# Real Vector Spaces (I) (Reminder)

**Definition (vector space).** Let $V$ be an arbitrary nonempty set of objects on which two opearations are defined: addition and multiplication by scalars. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and all scalars $k$ and $m$, then we call $V$ a vector space and we call the objects in $V$ **vectors**.

1. If $\mathbf{u} + \mathbf{v}$ are objects in $V$, then $\mathbf{u} + \mathbf{v}$ is in $V$.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. There is an object $\mathbf{0}$ in $V$ called a **zero vector** for $V$, such that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in $V$.
5. For each $\mathbf{u}$ in $V$, there is an object $-\mathbf{u}$ in $V$, called a **negative** of $\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.
6. If $k$ is any scalar and $\mathbf{u}$ is any object in $V$, the $k\mathbf{u}$ is in $V$.
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)(\mathbf{u})$
10. $1\mathbf{u} = \mathbf{u}$

**To show that a set with two operations is a vector space:**

Step 1. Identify the set $V$ of objects that will become vectors.

Step 2. Identify the addition and scalar multiplication operation on $V$.

Step 3. Verify Axioms 1 and 6; That is, adding two vectors in $V$ produces a vector in $V$, and multiplying a vector in $V$ by a scalar also produces a vector in $V$. Axiom 1 is called **closure under addition** and Axiom 6 is **called closure under scalar multiplication**.

Step 4. Confirm that Axioms 2, 3, 4, 5, 7, 8, 9 and 10 hold.

**Theorem (Properties of vectors).** Let $V$ be a vector space, $\mathbf{u}$ a vector in $V$, and $k$ a scalar; then

(a) $0\mathbf{u} = \mathbf{0}$

(b) $k\mathbf{0} = \mathbf{0}$

(c) $(-1)\mathbf{u} = -\mathbf{u}$

(d) If $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$

# Subspaces (I)

**Definition (subspace).** A subset $W$ of a vector space $V$ is called a subspace of $V$ if $W$ is itself a vector space under the addition and scalar multiplication defined on $V$.

**Comment:** To check that $W$ is itself a vector space we only need to verify Axioms 1 and 6 (next theorem). Axioms 4 and 5 hold in $W$ as a consequence of 1 and 6 and the rest are inherited from $V$.

**Theorem (subspace).** If $W$ is a set of one or more vectors in a vector space $V$, then $W$ is a subspace of $V$ if and only if the following conditions are satisfied:
(a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $W$, then $\mathbf{u} + \mathbf{v}$ is in $W$.
(b) If $k$ is a scalar and $\mathbf{u}$ is a vector in $W$, then $k\mathbf{u}$ is in $W$.

**Theorem (subspace from subspaces).** if $W_1$, $W_2$, ..., $W_r$ are subspaces of a vector space $V$, then the intersection of these subspaces is also a subspace.

**Definition (linear combination).** If $\mathbf{w}$ is a vector in a vector sapce $V$, then $\mathbf{w}$ is said to be a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in $V$ if $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the **coefficients** of the linear combination.

# Subspaces (II)

**Theorem (subspace from linear combinations).** If $\mathbf{S} = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space $V$, then:
(a) The set $W$ of all possible linear combinations of the vectors in $S$ is a subspace of $V$.
(b) The set $W$ in part (a) is the *smallest* subspace of $V$ that contains all of the vectors in $S$ in the sense that any other subspace that contains those vectors contains $W$.

**Definition (span).** If $\mathbf{S} = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space $V$ then the subspace $W$ of $V$ that consists of all possible linear combination of the vectors in $S$ is called the subspace of $V$ **generated** by $S$, and we say that the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ **span** $W$. We denote this supspace as

$$W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\} \qquad \text{or} \qquad W = \text{spans}(S)$$

**Theorem (solution spaces of HLS).** The solution set of a homogenous linear system $A\mathbf{x} = 0$ of $m$ equations in $n$ unknowns is a subspace of $R^n$.

This theorem can be viewed as a statement about matrix transformations by letting $T_A : R^n \to R^m$ be multiplication by the coefficient matrix $A$. The solution space of $A\mathbf{x} = \mathbf{0}$ is the set of vectors in $R^n$ that $T_A$ maps into the zero vector in $R^m$. This set is sometimes called the **kernel** of the transformation.

# Subspaces (III)

**Theorem (kernel).** If $A$ is an $m \times n$ matrix, then the kernel of the matrix transformation $T_A : R^n \to R^m$ is a subspace of $R^n$.

**Theorem (two sets of vectors spanning same space).** If $\mathbf{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $\mathbf{S}' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ are nonempty sets of vectors in a vector space $V$, then

$$\text{span}\,\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \text{span}\,\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$$

if and only if each vector in $S$ is a linear combination of those in $S'$, and each vector in $S'$ is a linear combination of those in $S$.

# Linear Independence (I)

**Definition (LI set).** If $\mathbf{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of two or more vectors in a vector space $V$, then $S$ is said to be a **linearly independent** set if no vector in $S$ can be expressed as a linear combination of the others. A set that is not linearly independent is said to be **linearly dependent**.

**Theorem (LI set)**. A nonempty set $\mathbf{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ in a vectors space $V$ is linearly independent if and only if the only coefficients satisfying the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = 0$$

are $k_1 = 0$, $k_2 = 0$, ..., $k_r = 0$.

**Theorem (LI sets with 1 or 2 vectors)**. The theorem has 3 parts:
(a) A finite set that contains $\mathbf{0}$ is linearly dependent.
(b) A set with exactly one vectors is linearly independent if and only if that vector is not $\mathbf{0}$.
(c) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

**Theorem (LI in $R^n$).** Let $\mathbf{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in $R^n$. If $r > n$, then $S$ is linearly dependent.

## Linear Independence (II)

**Definition (Wronskian).** If $\mathbf{f}_1 = f_1(x)$, $\mathbf{f}_2 = f_2(x)$, ..., $\mathbf{f}_n = f_n(x)$ are functions that are $n-1$ times differentiable on the interval $(-\infty, +\infty)$, then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the **Wronskian** of $f_1$, $f_2$, ..., $f_n$.

**Theorem (Wronskian and LI set of functions).** If the functions $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n$ have $n-1$ continuous derivatives on the interval $(-\infty, +\infty)$, and if the Wronskian of these functions is not identically zero on $(-\infty, +\infty)$, then these functions form a linearly independent set of vectors in $C^{(n-1)}(-\infty, +\infty)$.