# Orthogonality (I)

Last week we presented the following expression for the angles between two vectors $\mathbf{u}$ and $\mathbf{v}$ in $R^n$

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\,||\mathbf{v}||}\right)$$

which brings us to the fact that $\theta = \pi/2 \iff \mathbf{u}\cdot\mathbf{v} = 0$.

**Definition (Orthogonality).** Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $R^n$ are said to be **orthogonal** (or **perpendicular**) if $\mathbf{u}\cdot\mathbf{v} = 0$. We will also agree that the zero vector in $R^n$ is orthogonal to every vector in $R^n$.

**Example:** Each pair of the standard unit vectors in $R^3$ is orthogonal

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0$$

**Aplication of orthogonality.** One way of specifying the inclination of a plane is to use a nonzero vector $\mathbf{n}$ (called **normal**), which is orthogonal to the plane. Then, the plane is represented by the vector equation

$$\mathbf{n}\cdot\overrightarrow{P_0P} = 0$$

where $P_0 = (x_0, y_0, z_0)$, $P = (x, y, z)$ and $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$.

# Orthogonality (II)

**Theorem (Eq. for line and plane).** (a) If $a$ and $b$ are constants that are not both zero, then an equation of the form

$$ax + by + c = 0 \tag{1}$$

represents a line in $R^2$ with normal $\mathbf{n} = (a, b)$.
(b) If $a$, $b$ and $c$ are constant that are not all three zero, then an equation of the form

$$ax + by + cz + d = 0 \tag{2}$$

represents a plane in $R^3$ with normal $\mathbf{n} = (a, b, c)$.

**Comment:** if $c = 0$ (line through the origin), then equation (1) is homogenous and can be expressed in a vector form

$$\mathbf{n} \cdot \mathbf{x} = 0$$

If $d = 0$ we could make an equivalent statement for (2) (plane through the origin).

# Orthogonality (III)

**Theorem (Projection Theorem).** If $\mathbf{u}$ and $\mathbf{a}$ are vectors in $R^n$, and if $\mathbf{a} \neq 0$, then $\mathbf{u}$ can be expressed in exactly one way in the form $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is a scalar multiple of $\mathbf{a}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{a}$.

The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ in the Projection Theorem are called the *orthogonal projection of* $\mathbf{u}$ *on* $\mathbf{a}$ and the *component of* $\mathbf{u}$ *orthogonal to* $\mathbf{a}$, respectively, and have the form

$$\mathbf{w}_1 = \text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{||\mathbf{a}||^2}\mathbf{a}$$

$$\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{a}}{||\mathbf{a}||^2}\mathbf{a}$$

Sometimes we will be more interested in the norm of the *orthogonal projection of* $\mathbf{u}$ *on* $\mathbf{a}$ than the vector itself. It takes the form,

$$||\text{proj}_{\mathbf{a}}\mathbf{u}|| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{||\mathbf{a}||}$$

If $\theta$ is the angle between $\mathbf{u}$ and $\mathbf{a}$ then $(\mathbf{u}\cdot\mathbf{a} = ||\mathbf{u}||\,||\mathbf{a}||\,|\cos\theta|)$

$$||\text{proj}_{\mathbf{a}}\mathbf{u}|| = ||\mathbf{u}||\,|\cos\theta|$$

**Theorem (of Pythagoras in $R^n$).** If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in $R^n$ with the Euclidian inner product, then

$$||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

# Orthogonality (IV)

Orthogonal projections can be used to solve the following distance problems:

- Find the distance between a point and a line in $R^2$.
- Find the distance between a point and a plane in $R^3$.
- Find the distance between two parallel planes in $R^3$.

**Theorem (Distances).** (a) In $R^2$ the distance $D$ between the point $P_0(x_0, y_0)$ and the line $ax + by + c = 0$ is

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

(b) In $R^3$ the distance $D$ between the point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

# The Geometry of Linear Systems (I)

**Theorem (equation of line).** Let $L$ be the line in $R^2$ or $R^3$ that contains the point $\mathbf{x}_0$ and is parallel to the nonzero vector $\mathbf{v}$. Then the equation of the line through $\mathbf{x}_0$ that is parallel to $\mathbf{v}$ is

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$$

If $\mathbf{x}_0 = \mathbf{0}$, then the line passes through the origin and the equation has the form

$$\mathbf{x} = t\mathbf{v}$$

**Theorem (equation of plane).** Let $W$ be the plane in $R^3$ that contains the point $\mathbf{x}_0$ and is parallel to the noncollinear vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then an equation of the plane through $\mathbf{x}_0$ that is parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$ is given by

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

If $\mathbf{x}_0 = \mathbf{0}$, then the plane passes through the origin and the equation has the form

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

**Definition (line through $\mathbf{x}_0$).** If $\mathbf{x}_0$ and $\mathbf{v}$ are vectors in $R^n$, and if $\mathbf{v}$ is nonzero, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$$

defines the **line through $\mathbf{x}_0$ that is parallel to $\mathbf{v}$**. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the line is said to **pass through the origin**.

# The Geometry of Linear Systems (II)

**Definition (plane through $\mathbf{x}_0$).** If $\mathbf{x}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors in $R^n$, and if $\mathbf{v}_1$ and $\mathbf{v}_2$ are not collinear, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

defines the **plane through $\mathbf{x}_0$ that is parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$**. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the plane is said to **pass through the origin**.

**Definition (segment).** If $\mathbf{x}_0$ and $\mathbf{x}_1$ are distinct points in $R^n$, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0) \qquad \text{where} \qquad 0 \leq t \leq 1$$

defines the line segment from $\mathbf{x}_0$ to $\mathbf{x}_1$.

**Theorem (homogeneous LS).** If $A$ is an $m \times n$ matrix, then the solution set of the homogenous linear system $A\mathbf{x} = \mathbf{0}$ consists of all vectors in $R^n$ that are orthogonal to every row vector of $A$.

**Theorem (LS).** The general solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$ can be obtained by adding any specific solution of $A\mathbf{x} = \mathbf{b}$ to the general solution of $A\mathbf{x} = \mathbf{0}$.

# Cross Product (I)

**Definition (cross product).** If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the cross product $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$$

or, in determinant notation

$$\mathbf{u} \times \mathbf{v} = \left( \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

**Theorem (Rel. between cross and dot product).** If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in 3-space, then

(a) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ [$\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$]
(b) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ [$\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{v}$]
(c) $||\mathbf{u} \times \mathbf{v}||^2 = ||\mathbf{u}||^2 ||\mathbf{v}||^2 - (\mathbf{u} \cdot \mathbf{v})^2$ [Lagrange's identity]
(d) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ [vector triple product]
(e) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$ [vector triple product]

**Theorem (Prop's of cross product).** If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in 3-space:

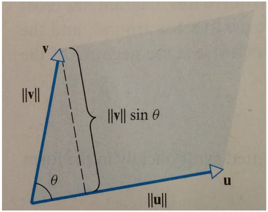
(a) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
(b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
(c) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$
(d) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$
(e) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = 0$
(f) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

**Determinant form of the cross product:**

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

**From the Lagrange's identity we can derive the following expression:**

$$||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| \, ||\mathbf{v}|| \sin\theta$$




**Theorem (Area of a Parallelogram).** If $\mathbf{u}$ and $\mathbf{v}$ are vectors in 3-space, then $||\mathbf{u} \times \mathbf{v}||$ is equal to the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

**Definition (scalar triple product).** If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in 3-space, then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

is called the **scalar triple product** of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

# Cross product (III)

**Theorem (Volume of Parallelogram/Parallelepiped).** (a) The absolute value of the determinant

$$\det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$

is equal to the area of the parallelogram in 2-space determined by the vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. (b) The absolute value of the determinant

$$\det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \qquad (= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|)$$

is equal to the volume of the parallelepiped in 3-space determined by the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$.

**Theorem (volume zero if vectors in same plane).** If the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ have the same initial point, then they lie in the same plane if and only if

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$$

# Real Vector Spaces (I)

**Definition (vector space).** Let $V$ be an arbitrary nonempty set of objects on which two opearations are defined: addition and multiplication by scalars. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and all scalars $k$ and $m$, then we call $V$ a vector space and we call the objects in $V$ **vectors**.

1. If $\mathbf{u}+\mathbf{v}$ are objects in $V$, then $\mathbf{u}+\mathbf{v}$ is in $V$.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There is an object $\mathbf{0}$ in $V$ called a **zero vector** for $V$, such that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in $V$.
5. For each $\mathbf{u}$ in $V$, there is an object $-\mathbf{u}$ in $V$, called a **negative** of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If $k$ is any scalar and $\mathbf{u}$ is any object in $V$, the $k\mathbf{u}$ is in $V$.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

# Real Vector Spaces (II)

**To show that a set with two operations is a vector space:**

Step 1. Identify the set $V$ of objects that will become vectors.

Step 2. Identify the addition and scalar multiplication operation on $V$.

Step 3. Verify Axioms 1 and 6; That is, adding two vectors in $V$ produces a vector in $V$, and multiplying a vector in $V$ by a scalar also produces a vector in $V$. Axiom 1 is called **closure under addition** and Axiom 6 is **called closure under scalar multiplication**.

Step 4. Confirm that Axioms 2, 3, 4, 5, 7, 8, 9 and 10 hold.