Eigenvalues and Eigenvectors (I) (Reminder)

Definition (Eigenvalue and Eigenvector). If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in R^n is called an eigenvector of A if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is,

$$\boxed{A\mathbf{x} = \lambda\mathbf{x}}$$

for some scalar λ. The scalar λ is called an **eigenvalue** of A, and $\mathbf{x}$ is said to be an **eigenvector corresponding to** λ.

Comment: Our next objective will be to find a procedure for finding Eigenvalues and Eigenvectors (they are very important in many applications) for a given matrix A. Note:

$$A\mathbf{x} = \lambda\mathbf{x} \quad \rightarrow \quad A\mathbf{x} = \lambda I\mathbf{x} \quad \rightarrow \quad (\lambda I - A)\mathbf{x} = 0$$

For λ to be an eigenvalue of A this equation must have a nonzero solution for $\mathbf{x}$. Equivalence Theorem says that this is only possible if $\det(\lambda I - A) = 0$.

Theorem (Characteristic equation of A). If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation

$$\boxed{\det(\lambda I - A) = 0}$$

This is called the **characteristic equation of** A.

Eigenvalues and Eigenvectors (II)

Comment: When the determinant $\det(\lambda I - A)$ is expanded, the characteristic equation of A takes the form

$$\lambda^n + c_1\lambda^{n-1} + \ldots + c_n = 0$$

which is a polynomial of degree n. The polynomial will have at most n roots, which represent the eigenvalues of A. The polynomial

$$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \ldots + c_n$$

is called the **characteristic polynomial** of A.

Theorem (A triangular $\implies$ entries eigenvalues). If A is an $n \times n$ triangular matrix, then the eigenvalues of A are the entries of the main diagonal of A.

Theorem (Equivalent statements for eigenval. and eigenvec.). If A is an $n \times n$ matrix, the following statements are equivalent:
(a) λ is an eigenvalue of A.
(b) λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$.
(c) The system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solution.
(d) There is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

Eigenvalues and Eigenvectors (III)

Eigenvectors, Eigenspaces and bases for Eigenspaces: All nonzero vectors solving

$$(\lambda I - A)\mathbf{x} = 0$$

for a given λ are called the **eigenspace** of A corresponding to λ. A **base** for a given eigenspace is a vector or vectors that (combined in a specific way called *linear combination*) allow to generate all the vectors in the eigenspace (we will formalize this definition in the future).

Theorem (A invertible $\iff \lambda \neq 0$). A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Theorem (Extension Extension Extension of Equivalence Theorem). If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false:

(a) A is invertible.
(b) $A\mathbf{x} = 0$ has only the trivial solution.
(c) The reduced row echelon form of A is I_n.
(d) A is expressable as a product of elementary matrices.
(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
(g) $\det(A) \neq 0$.
(h) $\lambda = 0$ is not an eigenvalue of A.

Diagonalization (I)

In this section we will study products of the form $P^{-1}AP$ in which A and P are $n \times n$ matrices and P is invertible. A transformation

$$A \to P^{-1}AP$$

is called a **similarity transformation**. These transformations preserve many properties of the matrix A:

Property	**Description**
Determinant	A and $P^{-1}AP$ have same determinant
Invertibility	A invertible $\iff$ $P^{-1}AP$ invertible
Trace	A and $P^{-1}AP$ have same trace
Characteristic polynomial	A and $P^{-1}AP$ have same characteristic polynomial
Eigenvalues	A and $P^{-1}AP$ have same eigenvalues
Eigenspace dimension	The dimension of the eigenspace of A corresponding to λ is the same for $P^{-1}AP$

Definition (B similar to A). If A and B are square matrices, then we say that B is similar to A if there is an invertible matrix P such that $B = P^{-1}AP$.

Definition (A diagonalizable). If a square matrix A is said to be diagonalizable if it is similar to some diagonal matrix; that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. In this case the matrix P is said to diagonalize A.

Diagonalization (II)

Theorem. If A is an $n \times n$ matrix, the following statements are equivalent.
(a) A is diagonalizable.
(b) A has n linearly independent eigenvectors.

Theorem.
(a) If $\lambda_1, \lambda_2, \dots \lambda_k$ are distinct eigenvalues of a matrix A, and if $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_k}$ are corresponding eigenvectors, then $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_k}\}$ is a linearly independent set.
(b) An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Procedure for Diagonalizing a Matrix A

1. Determine first whether A is actually diagonalizable by searching for n linearly independent eigenvectors. One way to do this is: find basis for each eigenspace and count total number of vectors. If there is n vectors then A is diagonalizable, otherwise it is not.
2. Form a matrix $P = [\mathbf{p_1} \ \mathbf{p_2} \ \dots \ \mathbf{p_n}]$ whose column vectors are the vectors obtained in Step 1.
3. $P^{-1}AP$ will be a diagonal matrix whose successive diagonal entries are the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ that correspond to the successive columns of P.

Diagonalization (III)

Theorem (λ eigenval. of $A \implies \lambda^k$ eigenval. of A^k). If k is a positive integer, λ is an eigenvalue of a matrix A, and $\mathbf{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.

Terminology: If λ_0 is an eigenvalue of a $n \times n$ matrix A, then the dimension of the eigenspace (number of vectors forming the base) corresponding to λ_0 is called the **geometric multiplicity**. The number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the **algebraic multiplicity**.

Theorem (Geometric and Algebraic Mutliplicity). If A is a square matrix, then:
(a) For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.
(b) A is diagonalizable if and only if the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity.