Properties of Matrices (I)

Theorem (Properties of Matrix Arithmetic). Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid:

(a) $A + B = B + A$
(b) $A + (B + C) = (A + B) + C$
(c) $A(BC) = (AB)C$
(d) $A(B \pm C) = AB \pm AC$
(e) $(B \pm C)A = BA \pm CA$
(f) $a(B \pm C) = aB \pm aC)$
(g) $(a \pm b)C = aC \pm bC$
(h) $a(bC) = (ab)C$
(i) $a(BC) = (aB)C = B(aC)$

Comment (I): In general, **multiplication of matrices is not commutative**, i. e. $AB \neq BA$. But, for certain cases, it can happen that $AB = BA$. Then, we say that AB and BA **commute**.
Comment (II): In general, $AB = AC$ (with $A \neq 0$) $\;\not\Rightarrow\;$ $B = C$.
Comment (III): In general, $AB = 0 \;\not\Rightarrow\; A = 0$ and/or $B = 0$.

Theorem (Properties of Zero Matrices). If c is a scalar, and if the sizes of the matrices are such that the operations can be performed, then:

(a) $A \pm 0 = 0 \pm A = A$
(b) $A - A = A + (-A) = 0$
(c) $0A = 0$
(d) If $cA = 0$, then $c = 0$ or $A = 0$

Identity Matrix and Inverse Matrix

The **Identity Matrix**, denoted by I or I_n (where n is the dimension), is a square matrix with 1's on the main diagonal and zeros elsewhere. Examples:

$$\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ldots$$

Comment: If A is a $m \times n$ matrix, then

$$AI_n = A \qquad \text{and} \qquad I_m A = A$$

Theorem. If R is the reduced row echelon form of a $n \times n$ (so, square) matrix A, then either R has a row of zeros or R is the identity matrix I_n.

Definition (Inverse Matrix). If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be **invertible** (or **nonsingular**) and B is called an **inverse** of A. If no such matrix B can be found, then A is said to be **singular**.

Comment: If A is invertible and B is the inverse of A, then it is also true that B is invertible, and A is an inverse of B. Thus, when $AB = BA = I$ we say that A and B are inverses of one another.

Properties of Inverses

Theorem (An invertible matrix has exactly one inverse). If B and C are both inverses of the matrix A, then $B = C$.

Notation: If A is invertible, then its inverse will be denoted by the symbol A^{-1}. Thus,

$$AA^{-1} = I \qquad \text{and} \qquad A^{-1}A = I$$

Theorem (Inverse of 2×2 matrix). The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - cd \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem. If A and B are invertible matrices with the same size, then AB is invertible and (note the reverse order)

$$(AB)^{-1} = B^{-1}A^{-1}$$

Comment: A product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverses in the reverse order.

Powers of a Matrix and Matrix Polynomials

Powers of a Matrix: If A is a square matrix, we define the **nonnegative integer powers** of A to be

$$A^0 = I \qquad \text{and} \qquad A^n = AA\cdots A \quad (n \text{ times})$$

and if A is invertible, then we define the **negative integer powers** of A to be

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1}\cdots A^{-1} \quad (n \text{ times})$$

In addition, the usual laws of nonnegative exponents hold:

$$A^rA^s = A^{r+s} \qquad \text{and} \qquad (A^r)^s = A^{rs}$$

Theorem. If A is invertible and n is a nonnegative integer, then:
(i) A^{-1} is invertible and $(A^{-1})^{-1} = A$.
(ii) A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.
(iii) kA is invertible for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$.

Matrix Polynomials: If A is a $n \times n$ square matrix and if

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$$

is any polynomial, then we define the $n \times n$ **matrix polynomial** in A, $p(A)$, to be

$$p(A) = a_0I + a_1A + a_2A^2 + \cdots + a_mA^m$$

Properties of the Transpose

Theorem (Properties of the Transpose). If the sizes of the matrices are such that the stated operations can be performed, then:

$$\begin{array}{ll} \text{(a) } (A^T)^T = A & \text{(c) } (kA)^T = kA^T \\ \text{(b) } (A \pm B)^T = A^T \pm B^T & \text{(d) } (AB)^T = B^T A^T \end{array}$$

Comment: The transpose of a product of any number of matrices is the product of the transposes in the reverse order.

Theorem. If A is an invertible matrix, then A^T is also invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Elementary Matrices and Finding the Inverse

Definition (Row equivalent matrices). Matrcies A and B are said to be **row equivalent** if either can be obtained from the other by sequence of elementary row operations.

Definition (Elemetary matrix). A matrix E is called an elementary matrix if it can be obtained from an identity matrix by performing a single elementary row operation.

Theorem (Row operation by matrix multiplication). If the elementary matrix E results from performing a certain row operation on I_m and if A is a $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

Theorem (Elementary matrix invertible). Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Theorem (Equivalence theorem). If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false:
(a) A is invertible.
(b) $A\mathbf{x} = 0$ has only the trivial solution.
(c) The reduced row echelon form of A is I_n.
(d) A is expressable as a product of elementary matrices.

Method for inverting matrices: Inversion Algorithm

Comment: The same sequence of row operations that reduces A to I_n will transform I_n to A^{-1}:

$$A^{-1} = E_k \dots E_2 E_1 I_n$$

Inversion Algorithm. To find the inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to the identity and then perform that same sequence of operations on I_n to obtain A^{-1}.

Example:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \to \text{Inversion Algorithm} \to \quad A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to [I|A^{-1}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$