Improper Integrals (I)

Definition (Improper Integral - Infinite Limits). Assume that the definite integral $\int_a^T f(x)dx$ exists for every $T \geq a$. Then we define the **improper integral of** $f(x)$ **on** (a, ∞) by

$$\int_a^\infty f(x)dx = \lim_{T \to \infty} \int_a^T f(x)dx \tag{1}$$

provided that the limit exists. If the definite integral $\int_a^T f(x)dx$ exists for every $T \leq a$, we define the **improper integral of** $f(x)$ **on** $(-\infty, a)$ by

$$\int_{-\infty}^a f(x)dx = \lim_{T \to -\infty} \int_T^a f(x)dx \tag{2}$$

assuming that the limit exists.

Comment: If the limit on the right hand side exists (i.e., is equal to a real number) then the integral is called **convergent**. Otherwise, the integral is called **divergent**.

Improper Integrals (II)

Definition (Improper Integral - Infinite Integrands).

- Assume that $f(x)$ is continuous on $[a, b)$ but not continuous at $x = b$. Then, we define
$$\int_a^b f(x)dx = \lim_{T \to b^-} \int_a^T f(x)dx \tag{3}$$
if the limit exists (i.e., integral convergent).
- If $f(x)$ is continuous on $(a, b]$, but not continuous at $x = a$, then
$$\int_a^b f(x)dx = \lim_{T \to a^+} \int_T^b f(x)dx \tag{4}$$
if the limit exists (i.e., the integral is convergent).
- If $f(x)$ is continuous on $[a, b]$ except at $x = c$, $a < c < b$, and if the improper integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge, then the improper integral $\int_a^b f(x)dx$ converges and
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\,. \tag{5}$$