-1}, \dots).$$

□

Remark. The random variable S_α is Pitman's α -diversity ([32]). Consider the random partition Π_∞ of the set $\{1, 2, \dots\}$ with asymptotic frequencies following the $PD(\alpha, 0)$ distribution. For any $n \geq 2$, let Π_n denote the restriction of Π_∞ on $\{1, 2, \dots, n\}$ and K_n denote number of blocks in the random partition Π_n . The random variable S_α is the asymptotic limit of K_n/n^α as n tends to infinity (cf. Theorem 3.8 in [32]). When α converges to 1, S_α converges to 1. Thus all blocks will become singletons.

3.2. Large Deviations

In this section we consider the large deviations associated with the deterministic limits obtained in Theorem 2. In comparison with the large deviations for $\mathbf{P}(\alpha, 0)$

these results may be viewed as moderate deviations for $\log \mathbf{P}(\alpha, 0) = (\log P_1(\alpha, 0), \dots)$.

We prove these results through a series of lemmas.

For any $n \geq 1$ let

$$R_n = \frac{P_{n+1}(\alpha, 0)}{P_n(\alpha, 0)}.$$

Then $\{R_n : n \geq 1\}$ is a sequence of independent beta random variables with each R_n having the $Beta(n\alpha, 1)$ distribution (Proposition 8 in [34]).

Lemma 1. *Let $\mathbf{R}^{\gamma(\alpha)} = (R_1^{\gamma(\alpha)}, R_2^{\gamma(\alpha)}, \dots)$. As α converges to 0, large deviation principles hold for $\mathbf{R}^{\gamma(\alpha)}$ on space $[0, 1]^\infty$ with respective speeds and rate functions $(\frac{\alpha}{\gamma(\alpha)}, \tilde{I}_1(\cdot))$ and $(\log \frac{\gamma(\alpha)}{\alpha}, \tilde{I}_2(\cdot))$ depending on whether $c_1 = 0$ or $c_1 = \infty$, where*

$$\tilde{I}_1(\mathbf{x}) = \begin{cases} \sum_{n=1}^{\infty} n \log \frac{1}{x_n}, & x_n > 0 \text{ for all } n > 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

and

$$\tilde{I}_2(\mathbf{x}) = \#\{n \geq 1 : x_n > 0\}.$$

Proof: First note that for any $0 \leq a < b \leq 1$, we have that for any $n \geq 1$

$$\mathbb{P}\{R_n^{\gamma(\alpha)} \in (a, b)\} = \mathbb{P}\{R_n^{\gamma(\alpha)} \in [a, b]\} = b^{n \frac{\alpha}{\gamma(\alpha)}} - a^{n \frac{\alpha}{\gamma(\alpha)}}. \quad (10)$$

Assume that $c_1 = 0$. Then we have $\lim_{\alpha \rightarrow 0} \frac{\alpha}{\gamma(\alpha)} = \infty$. For any $n \geq 1$ and any x in $[0, 1]$, one has by applying (10) that

$$\begin{aligned} \lim_{\delta \rightarrow 0} \liminf_{\alpha \rightarrow 0} \frac{\gamma(\alpha)}{\alpha} \log \mathbb{P}\{|R_n^{\gamma(\alpha)} - x| < \delta\} &= \lim_{\delta \rightarrow 0} \limsup_{\alpha \rightarrow 0} \frac{\gamma(\alpha)}{\alpha} \log \mathbb{P}\{|R_n^{\gamma(\alpha)} - x| \leq \delta\} \\ &= n \log x, \end{aligned}$$

where $\log x = -\infty$ for $x = 0$. This combined with the compactness of $[0, 1]$ implies that $R_n^{\gamma(\alpha)}$ satisfies a large deviation principle on $[0, 1]$ with speed $\frac{\alpha}{\gamma(\alpha)}$ and rate function $n \log x$. Similarly for $c_1 = \infty$, we have

$$\begin{aligned} &\lim_{\delta \rightarrow 0} \liminf_{\alpha \rightarrow 0} (\log \frac{\gamma(\alpha)}{\alpha})^{-1} \log \mathbb{P}\{|R_n^{\gamma(\alpha)} - x| < \delta\} \\ &= \lim_{\delta \rightarrow 0} \limsup_{\alpha \rightarrow 0} (\log \frac{\gamma(\alpha)}{\alpha})^{-1} \log \mathbb{P}\{|R_n^{\gamma(\alpha)} - x| \leq \delta\} \\ &= -\chi_{(0, \infty)} \end{aligned}$$

where $\chi_{(0,\infty)}$ is the indicator function of set $(0, \infty)$. These combined with the independence of R_1, R_2, \dots imply the large deviations for $\mathbf{R}^{\gamma(\alpha)}$. \square

Lemma 2. *There exists $\delta \geq 1$ such that for any $\lambda < \delta$*

$$\mathbb{E}[\exp\{\lambda(1-\alpha)(P_1^{-1}(\alpha, 0) - 1)\}] = (1 + A_{\lambda,\alpha})^{-1} < \infty \quad (11)$$

where

$$A_{\lambda,\alpha} = \alpha \int_0^1 (1 - e^{\lambda(1-\alpha)z}) z^{-(1+\alpha)} dz.$$

Proof: Clearly $A_{\lambda,\alpha}$ is nonnegative for $\lambda \leq 0$, and converges to negative infinity as λ tends to positive infinity. It is known (equation (77) in [27]) that

$$\mathbb{E}[\exp\{\lambda(1-\alpha)(P_1^{-1}(\alpha, 0) - 1)\}] = (1 + A_{\lambda,\alpha})^{-1} < \infty \quad (12)$$

for $\lambda \leq 0$. For $\lambda > 0$, we have

$$\begin{aligned} A_{\lambda,\alpha} &= (1-\lambda)e^{\lambda(1-\alpha)} - 1 + \lambda^2(1-\alpha) \int_0^1 z^{1-\alpha} e^{\lambda(1-\alpha)z} dz \\ &\geq (1-\lambda)e^{\lambda(1-\alpha)} - 1 + \lambda^2(1-\alpha) \int_0^1 z^{1-\alpha} e^{\lambda(1-\alpha)z} dz. \end{aligned} \quad (13)$$

If we define

$$\lambda_\alpha = \sup\{\lambda > 0 : A_{\lambda,\alpha} + 1 > 0\},$$

then $\lambda_\alpha \geq 1$ by (13) and

$$\delta = \inf\{\lambda_\alpha : 0 < \alpha < 1\} \geq 1.$$

By Campbell's theorem (11) holds for any $\lambda < \delta$. \square

Lemma 3. *Let $\epsilon > 0$ be arbitrarily given. If $c_1 = 0$, then*

$$\limsup_{\alpha \rightarrow 0} \frac{\gamma(\alpha)}{\alpha} \log \mathbb{P}\{|P_1^{\gamma(\alpha)}(\alpha, 0) - 1| > \epsilon\} = -\infty. \quad (14)$$

If $c_1 = \infty$ and

$$\lim_{\alpha \rightarrow 0} \gamma(\alpha) = 0, \quad (15)$$

then

$$\limsup_{\alpha \rightarrow 0} \frac{1}{\log \frac{\gamma(\alpha)}{\alpha}} \log \mathbb{P}\{|P_1^{\gamma(\alpha)}(\alpha, 0) - 1| > \epsilon\} = -\infty. \quad (16)$$

Proof: Since the limit involves only small α , we may assume that $0 < \alpha < 1/2$ and $0 < \epsilon < 1/2$. Let δ be as in Lemma 2 and set $\delta_1 = \delta/4$. By direct calculation we obtain that

$$\begin{aligned} \mathbb{P}\{|P_1^{\gamma(\alpha)}(\alpha, 0) - 1| > \epsilon\} &= \mathbb{P}\{P_1^{-1}(\alpha, 0) - 1 \geq (1 - \epsilon)^{-1/\gamma(\alpha)} - 1\} \\ &\leq \mathbb{E}[e^{\delta_1(P_1^{-1}(\alpha, 0) - 1)}]e^{-\delta_1[(1 - \epsilon)^{-1/\gamma(\alpha)} - 1]} \\ &\leq (1 + A_{\delta_1, \alpha})^{-1}e^{-\delta_1[(1 - \epsilon)^{-1/\gamma(\alpha)} - 1]}. \end{aligned} \quad (17)$$

It follows from (13) that

$$\lim_{\alpha \rightarrow 0} (1 + A_{\delta_1, \alpha}) = 1. \quad (18)$$

If $c_1 = 0$, then

$$\begin{aligned} \limsup_{\alpha \rightarrow 0} \frac{(1 - \epsilon)^{-1/\gamma(\alpha)} - 1}{\frac{\alpha}{\gamma(\alpha)}} &= \limsup_{\alpha \rightarrow 0} \frac{(1 - \epsilon)^{-1/\gamma(\alpha)}}{\frac{\alpha}{\gamma(\alpha)}} \\ &= \limsup_{\alpha \rightarrow 0} \exp\left\{\frac{1}{\gamma(\alpha)}\left[\log \frac{1}{(1 - \epsilon)} + \gamma(\alpha) \log \gamma(\alpha) - \frac{\gamma(\alpha)}{\alpha} \alpha \log \alpha\right]\right\} \\ &= \infty. \end{aligned} \quad (19)$$

Next assume that $c_1 = \infty$ and (15) hold. For any $0 < \epsilon < 1/2$, $(1 - \epsilon)^{1/\gamma(\alpha)}$ converges to zero as α tends to zero. Hence for any $k \geq 1$, one can find $\alpha_k > 0$ such that for all $0 < \alpha < \alpha_k$

$$\mathbb{P}\{|P_1^{\gamma(\alpha)}(\alpha, 0) - 1| > \epsilon\} \leq \mathbb{P}\{P_1(\alpha, 0) < \frac{1}{k}\}.$$

By the large deviation principle for $P_1(\alpha, 0)$ (cf. Lemma 2.3 in [15]), we obtain that

$$\begin{aligned} \limsup_{\alpha \rightarrow 0} \frac{1}{\log \frac{1}{\alpha}} \log \mathbb{P}\{|P_1^{\gamma(\alpha)}(\alpha, 0) - 1| > \epsilon\} &\leq \limsup_{\alpha \rightarrow 0} \frac{1}{\log \frac{1}{\alpha}} \log \mathbb{P}\{P_1(\alpha, 0) < \frac{1}{k}\} \\ &\leq -(k - 1). \end{aligned}$$

Noting that $\gamma(\alpha) < 1$ and k is arbitrary it follows that

$$\begin{aligned} \limsup_{\alpha \rightarrow 0} \frac{1}{\log \frac{\gamma(\alpha)}{\alpha}} \log \mathbb{P}\{|P_1^{\gamma(\alpha)}(\alpha, 0) - 1| > \epsilon\} &\leq \limsup_{\alpha \rightarrow 0} \frac{1}{\log \frac{1}{\alpha}} \log \mathbb{P}\{|P_1^{\gamma(\alpha)}(\alpha, 0) - 1| > \epsilon\} \\ &\leq \lim_{k \rightarrow \infty} \limsup_{\alpha \rightarrow 0} \frac{1}{\log \frac{1}{\alpha}} \log \mathbb{P}\{P_1(\alpha, 0) \leq \frac{1}{k}\} \\ &= -\infty. \end{aligned} \quad (20)$$

Putting together (17)-(20), we get (14) and (16).

□

Theorem 4. *Let $\gamma(\alpha)$ satisfy (6), and set*

$$\nabla = \{\mathbf{x} = (x_1, x_2, \dots) : 1 \geq x_1 \geq x_2 \geq \dots \geq 0\}.$$

Then the followings hold as α converges to 0.

(i) *If $c_1 = 0$, then the family $\{\mathbf{P}^{\gamma(\alpha)}(\alpha, 0) : 0 < \alpha < 1\}$ satisfies a large deviation principle on space ∇ with speed $\frac{\alpha}{\gamma(\alpha)}$ and rate function*

$$I_1(\mathbf{x}) = \begin{cases} \sum_{n=1}^{\infty} n \log \frac{x_n}{x_{n+1}}, & x_1 = 1, x_n > 0 \text{ for all } n > 1, \\ +\infty, & \text{otherwise.} \end{cases} \quad (21)$$

(ii) *If $c_1 = \infty$ and (15) holds, then the family $\{\mathbf{P}^{\gamma(\alpha)}(\alpha, 0) : 0 < \alpha < 1\}$ satisfies a large deviation principle on space ∇ with speed $\log \frac{\gamma(\alpha)}{\alpha}$ and the rate function*

$$I_2(\mathbf{x}) = \begin{cases} n - 1, & x_1 = 1, x_n > 0, x_k = 0, k > n, \\ +\infty, & \text{otherwise.} \end{cases} \quad (22)$$

Proof: Writing $\mathbf{P}^{\gamma(\alpha)}$ in terms of $\mathbf{R}^{\gamma(\alpha)}$ we have

$$\mathbf{P}^{\gamma(\alpha)} = P_1^{\gamma(\alpha)}(\alpha, 0)(1, R_1^{\gamma(\alpha)}, R_1^{\gamma(\alpha)} R_2^{\gamma(\alpha)}, \dots).$$

By Lemma 3, $P_1^{\gamma(\alpha)}(\alpha, 0)$ is exponentially equivalent to one. Hence by lemma 2.1 in [17] $(1, R_1^{\gamma(\alpha)}, R_1^{\gamma(\alpha)} R_2^{\gamma(\alpha)}, \dots)$ and $\mathbf{P}^{\gamma(\alpha)}$ have the same large deviation principle. Define

$$\psi : [0, 1]^\infty \longrightarrow \nabla, \quad (x_1, x_2, \dots) \rightarrow (1, x_1, x_1 x_2, \dots).$$

Then ψ is clearly continuous and $(1, R_1^{\gamma(\alpha)}, R_1^{\gamma(\alpha)} R_2^{\gamma(\alpha)}, \dots) = \psi(\mathbf{R}^{\gamma(\alpha)})$. Noting that

$$I_i(\mathbf{x}) = \inf\{\tilde{I}_i(\mathbf{y}) : \psi(\mathbf{y}) = \mathbf{x}\}, i = 1, 2,$$

the theorem follows from Lemma 1 and the contraction principle.

□

4. Asymptotic Behaviour of $\Xi_{\alpha,0,\nu}$

Recall that the REM has configuration space $S_N = \{-1, 1\}^N$ and the Hamiltonian given by a family of *i.i.d.* normal random variables with mean 0 and variance N

$$\{H_N(\sigma) \mid \sigma \in S_N\}.$$

The Gibbs measure $G_N(\sigma)$ at temperature T is given by

$$Z_N^{-1} \exp\{-\beta H_N(\sigma)\},$$

where $\beta = 1/T$ and $Z_N = \sum_{\sigma \in S_N} \exp\{-\beta H_N(\sigma)\}$. By making the change of variable

$$r_N(\sigma) = 1 - \sum_{i=1}^N (1 - \sigma_i) 2^{-i-1},$$

we can regard $[0, 1]$ as the new configuration space. The corresponding Gibbs measure has the form

$$\mu_N^T(dx) = \sum_{\sigma \in S_N} \delta_{r_N}(\sigma)(dx) G_N(\sigma).$$

As $N \rightarrow \infty$, the limiting Gibbs measure $\mu^T = \lim_{N \rightarrow \infty} \mu_N^T$ exhibits phase transition at the critical temperature $T_c = \sqrt{2 \log 2}$. More specifically, by Theorems 9.3.1 and 9.3.4 in [5], we have

$$\mu^T = \begin{cases} \nu, & \text{if } T \geq T_c \\ \Xi_{\alpha,0,\nu}, & \text{if } T < T_c. \end{cases}$$

Thus a phase transition occurs when the temperature crosses the critical value between high temperature and low temperature regimes. The low temperature regime has a rich structure. The transition from the low temperature regime to the critical temperature regime corresponds to α tending to one from below. The goal of this section is to understand the microscopic behaviour of this transition through the establishment of a large deviation principle for $\Xi_{\alpha,0,\nu}$.

4.1. Estimates for Stable Subordinator

Recall that ρ_t be the stable subordinator with index $0 < \alpha < 1$. For $t = 1$, the following holds.

Lemma 4. ([35], [26]) *The distribution function of $\rho_1^{\frac{\alpha}{1-\alpha}}$ has two integral representations:*

$$F(x) = \mathbb{P}\{\rho_1^{\frac{\alpha}{1-\alpha}} \leq x\} = \frac{1}{\pi} \int_0^\pi e^{-\frac{A(u)}{x}} du, \quad (23)$$

where $A(u)$ is the Zolotarev's function defined as

$$A(u) = \left\{ \frac{\sin^\alpha(\alpha u) \sin^{1-\alpha}((1-\alpha)u)}{\sin u} \right\}^{\frac{1}{1-\alpha}}.$$

The distribution function of ρ_1 is thus $F(x^{\frac{\alpha}{1-\alpha}})$. The density function of ρ_1 has the following representation

$$\phi_\alpha(t) = \frac{1}{\pi} \int_0^\infty e^{-tu} e^{-u^\alpha \cos \pi \alpha} \sin(u^\alpha \sin \pi \alpha) du \quad (24)$$

Applying these representations, we obtain the following estimations.

Theorem 5. *For any given $1 > \delta > 0$, we have*

$$\lim_{\alpha \rightarrow 1} (1-\alpha) \log \log \frac{1}{\mathbb{P}\{\rho_1 < 1-\delta\}} = \lim_{\alpha \rightarrow 1} (1-\alpha) \log \log \frac{1}{\mathbb{P}\{\rho_1 \leq 1-\delta\}} = \log \frac{1}{1-\delta} \quad (25)$$

and

$$\lim_{\alpha \rightarrow 1} \frac{1}{\log \frac{1}{1-\alpha}} \log \mathbb{P}\{\rho_1 > 1+\delta\} = \lim_{\alpha \rightarrow 1} \frac{1}{\log \frac{1}{1-\alpha}} \log \mathbb{P}\{\rho_1 \geq 1+\delta\} = -1. \quad (26)$$

Proof: For any $u \in (0, \pi), v \in (0, 1)$, one has

$$\begin{aligned} \frac{d[v \cot(vu) - \cot u]}{dv} &= \frac{1}{2 \sin^2(vu)} (\sin(vu) - 2vu) \\ &\leq \frac{1}{2 \sin^2(vu)} (\sin(vu) - vu) \leq 0 \end{aligned}$$

which implies that

$$\frac{d \log \frac{\sin(vu)}{\sin u}}{du} = v \cot(vu) - \cot u \geq 0.$$

Hence

$$A(u) = \exp\left\{ \alpha \log \frac{\sin(\alpha u)}{\sin u} + (1-\alpha) \log \frac{\sin((1-\alpha)u)}{\sin u} \right\}$$

is nondecreasing in u . Further more it follows from direct calculation that

$$A(0+) = \lim_{u \rightarrow 0} A(u) = (1-\alpha) \alpha^{\frac{\alpha}{1-\alpha}} \quad \lim_{u \rightarrow \pi} A(u) = \infty.$$

Therefore, applying the representation (23) we get that for any $\epsilon > 0$

$$\begin{aligned} & \frac{\pi - \epsilon}{\pi} \exp \left\{ -\frac{A(\pi - \epsilon)}{(1 - \delta)^{\frac{\alpha}{1 - \alpha}}} \right\} \\ & \leq \frac{1}{\pi} \int_0^{\pi - \epsilon} e^{-\frac{A(u)}{(1 - \delta)^{\frac{\alpha}{1 - \alpha}}}} du = \mathbb{P}\{\rho_1 \leq 1 - \delta\} \\ & \leq \exp \left\{ -\frac{A(0+)}{(1 - \delta)^{\frac{\alpha}{1 - \alpha}}} \right\}. \end{aligned}$$

This implies that

$$\frac{1}{\mathbb{P}\{\rho_1 \leq 1 - \delta\}} \geq \exp \left\{ \exp \left\{ \log A(0+) + \frac{\alpha}{1 - \alpha} \log \frac{1}{1 - \delta} \right\} \right\}$$

and

$$\frac{1}{\mathbb{P}\{\rho_1 \leq 1 - \delta\}} \leq \frac{\pi}{\pi - \epsilon} \exp \left\{ \exp \left\{ \log A(\pi - \epsilon) + \frac{\alpha}{1 - \alpha} \log \frac{1}{1 - \delta} \right\} \right\}.$$

Taking the double logarithm and letting α converge to 1 we have

$$\begin{aligned} \log \frac{1}{1 - \delta} & \leq \liminf_{\alpha \rightarrow 1} (1 - \alpha) \log \log \frac{1}{\mathbb{P}\{\rho_1 \leq 1 - \delta\}} \\ & = \liminf_{\alpha \rightarrow 1} (1 - \alpha) \log \log \frac{1}{\mathbb{P}\{\rho_1 < 1 - \delta\}} \\ & \leq \limsup_{\alpha \rightarrow 1} (1 - \alpha) \log \log \frac{1}{\mathbb{P}\{\rho_1 < 1 - \delta\}} \\ & = \limsup_{\alpha \rightarrow 1} (1 - \alpha) \log \log \frac{1}{\mathbb{P}\{\rho_1 \leq 1 - \delta\}} \\ & \leq \log \left(\frac{1}{1 - \delta} \right). \end{aligned}$$

which is (25).

To prove (26), we apply (24) and get

$$\begin{aligned} \mathbb{P}\{\rho_1 > 1 + \delta\} & = \mathbb{P}\{\rho_1 \geq 1 + \delta\} \\ & = \frac{1}{\pi} \int_{1 + \delta}^{\infty} \int_0^{\infty} u^{-1} e^{-(1 + \delta)u} e^{-u^\alpha \cos \pi \alpha} \sin(u^\alpha \sin \pi \alpha) du dt \\ & = \frac{\sin \pi \alpha}{\pi} \int_0^{\infty} u^{-(1 - \alpha)} e^{-\delta u} \left[e^{-u - u^\alpha \cos \pi \alpha} \frac{\sin(u^\alpha \sin \pi \alpha)}{u^\alpha \sin \pi \alpha} \right] du \end{aligned}$$

Noting that $\frac{\sin(u^\alpha \sin \pi \alpha)}{u^\alpha \sin \pi \alpha}$ is bounded and

$$\lim_{\alpha \rightarrow 1} \frac{\sin \pi \alpha}{\pi(1 - \alpha)} = 1,$$

it follows that (26) holds. □

Theorem 6. *The family $\{\rho_1 : 0 < \alpha < 1\}$ satisfies a large deviation principle on $(0, \infty)$ as α tends to one with speed $-\log(1 - \alpha)$ and rate function (not good in this case)*

$$J(x) = \begin{cases} 1, & x > 1, \\ 0, & x = 1 \\ +\infty, & \text{otherwise.} \end{cases} \quad (27)$$

Proof: Let A be a closed set in $(0, \infty)$. If A contains 1, then $\inf_{x \in A} J(x) = 0$ and the upper estimate holds. If A does not contain 1, then one can find $0 < a < 1 < b$ such that A is either a subset of $(0, a]$, a subset of $[b, \infty)$ or a subset of $(0, a] \cup [b, \infty)$. For each case we can apply Theorem 5 to obtain the upper estimate.

The proof for lower estimates goes as follows. Let B be any open set. If B intersects with $[0, 1)$, then the lower estimates are trivial. If B does not intersect with $[0, 1)$, then B can not contain 1. Hence one can find $1 < a < b < \infty$ such that $(a, b) \subset B$ and

$$\begin{aligned} \mathbb{P}\{\rho_1 \in B\} &\geq \mathbb{P}\{\rho_1 \in (a, b)\} \\ &\geq \frac{b-a}{\pi} \int_0^\infty u^{-1} e^{-bu} e^{-u^\alpha \cos \pi \alpha} \sin(u^\alpha \sin \pi \alpha) du \end{aligned}$$

which implies that

$$\liminf_{\alpha \rightarrow 1} \frac{1}{-\log(1 - \alpha)} \log \mathbb{P}\{\rho_1 \in B\} \geq -1 = -\inf_{x \in B} J(x)$$

□

For any $n \geq 1$, let $\tau_1, \dots, \tau_{n+1}$ be independent copies of ρ_1 . Set

$$\sigma_i = \frac{\tau_i}{\tau_1}, \quad i = 2, \dots, n+1. \quad (28)$$

Set

$$\tilde{\sigma}_n = \min\{\sigma_i : 2 \leq i \leq n+1\}$$

and let r_n denote the frequency of $\tilde{\sigma}_n$ among $\{\sigma_i\}_{i=2, \dots, n+1}$. Define

$$J_n(u_1, \dots, u_n) = \begin{cases} n+1 - r_n, & \tilde{\sigma}_n < 1, \\ n - r_n, & \tilde{\sigma}_n = 1, \\ n, & \tilde{\sigma}_n > 1. \end{cases} \quad (29)$$

Clearly $J_n(\cdot)$ is a rate function on $(0, \infty)^n$.

Theorem 7. *The family $\{(\sigma_2, \dots, \sigma_{n+1}) : 0 < \alpha < 1\}$ satisfies a large deviation principle on $(0, \infty)^n$ with speed $-\log(1 - \alpha)$ and rate function $J_n(\cdot)$ as α tends to one.*

Proof: Note that the map

$$\Phi : (0, \infty)^{n+1} \rightarrow (0, \infty)^n, (x_1, \dots, x_{n+1}) \rightarrow \left(\frac{x_2}{x_1}, \dots, \frac{x_{n+1}}{x_1}\right)$$

is clearly continuous. It follows from the contraction principle that large deviation upper and lower estimates hold for the family $\{(\sigma_2, \dots, \sigma_{n+1}) : 0 < \alpha < 1\}$ with the bounds given by the function

$$\tilde{J}_n(u_1, \dots, u_n) = \inf\left\{\sum_{i=1}^{n+1} J(x_i) : x_{j+1} = u_j x_1, j = 1, \dots, n\right\}.$$

Since $J(x) = \infty$ for x in $(0, 1)$, it follows that

$$\begin{aligned} \tilde{J}_n(u_1, \dots, u_n) &= \inf\left\{\sum_{i=1}^{n+1} J(x_i) : x_1 \geq 1, x_{j+1} = u_j x_1 \geq 1, j = 1, \dots, n\right\} \\ &= J_n(u_1, \dots, u_n) \end{aligned}$$

and the theorem follows. \square

Remark. The contraction principle used in Theorem 7 does not lead to a large deviation principle in general due to the fact that the starting rate function is not good. But here and later on, direct calculations show that the upper and lower bounds are all given by rate functions.

4.2. Large Deviations for $\Xi_{\alpha,0,\nu}$

Let $M_1([0, 1])$ denote the space of probabilities on $[0, 1]$ equipped with the weak topology. For any μ in $M_1([0, 1])$ define

$$\mathcal{I}(\mu) = \begin{cases} 0, & \mu = \nu \\ n, & \mu = \sum_{i=1}^n p_i \delta_{x_i} + (1 - \sum_{i=1}^n p_i) \nu \\ \infty, & \text{otherwise.} \end{cases}$$

The main result of this subsection is

Theorem 8. *The family $\{\Xi_{\alpha,0,\nu} : 0 < \alpha < 1\}$ satisfies a large deviation principle on $M_1([0, 1])$ with speed $-\log(1 - \alpha)$ and good rate function $\mathcal{I}(\cdot)$ as α tends to one.*

We prove this theorem through a series of lemmas.

Lemma 5. *For any $n \geq 1$, let $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = 1$ and B_1, \dots, B_{n+1} be a measurable partition of $[0, 1]$ such that $\nu(B_i) = t_i - t_{i-1}$. Then*

$$\begin{aligned} & (\Xi_{\alpha,0,\nu}(B_1), \dots, \Xi_{\alpha,0,\nu}(B_{n+1})) \\ & \stackrel{\text{D}}{=} \rho_1^{-1}(\rho_{t_1}, \rho_{t_2} - \rho_{t_1}, \dots, \rho_{t_k} - \rho_{t_{k-1}}, \rho_1 - \rho_{t_k}) \\ & \stackrel{\text{D}}{=} (t_1^{1/\alpha} + \sum_{k=2}^{n+1} (t_k - t_{k-1})^{1/\alpha} \sigma_k)^{-1} (t_1^{1/\alpha}, (t_2 - t_1)^{1/\alpha} \sigma_2, \dots, (1 - t_n)^{1/\alpha} \sigma_{n+1}) \end{aligned}$$

where $\sigma_2, \dots, \sigma_{n+1}$ are defined in (28), and $\stackrel{\text{D}}{=}$ denotes equality in distribution.

Proof: The first equality is from [31] and the second equality follows from the independent increments of the stable subordinator and the equality $\rho_t \stackrel{\text{D}}{=} t^{1/\alpha} \rho_1$. □

Lemma 6. *Let*

$$\Delta_{n+1} := \{(y_1, \dots, y_{n+1}) : y_i \geq 0, \sum_{k=1}^{n+1} y_k = 1\}.$$

Then the family $\{(\Xi_{\alpha,0,\nu}(B_1), \dots, \Xi_{\alpha,0,\nu}(B_{n+1})) : 0 < \alpha < 1\}$ satisfies a large deviation principle on Δ_{n+1} with speed $-\log(1 - \alpha)$ and good rate function $\mathcal{I}_n(\cdot)$ as α tends to one, where

$$\mathcal{I}_n(y_1, \dots, y_{n+1}) = (n + 1) - \gamma(y_1, \dots, y_{n+1})$$

with

$$\gamma(y_1, \dots, y_{n+1}) = \#\{1 \leq i \leq n + 1 : \frac{y_i}{t_i - t_{i-1}} = \min\{\frac{y_k}{t_k - t_{k-1}} : 1 \leq k \leq n + 1\}\}.$$

Proof: First note that the map

$$\begin{aligned} H : [0, 1]^n \times (0, \infty)^n & \rightarrow [0, 1], \\ (v_1, \dots, v_{n+1}; u_1, \dots, u_n) & \rightarrow (v_1 + \sum_{k=2}^{n+1} v_k u_{k-1})^{-1} (v_1, v_2 u_1, \dots, v_{n+1} u_n) \end{aligned}$$

is continuous and $(\Xi_{\alpha,0,\nu}(B_1), \dots, \Xi_{\alpha,0,\nu}(B_{n+1}))$ has the same distribution as

$$H(t_1^{1/\alpha}, \dots, (1 - t_n)^{1/\alpha}; \sigma_2, \dots, \sigma_{n+1}).$$

Noting that $(t_1^{1/\alpha}, \dots, (1 - t_n)^{1/\alpha})$ satisfies a full large deviation principle with effective domain $(t_1, \dots, (1 - t_n))$. It follows from Theorem 7, the independence

between $(t_1^{1/\alpha}, \dots, (1-t_n)^{1/\alpha})$ and $(\sigma_2, \dots, \sigma_{n+1})$ and the contraction principle that large deviation estimates hold for $(\Xi_{\alpha,0,\nu}(B_1), \dots, \Xi_{\alpha,0,\nu}(B_{n+1}))$ with upper and lower bounds given by the function

$$\begin{aligned} \tilde{\mathcal{I}}_n(y_1, \dots, y_{n+1}) &= \inf\{J_n(u_1, \dots, u_n) : u_i \in (0, \infty), u_i = \frac{t_1}{y_1} \frac{y_{i+1}}{t_{i+1} - t_i}, i = 1, \dots, n\} \\ &= \begin{cases} n+1 - \tilde{r}_n, & \min_{2 \leq i \leq n+1} \left\{ \frac{y_i}{t_i - t_{i-1}} \right\} < \frac{y_1}{t_1}, \\ n - \tilde{r}_n, & \min_{2 \leq i \leq n+1} \left\{ \frac{y_i}{t_i - t_{i-1}} \right\} = \frac{y_1}{t_1}, \\ n, & \min_{2 \leq i \leq n+1} \left\{ \frac{y_i}{t_i - t_{i-1}} \right\} > \frac{y_1}{t_1}. \end{cases} \end{aligned}$$

where \tilde{r}_n is the frequency of $\min_{2 \leq i \leq n+1} \left\{ \frac{y_i}{t_i - t_{i-1}} \right\}$ among $\frac{y_2}{t_2 - t_1}, \dots, \frac{y_{n+1}}{1 - t_n}$. On the other hand,

$$\gamma(y_1, \dots, y_{n+1}) = \begin{cases} \tilde{r}_n, & \min_{2 \leq i \leq n+1} \left\{ \frac{y_i}{t_i - t_{i-1}} \right\} < \frac{y_1}{t_1}, \\ \tilde{r}_n + 1, & \min_{2 \leq i \leq n+1} \left\{ \frac{y_i}{t_i - t_{i-1}} \right\} = \frac{y_1}{t_1}, \\ 1, & \min_{2 \leq i \leq n+1} \left\{ \frac{y_i}{t_i - t_{i-1}} \right\} > \frac{y_1}{t_1}. \end{cases}$$

Hence we obtain that $\tilde{\mathcal{I}}_n(\cdot) = \mathcal{I}_n(\cdot)$. It remains to show that $\mathcal{I}_n(\cdot)$ is a good rate function. Since Δ_{n+1} is compact, it suffices to verify the lower semicontinuity of the $\mathcal{I}_n(\cdot)$. For any point (y_1, \dots, y_{n+1}) in Δ_{n+1} , let $\gamma(y_1, \dots, y_{n+1}) = m$. If the neighbourhood of (y_1, \dots, y_{n+1}) is small enough, then the frequency of the minimum in each point inside the neighbourhood is at least m . Hence $\mathcal{I}(\cdot)$ is lower semicontinuous. \square

Lemma 7.

$$\begin{aligned} \mathcal{I}(\mu) &= \sup\{\mathcal{I}_n(\mu([0, t_1]), \mu((t_1, t_2]), \dots, \mu((t_n, 1])) : \\ &0 = t_0 < t_1 < \dots < t_n < t_{n+1} = 1, n = 1, 2, \dots\}. \end{aligned} \quad (30)$$

The supremum can be taken over all continuity points t_1, \dots, t_n of μ .

