More on Linear Systems and Invertible Matrices (I)

Theorem (0, 1 or ∞ solutions). A system of linear equations has zero, one, or infinitely many solutions. There are no other options. (Reminder)

Theorem (Solution of LS by matrix inversion). If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely,

$$\boxed{\mathbf{x} = A^{-1}\mathbf{b}}$$

Linear systems with a common coefficient matrix

$$A\,\mathbf{x_1} = \mathbf{b}_1, \quad A\,\mathbf{x_2} = \mathbf{b}_2, \quad \ldots \quad A\,\mathbf{x_k} = \mathbf{b_k}$$

have solutions

$$\mathbf{x_1} = A^{-1}\mathbf{b}_1, \quad \mathbf{x_2} = A^{-1}\mathbf{b}_2, \quad \ldots \quad \mathbf{x_k} = A^{-1}\mathbf{b_k}$$

In this situation the k systems can be solved at once by applying the Gauss-Jordan algorithm to the following augmented matrix

$$[A|\mathbf{b_1}|\mathbf{b_2}|\ldots|\mathbf{b_k}]$$

Theorem. Let A be a square matrix. Then,
(i) If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$.
(ii) If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.

More on Linear Systems and Invertible Matrices (II)

Theorem (Extension of Equivalence Theorem). If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false:
(a) A is invertible.
(b) $A\mathbf{x} = 0$ has only the trivial solution.
(c) The reduced row echelon form of A is I_n.
(d) A is expressable as a product of elementary matrices.
(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

Theorem (AB invertible $\implies$ A, B invertible). Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

Fundamental Problem. Let A be a fixed $n \times m$ matrix. Find all $m \times 1$ matrices $\mathbf{b}$ such that the system of equations $A\mathbf{x} = \mathbf{b}$ is consistent. Solution:

- If A is invertible (so, square): Theorem *Solution of LS by matrix inversion* from previous slide ensures that the system has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.
- If A is square but not invertible or not square: Then $\mathbf{b}$ must usually satisfy certain conditions for $Ax = b$ to be consistent. This conditions can be found by using Gauss-Jordan algorithm, for instance. (see example)

Diagonal, Triangular and Symmetric Matrices (I)

Diagonal Matrix: Matrix in which all the entries off the main diagonal are zeros (Example: I_n). A general $n \times n$ diagonal matrix can be written as

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix}$$

Inverse and powers of a Diagonal Matrix:

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \dots & 0 \\ 0 & \frac{1}{d_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \frac{1}{d_n} \end{bmatrix}, \quad D^k = \begin{bmatrix} d_1^k & 0 & \dots & 0 \\ 0 & d_2^k & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_n^k \end{bmatrix}$$

Multiplying a matrix by a Diagonal Matrix from the left and from the right:

$$\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_1 a_{12} & d_1 a_{13} \\ d_2 a_{21} & d_2 a_{22} & d_2 a_{23} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_2 a_{12} \\ d_1 a_{21} & d_2 a_{22} \end{bmatrix}$$

Diagonal, Triangular and Symmetric Matrices (II)

Triangular Matrix: A square matrix in which all the entries above or below the main diagonal are zero (If zeros above: **lower triangular**, if zeros below: **upper diagonal**). Examples:

$$\text{Lower:} \quad L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, \quad \text{Upper:} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Theorem (Properties of Triangular Matrices):
(a) The transpose of a lower triangular matrix is upper triangular (and vice versa).
(b) The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.
(c) A triangular matrix is invertible if and only if its diagonal entries are all nonzero.
(d) The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

Diagonal, Triangular and Symmetric Matrices (III)

Symmetric Matrices (SM): A square matrix A is said to be symmetric if $A = A^T$. In compact form this can be expressed as

$$(A)_{ij} = (A)_{ji}$$

Theorem (Properties of SM (i)): If A and B are symmetric matrices with the same size, and if k is any scalar, then:
(a) A^T is symmetric.
(b) $A + B$ and $A - B$ are symmetric.
(c) kA is symmetric.

Theorem (Properties of SM (ii)): The product of two symmetric matrices is symmetric if and only if the matrices commute.

Theorem (Properties of SM (iii)): If A is an invertible symmetric matrix, then A^{-1} is symmetric.

Comment: Matrix products of the form AA^T and A^TA are always symmetric since (Using Theorem "Properties of the Transpose" part (d) from last week)

$$(AA^T)^T = (A^T)^T A^T = AA^T \qquad \text{and} \qquad (A^TA)^T = A^T(A^T)^T = A^TA$$

Theorem: If A is an invertible matrix, then AA^T and A^TA are also invertible.

Determinants by Cofactor Expansion (I)

Recall

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \rightarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Comment: the factor $ad - bc$ is the **determinant** of A, i. e.

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The following definitions and theorems will allow us to learn how to compute determinants of higher order matrices.

Definition (Minor & Cofactor): If A is a square matrix, then the **minor of entry** a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A. The number $(-1)^{i+j} M_{ij}$ is denoted by C_{ij} and is called the **cofactor of entry** a_{ij}.

Theorem: If A is a $n \times n$ matrix, then regardeless of which row or column of A is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

Determinants by Cofactor Expansion (II)

Definition (Determinant): If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the determinant of A, and the sums themselves are called cofactor expansion of A. That is,
Cofactor expansion along the jth column:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

Cofactor expansion along the ith row :

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$