Determinants by Cofactor Expansion (III)

Comment: (Reminder) If A is an $n \times n$ matrix, then the determinant of A can be computed as a cofactor expansion along the jth column

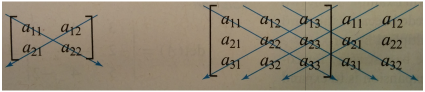
$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

or as a cofactor expansion along the ith row

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

Theorem (A triangular $\implies \det(A) = a_{11}a_{22}\ldots a_{nn}$). If A is an $n \times n$ triangular or diagonal matrix, then $\det(A)$ is the product of the entries on the main diagonal of the matrix, that is, $\det(A) = a_{11}a_{22}\ldots a_{nn}$.

Note: Determinants of 2×2 and 3×3 can be evaluated very efficiently (see pattern suggested below) by summing the products of the entries on the rightward arrows and substracting the products on the leftwards arrows.



Evaluating Determinants by Row Reduction (I)

Theorem (A row/column of zeros $\Longrightarrow$ $\det(A) = 0$). Let A be a square matrix. If A has a row of zeros or a column of zeros, then $\det(A) = 0$.

Theorem ($\det(A) = \det(A^T)$). Let A be a square matrix. Then $\det(A) = \det(A^T)$.

Theorem (Det. of row equiv. matrices). Let A be an $n \times n$ matrix:
(a) If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then $\det(B) = k \det(A)$.
(b) If B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = -\det(A)$.
(c) If B is the matrix that results when a multiple of one row of A is added to another or when a multiple of one column is added to another, then $\det(B) = \det(A)$.

Theorem (Det. of element. matrices). Let E be an $n \times n$ elementary matrix:
(a) If E results from multiplying a row of I_n by a nonzero number k, $\det(E) = k$.
(b) If E results from interchanging two rows of I_n, then $\det(E) = -1$.
(c) If E results from adding a multiple of one row of I_n to another, then $\det(E) = 1$.

Relationship	Operation
$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = k\det(A)$	In the matrix B the first row of A was multiplied by k.
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = -\det(A)$	In the matrix B the first and second rows of A were interchanged.
$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = \det(A)$	In the matrix B a multiple of the second row of A was added to the first row.

Evaluating Determinants by Row Reduction (II)

Theorem (A proportional row/columns $\implies \det(A) = 0$). If A is a square matrix with two proportional rows or two proportional columns, then $\det(A) = 0$.

Evaluating Determinants by Row Reduction: Reduce the given matrix to upper triangular form by elementary row operations, then compute the determinant of the upper triangular matrix, and then relate that determinant to that of the original matrix.

Properties of Determinants (I)

Determinant of kA: If A is a $n \times n$ matrix and k a scalar, then

$$\boxed{\det(kA) = k^n \det(A)}$$

Comment: In general $\det(A + B) \neq \det(A) + \det(B)$.

Theorem ($\det(C) = \det(A) + \det(B)$). Let A, B and C be $n \times n$ matrices that differ only in a single row, say the rth, and assume that the rth row of C can be obtianed by adding correspoinding entries in the rth rows of A and B. Then,

$$\det(C) = \det(A) + \det(B)$$

The same results holds for columns.

Theorem ($\det(EB) = \det(E)\det(B)$). If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det(EB) = \det(E)\det(B)$$

Comment: The previous result can be extended for multiple elementary matrices

$$\det(E_1 E_2 \ldots E_r B) = \det(E_1)\det(E_2)\ldots\det(E_r)\det(B)$$

Theorem (A invertible $\iff \det(A) \neq 0$). A square matrix A is invertible if and only if $\det(A) \neq 0$.

Properties of Determinants (II)

Theorem (A,B square $\implies$ $\det(AB) = \det(A)\det(B)$). If A and B are square matrices of the same size, then

$$\boxed{\det(AB) = \det(A)\det(B)}$$

Theorem ($\det(A^{-1}) = 1/\det(A)$). If A is invertible, then

$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

Definition (Adjoint of A). If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & & & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

is called the matrix of cofactors from A. The traspose of this matrix is called the adjoint of A and is denoted by $\mathrm{adj}(A)$.

Theorem ($A^{-1} = \mathrm{adj}(A)/\det(A)$). If A is an invertible matrix, then

$$\boxed{A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)}$$

Properties of Determinants (III)

Theorem (Extension Extension of Equivalence Theorem). If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false:

(a) A is invertible.
(b) $A\mathbf{x} = 0$ has only the trivial solution.
(c) The reduced row echelon form of A is I_n.
(d) A is expressable as a product of elementary matrices.
(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
(g) $\det(A) \neq 0$

Eigenvalues and Eigenvectors (I)

Definition (Eigenvalue and Eigenvector). If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in R^n is called an eigenvector of A if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is,

$$\boxed{A\mathbf{x} = \lambda\mathbf{x}}$$

for some scalar λ. The scalar λ is called an **eigenvalue** of A, and $\mathbf{x}$ is said to be an **eigenvector corresponding to** λ.

Comment: Our next objective will be to find a procedure for finding Eigenvalues and Eigenvectors (they are very important in many applications) for a given matrix A. Note:

$$A\mathbf{x} = \lambda\mathbf{x} \quad \rightarrow \quad A\mathbf{x} = \lambda I\mathbf{x} \quad \rightarrow \quad (\lambda I - A)\mathbf{x} = 0$$

For λ to be an eigenvalue of A this equation must have a nonzero solution for $\mathbf{x}$. Equivalence Theorem says that this is only possible if $\det(\lambda I - A) = 0$.

Theorem (Characteristic equation of A). If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation

$$\boxed{\det(\lambda I - A) = 0}$$

This is called the **characteristic equation of** A.