Proof: We divide the proof into several cases. Let μ be any probability in $M_1([0, 1])$. By Lebesgue's Decomposition Theorem, one can write

$$\mu = \lambda_1 \mu_a + \lambda_2 \mu_s + \lambda_3 \mu_{ac}$$

where μ_a is atomic, μ_s is diffuse and singular with respect to ν , μ_{ac} is absolutely continuous with respect to ν , and

$$\lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_i \geq 0, i = 1, 2, 3.$$

Set

$$F_s(x) = \mu_s([0, x]), \quad f(x) = \frac{d\mu_{ac}}{d\nu}(x).$$

Case 1: The probability μ has countable number of atoms.

Since the total mass of μ_a is equal to one, there exists a countable infinite number of atoms with all different value of masses. Let the masses of these atoms be ranked in descending order and the corresponding atoms are x_1, x_2, \dots . Clearly $\mu_s(\{x_i\}) = \mu_{ac}(\{x_i\}) = 0$ for all $i \geq 1$. For any $m \geq 2$, by the continuity of probabilities, one can choose small positive numbers $\epsilon_1, \epsilon_2, \dots, \epsilon_m$ such that $x_i \pm \epsilon_i, 1 \leq i \leq m$ are the continuity points of μ , $(x_i - \epsilon_i, x_i + \epsilon_i] \subset [0, 1], 1 \leq i \leq m$ are disjoint, and

$$\mu((x_1 - \epsilon_1, x_1 + \epsilon_1]) > \mu((x_2 - \epsilon_2, x_2 + \epsilon_2]) > \dots > \mu((x_m - \epsilon_m, x_m + \epsilon_m]).$$

The partition based on the points $\{x_i \pm \epsilon_i, i = 1, 2, \dots, m\}$ clearly gives a lower bound $m - 1$ for $\mathcal{I}(\cdot)$. Since m is arbitrary, the supremum taken over continuity points of μ gives the value of infinity which is the same as $\mathcal{I}(\cdot)$.

Case 2: The probability μ has at most finite number of atoms and $\nu(\{f(x) \neq 1\}) > 0$.

Let $A = \{x \in [0, 1] : f(x) < 1\}$, $B = \{x \in [0, 1] : f(x) > 1\}$, and $C = \{x \in [0, 1] : f(x) = 1\}$. Then we have

$$\mu_{ac}(A) < \nu(A), \mu_{ac}(B) > \nu(B), \mu_{ac}(C) = \nu(C)$$

and

$$\nu(A) - \mu_{ac}(A) = \mu_{ac}(B) - \nu(B)$$

The fact that $\nu\{C\} < 1$ thus implies that $\nu(A) > 0, \nu(B) > 0$. For any $m \geq 1$ we can find $0 < s_1 < \dots < s_m < 1, 0 < t_1 < \dots < t_m < 1$ such that

$$\{s_i\}_{1 \leq i \leq m} \subset A, \{t_i\}_{1 \leq i \leq m} \subset B$$

$$\{s_i, t_i\}_{i \geq 1} \text{ does not contain atoms of } \mu$$

$$\text{when } \lambda_2 > 0, F'_s(x) = 0 \text{ for } x = s_i \text{ or } t_i, i \geq 1.$$

For any $i, j \geq 1$, we then have

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{\mu((s_i - \epsilon, s_i + \epsilon])}{2\epsilon} &= \lambda_3 \lim_{\epsilon \rightarrow 0} \frac{\mu_{ac}((s_i - \epsilon, s_i + \epsilon])}{2\epsilon} = \lambda_3 f(s_i) \\ &< \lambda_3 f(t_j) = \lambda_3 \lim_{\epsilon \rightarrow 0} \frac{\mu_{ac}((t_j - \epsilon, t_j + \epsilon])}{2\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\mu((t_j - \epsilon, t_j + \epsilon])}{2\epsilon}. \end{aligned}$$

This makes it possible to choose $\epsilon_i > 0$ such that $s_i \pm \epsilon_i, t_j \pm \epsilon_j$ are all continuity points of μ and

$$\frac{\mu((s_i - \epsilon_i, s_i + \epsilon_i])}{\nu(s_i - \epsilon_i, s_i + \epsilon_i]} < \frac{\mu((t_j - \epsilon_j, t_j + \epsilon_j])}{\nu(t_j - \epsilon_j, t_j + \epsilon_j]}.$$

This provides a lower bound of m for $\mathcal{I}(\mu)$. Since m is arbitrary, we established (30) in this case.

Case 3: The probability μ has at most finite number of atoms, $\lambda_2 > 0$ and $\nu(\{f(x) \neq 1\}) = 0$.

It is clear that we have $\mu_{ac} = \nu$ in this case. For any $m \geq 1$, the singularity guarantees the existence of $0 < s_1 < \dots < s_m < 1, 0 < t_1 < \dots < t_m < 1$ such that the derivative of $F_s(x)$ is zero for $x = t_i$ while the derivative at s_i is either infinity or does not exist. Additionally we can choose s_i, t_i so that none of them are atoms of μ_a . Let ϵ be small enough so that all intervals $(s_i - \epsilon, s_i + \epsilon]$ and $(t_i - \epsilon, t_i + \epsilon]$ $i = 1, \dots, m$ are disjoint. Let \mathcal{J} denote the partition of $[0, 1]$ using $\{t_i \pm \epsilon, s_i \pm \epsilon : i = 1, \dots, m\}$. One can then find a refined partition, using subsequence if necessary, $\tilde{\mathcal{J}}$ of \mathcal{J} , and positive numbers ϵ_0, δ_0 such that $s_i \pm \epsilon_0, t_i \pm \epsilon_0$ are continuity points of μ and the value of $(2\epsilon_0)^{-1} \mu_s$ on each interval containing one of the t_i 's is less than δ_0 while its value on each interval containing one of the s_i 's is greater than δ_0 . In other words, we can have for any $1 \leq i, j \leq m$

$$\frac{\mu((s_i - \epsilon_0, s_i + \epsilon_0])}{\nu((s_i - \epsilon_0, s_i + \epsilon_0])} \neq \frac{\mu((t_j - \epsilon_0, t_j + \epsilon_0])}{\nu((t_j - \epsilon_0, t_j + \epsilon_0])}.$$

This implies that

$$\sup\{\mathcal{I}_n(\mu([0, t_1]), \mu((t_1, t_2]), \dots, \mu((t_n, 1])) : 0 = t_0 < \dots < t_{n+1} = 1, n \geq 1\} \geq m.$$

The arbitrary selection of m leads to (30) in this case.

Case 4: The probability μ has at most finite number of atoms, $\lambda_2 = 0$ and $\nu(\{f(x) \neq 1\}) = 0$.

In this case we have $\mu = \lambda_1\mu_a + \lambda_3\nu$. If $\lambda_1 = 0$, then $\mu = \nu$ and $\mathcal{I}(\mu)$ is clearly zero. Assume that $\lambda_1 > 0$ and the number of atoms is r . Let $F(x) = \mu([0, x])$. Since r is finite, any partition \mathcal{J} of $[0, 1]$ will have at most r disjoint intervals covering these atoms. The maximum

$$\sup\{\mathcal{I}_n(\mu([0, t_1]), \mu((t_1, t_2]), \dots, \mu((t_n, 1])) : 0 = t_0 < t_1 < \dots < t_n < t_{n+1} = 1, n \geq 1\}$$

is achieved at any partition with exactly r disjoint intervals covering the r atoms

□

Proof of Theorem 8: Let $C([0, 1])$ be the space of all continuous function on $[0, 1]$ equipped with the supremum norm, and $\{g_j(x) : j = 1, 2, \dots\}$ be a countable dense subset of $C([0, 1])$. The set $\{g_j(x) : j = 1, 2, \dots\}$ is clearly convergence determining on $M_1([0, 1])$. Let $|g_j| = \sup_{x \in [0, 1]} |g_j(x)|$ and $\{h_j(x) = \frac{g_j(x)}{|g_j| \sqrt{1}} : j = 1, \dots\}$ is also convergence determining.

For any μ, ν in $M_1([0, 1])$, define

$$d(\mu, \nu) = \sum_{j=1}^{\infty} \frac{1}{2^j} |\langle \mu, h_j \rangle - \langle \nu, h_j \rangle|. \quad (31)$$

Then d is a metric generating the weak topology on $M_1([0, 1])$.

For any $\delta > 0, \mu \in M_1([0, 1])$, let

$$B(\mu, \delta) = \{\nu \in M_1([0, 1]) : d(\nu, \mu) < \delta\}, \quad \bar{B}(\mu, \delta) = \{\nu \in M_1([0, 1]) : d(\nu, \mu) \leq \delta\}.$$

Since $M_1([0, 1])$ is compact, the family of the laws of $\Xi_{\alpha, 0, \nu}$ is exponentially tight. By theorem (P) in [36], to prove the theorem it suffices to verify that

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \liminf_{\alpha \rightarrow 1} \frac{1}{-\log(1 - \alpha)} \log \mathbb{P}\{B(\mu, \delta)\} \\ &= \lim_{\delta \rightarrow 0} \limsup_{\alpha \rightarrow 1} \frac{1}{-\log(1 - \alpha)} \log \mathbb{P}\{\bar{B}(\mu, \delta)\} = -\mathcal{I}(\mu). \end{aligned} \quad (32)$$

Let m be large enough so that

$$\{\nu \in M_1([0, 1]) : |\langle \mu, h_j \rangle - \langle \nu, h_j \rangle| < \delta/2 : j = 1, \dots, m\} \subset B(\nu, \delta). \quad (33)$$

Consider $0 = t_0 < t_1 < \dots < t_{n+1} = 1$ with $A_i = (t_{i-1}, t_i], i = 1, \dots, n+1$ such that

$$\sup\{|h_j(x) - h_j(y)| : x, y \in A_i, i = 1, \dots, n; j = 1, \dots, m\} < \delta/8.$$

Choosing $0 < \delta_1 < \frac{\delta}{4n}$, and define

$$V_{t_1, \dots, t_n}(\mu, \delta_1) = \{(y_1, \dots, y_n) \in \Delta_n : |y_i - \mu(A_i)| < \delta_1, i = 1, \dots, n\}.$$

For any v in $M_1([0, 1])$, let

$$\Psi(v) = (v(A_1), \dots, v(A_{n+1})).$$

If $\Psi(v)$ belongs to $V_{t_1, \dots, t_n}(\mu, \delta_1)$, then for $j = 1, \dots, m$

$$\begin{aligned} |\langle v, h_j \rangle - \langle \mu, h_j \rangle| &= \left| \sum_{i=1}^{n+1} \int_{A_i} h_j(x)(v(dx) - \mu(dx)) \right| \\ &< \frac{\delta}{4} + n\delta_1 < \delta/2, \end{aligned}$$

which implies that

$$\Psi^{-1}(V_{t_1, \dots, t_n}(\mu, \delta_1)) \subset \{v \in M_1([0, 1]) : |\langle v, h_j \rangle - \langle \mu, h_j \rangle| < \delta/2 : j = 1, \dots, m\}.$$

This combined with (33) implies that

$$\Psi^{-1}(V_{t_1, \dots, t_n}(\mu, \delta_1)) \subset B(\mu, \delta).$$

Since $V_{t_1, \dots, t_n}(\mu, \delta_1)$ is open in Δ_n , it follows from Lemma 6 that

$$\begin{aligned} &\lim_{\delta \rightarrow 0} \liminf_{\alpha \rightarrow 1} \frac{1}{-\log(1 - \alpha)} \log \mathbb{P}\{B(\mu, \delta)\} \\ &\geq \lim_{\delta \rightarrow 0} \liminf_{\alpha \rightarrow 1} \frac{1}{-\log(1 - \alpha)} \log \mathbb{P}\{\Psi^{-1}(V_{t_1, \dots, t_n}(\mu, \delta_1))\} \\ &= \lim_{\delta \rightarrow 0} \liminf_{\alpha \rightarrow 1} \frac{1}{-\log(1 - \alpha)} \log \mathbb{P}\{(\Xi_{\alpha, 0, \nu}(A_1), \dots, \Xi_{\alpha, 0, \nu}(A_{n+1})) \in V_{t_1, \dots, t_n}(\mu, \delta_1)\} \\ &\geq -\mathcal{I}_{n+1}(\mu(A_1), \dots, \mu(A_{n+1})) \geq -\mathcal{I}(\mu). \end{aligned} \tag{34}$$

Next we assume that t_1, \dots, t_n are continuity points of μ . We denote the collection of all partitions from these points by \mathcal{J}_μ . This implies that $\Psi(v)$ is continuous at μ . Hence for any $\delta_2 > 0$, one can choose $\delta > 0$ small enough such that

$$\overline{B}(\mu, \delta) \subset \Psi^{-1}(V_{t_1, \dots, t_k}(\mu, \delta_2)).$$

Let

$$\overline{V}_{t_1, \dots, t_k}(\mu, \delta_2) = \{(y_1, \dots, y_n) \in \Delta_n : |y_i - \mu(A_i)| \leq \delta_2, i = 1, \dots, n\}.$$

Then we have

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \limsup_{\alpha \rightarrow 1} \frac{1}{-\log(1-\alpha)} \log \mathbb{P}\{\bar{B}(\mu, \delta)\} \\ & \leq \limsup_{\alpha \rightarrow 1} \frac{1}{-\log(1-\alpha)} \log \mathbb{P}\{(\Xi_{\alpha,0,\nu}(A_1), \dots, \Xi_{\alpha,0,\nu}(A_{n+1})) \in \bar{V}_{t_1, \dots, t_n}(\mu, \delta_2)\}. \end{aligned} \quad (35)$$

Letting δ_2 go to zero and applying Lemma 6 again, one gets

$$\lim_{\delta \rightarrow 0} \limsup_{\theta \rightarrow \infty} \frac{1}{\theta} \log P\{\bar{B}(\mu, \delta)\} \leq -\mathcal{I}_{n+1}(\mu(A_1), \dots, \mu(A_{n+1})).$$

Finally, taking supremum over \mathcal{J}_μ and applying Lemma 7, one gets

$$\lim_{\delta \rightarrow 0} \limsup_{\alpha \rightarrow 1} \frac{1}{-\log(1-\alpha)} \log \mathbb{P}\{\bar{B}(\mu, \delta)\} \leq -\mathcal{I}(\mu),$$

which combined with (34) leads to the theorem. □

5. Concluding Remarks

A comparison between α converging to 1 and θ converging to infinity reveals fundamental differences. Under these limiting procedures, we have both $\mathbf{P}(\alpha, 0)$ and $\mathbf{P}(0, \theta)$ converge to $(0, 0, \dots)$. This can be seen from the distributions of $\varphi_2(\mathbf{P}(\alpha, 0))$ and $\varphi_2(\mathbf{P}(0, \theta))$.

It is shown in [22] and [25] that

$$\sqrt{\theta/2}[\theta\varphi_2(\mathbf{P}(0, \theta)) - 1] \implies Z, \quad \theta \rightarrow \infty,$$

where Z is the standard normal random variable. By Ewens sampling formula, we have

$$\begin{aligned} \mathbb{E}[\varphi_2(\mathbf{P}(0, \theta))] &= \frac{1}{\theta + 1} \\ \mathbb{E}[\varphi_2^2(\mathbf{P}(0, \theta))] &= \frac{3! + \theta}{(\theta + 1)(\theta + 2)(\theta + 3)} \end{aligned}$$

and

$$\mathbb{E}[\varphi_2^3(\mathbf{P}(0, \theta))] = \frac{1}{(\theta + 1)_{(5)}} (5! + 3 \cdot 3! \theta + \theta^2).$$

The skewness of $\varphi_2(\mathbf{P}(0, \theta))$ is given by

$$\begin{aligned} & \frac{\mathbb{E}[\varphi_2^3(\mathbf{P}(0, \theta))] - 3\mathbb{E}[\varphi_2(\mathbf{P}(0, \theta))]\mathbb{E}[\varphi_2^2(\mathbf{P}(0, \theta))] + 2(\mathbb{E}[\varphi_2(\mathbf{P}(0, \theta))])^3}{(\mathbb{E}[\varphi_2^2(\mathbf{P}(0, \theta))] - (\mathbb{E}[\varphi_2(\mathbf{P}(0, \theta))])^2)^{3/2}} \\ &= \frac{O(\theta^{-5})}{O(\theta^{-4.5})} \rightarrow 0, \quad \theta \rightarrow \infty \end{aligned}$$

which is consistent with the Gaussian limit.

On the other hand, for $\varphi_2(\mathbf{P}(\alpha, 0))$ one has

$$\begin{aligned} \mathbb{E}[\varphi_2(\mathbf{P}(\alpha, 0))] &= 1 - \alpha \\ \mathbb{E}[\varphi_2^2(\mathbf{P}(\alpha, 0))] &= \frac{(1 - \alpha)(2 - \alpha)(3 - \alpha) + \alpha(1 - \alpha)^2}{6}, \end{aligned}$$

and

$$\begin{aligned} \text{Var}(\varphi_2(\mathbf{P}(\alpha, 0))) &= \frac{\alpha(1 - \alpha)}{3} \\ \mathbb{E}[\varphi_2^3(\mathbf{P}(\alpha, 0))] &= \frac{1}{5!} [(1 - \alpha)_{(5)} + 3\alpha(1 - \alpha)^2(2 - \alpha)(3 - \alpha) + \alpha^2(1 - \alpha)^3]. \end{aligned}$$

This means that the skewness of $\varphi_2(\mathbf{P}(\alpha, 0))$ is of order $O((1 - \alpha)/O((1 - \alpha)^{3/2}))$ which goes to infinity as α converges to 1. Thus the distribution of $\varphi_2(\mathbf{P}(\alpha, 0))$ is skewed strongly to the right and a Gaussian limit is unlikely. The cause for this is the fact that as α increases the frequencies become more even, and tail becomes more heavier.

Another difference is reflected from the large deviation behaviour of the Pitman sampling formula. For any $n \geq 1$, a partition $\boldsymbol{\eta}$ of n with length l , the conditional Pitman sampling formula given $\mathbf{P}(\alpha, \theta) = \mathbf{p}$ is

$$F_{\boldsymbol{\eta}}(\mathbf{p}) = C(n, \boldsymbol{\eta}) \sum_{\text{distinct } i_1, \dots, i_l} p_{i_1}^{\eta_1} \cdots p_{i_l}^{\eta_l}$$

where

$$C(n, \boldsymbol{\eta}) = \frac{n!}{\prod_{k=1}^l \eta_k! \prod_{j=1}^n a_j(\boldsymbol{\eta})}.$$

Assuming $\eta_i \geq 2$ for all i . Then $F_{\boldsymbol{\eta}}(\mathbf{p})$ is continuous on ∇_{∞} . By contraction principle, large deviation principles hold for the image laws of $PD(0, \theta)$ and $PD(\alpha, 0)$ under $F_{\boldsymbol{\eta}}(\mathbf{p})$ with respective speed θ and $-\log(1 - \alpha)$.

Integrating $F_{\boldsymbol{\eta}}(\mathbf{p})$ with respect to $PD(\alpha, \theta)$ leads to the unconditional Pitman sampling formula. The large deviation speed is shown in [14] to be $\log \theta$ under $PD(0, \theta)$.

In [19], the large deviation speed under $PD(\alpha, 0)$ is shown to be $-\log(1 - \alpha)$. In other words, under $PD(0, \theta)$ the conditional and unconditional Pitman sampling formulae have different large deviation speeds due to averaging and finite sample size, while under $PD(\alpha, 0)$ the corresponding speeds are the same.

In Bayesian context, the limiting procedure $\theta \rightarrow \infty$ corresponds to the sample size tending to infinity and the large deviations for $\Xi_{0, \theta, \nu}$ becomes the large deviations for the posteriors of Dirichlet prior when the sample size converges to infinity ([21]). It is not clear what the natural Bayesian counterpart is for the limiting procedure $\alpha \rightarrow 1$.

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