Diagonalization (III) (Reminder)

Theorem (λ eigenval. of $A \implies \lambda^k$ eigenval. of A^k). If k is a positive integer, λ is an eigenvalue of a matrix A, and $\mathbf{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.

Terminology: If λ_0 is an eigenvalue of a $n \times n$ matrix A, then the dimension of the eigenspace (number of vectors forming the base) corresponding to λ_0 is called the **geometric multiplicity**. The number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the **algebraic multiplicity**.

Theorem (Geometric and Algebraic Mutliplicity). If A is a square matrix, then:
(a) For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.
(b) A is diagonalizable if and only if the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity.

Dynamical Systems and Markov Chains (I)

- **Dynamical System (DS):** Finite set of variables whose values change with time.
- **State of a variable and state vector:** The value of a variable in a DS at a point in time. The state vector is a vector formed by the state of the variables.
- **Stochastic processes :** A DS where the state of the variables are not known with certainty but can be expressed as probabilities.
- **Probability vector:** Vector with non-negative entries that add up to 1.
- **Stochastic matrix:** Square matrix each of whose columns is a probability vector.

Definition (Markov Chain). A **Markov chain** is a dynamical system whose state vectors at a succession of equally spaced times are probability vectors and for which the state vectors at successive times are related by an equation of the form

$$\mathbf{x}(k+1) = P\,\mathbf{x}(k)$$

in which $P = [p_{ij}]$ is a stochastic matrix and p_{ij} is the probability that the system will be in state i at time $t = k+1$ if it is in state j at time $t = k$. The matrix P is called the **transition matrix** for the system.

Markov chains in terms of powers of the transition matrix: In a Markov chain with an initial state $\mathbf{x}(0)$, the successive state vectors are

$$\mathbf{x}(1) = P\mathbf{x}(0), \quad \mathbf{x}(2) = P\mathbf{x}(1), \quad \mathbf{x}(3) = P\mathbf{x}(2), \quad \ldots$$

It is common to denote $\mathbf{x}(k)$ by $\mathbf{x}_k$, which allows us to write the successive state vectors as

$$\mathbf{x}_1 = P\mathbf{x}_0, \quad \mathbf{x}_2 = P\mathbf{x}_1, \quad \mathbf{x}_3 = P\mathbf{x}_2, \quad \ldots$$

In addition, these state vectors can be expressed in terms of the initial state vector $\mathbf{x}_k$ as

$$\mathbf{x}_1 = P\mathbf{x}_0, \quad \mathbf{x}_2 = P^2\mathbf{x}_0, \quad \mathbf{x}_3 = P^3\mathbf{x}_0, \quad \ldots$$

from which it follows that

$$\boxed{\mathbf{x}_k = P^k\mathbf{x}_0}$$

Convergence: We say that a sequence of vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_k, \ldots$ converges to $\mathbf{q}$ if $\mathbf{x}_k \to \mathbf{q}$ as $k \to \infty$. Similarly, we say that a sequence of matrices $P_1, P_2, P_3, \ldots, P_k, \ldots$ converges to a matrix Q if $P_k \to Q$ as $k \to \infty$.

Dynamical Systems and Markov Chains (III)

Definition (Regular Markov chain). A stochastic matrix P is said to be **regular** if P or some positive power of P has all positive entries, and a Markov chain whose transition matrix is regular is said to be a **regular Markov chain**.

Theorem (Regular Markov chains). If P is the transition matrix of a regular Markov chain, then:

(i) There is a unique probability vector $\mathbf{q}$ with positive entries such that $P\mathbf{q} = \mathbf{q}$.

(ii) For any initial probability vector $\mathbf{x}_0$, the sequence of state vectors

$$\mathbf{x}_0,\ P\mathbf{x}_0,\ \ldots,\ P^k\mathbf{x}_0,\ \ldots$$

converges to $\mathbf{q}$.

(iii) The sequence P, P^2, P^3, ..., P^k, ... converges to the matrix Q each of whose column vectors is $\mathbf{q}$.

Steady-state vector: The vector $\mathbf{q}$, called the stady-state vector of the Markov chain, is an eigenvector of the eigenvalue $\lambda = 1$ of P. Thus, it can be found solivng

$$(I - P)\mathbf{q} = 0$$

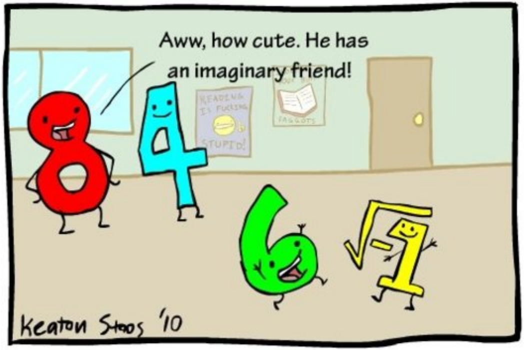
subject to the requirement that the sum of all the entries of $\mathbf{q}$ equals 1.

Motivation: What is the square root of a negative number? To answer this question we need to introduce the **imaginary number**

$$i = \sqrt{-1}$$

which has the property $i^2 = (\sqrt{-1})^2 = -1$.

Example: The solution to the equation $x^2 = -1$ is $x = \pm i$.

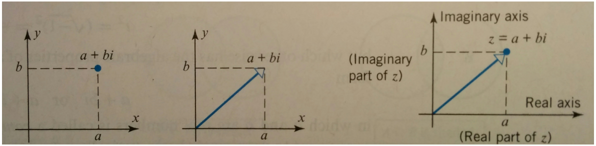


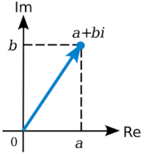
Definition (complex numbers). A **complex number** is an ordered pair of real numbers, denoted by (a, b) or $a + bi$, where $i^2 = -1$.

Complex Plane: Complex numbers can be viewed as a point or a vector in the xy-plane (the complex plane). Typically complex numbers are denoted by a single letter z

$$z = a + bi$$

where a is called the **real part** of z (denoted by $\mathrm{Re}(z)$) and b is the **imaginary part** of z (denoted by $\mathrm{Im}(z)$). [Plot on the right is called Argand diagram]





Definition (equality). Two complex numbers, $a + bi$ and $c + di$, are defined to be equal, written

$$a + bi = c + di$$

if $a = c$ and $b = d$.

Complex Numbers (III)

Basic operations with complex numbers:

- Addition:
$$(a+bi)+(c+di)=(a+c)+(b+d)i$$
- Substraction:
$$(a+bi)-(c+di)=(a-c)+(b-d)i$$
- Multiplication by a scalar k:
$$k(a+bi)=(ka)+(kb)i$$
- Multiplication:
$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i$$

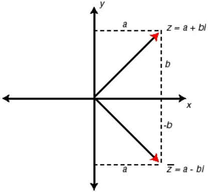
Comment: Similary, it is possible to add, substract, and multiply matrices with complex entries and to multiply a matrix by a complex number. Matrix operations and terminology discussed in previous sections carry over without change to matrices with complex entries.

Division of Complex Numbers (I)

Complex Conjugates: If $z = a + bi$ is a complex number, then the complex conjugate (or just conjugate) of z is denoted by $\bar{z}$ and defined by

$$\bar{z} = a - bi$$

The conjugate is obtained by reversing the sign of $\mathrm{Im}(z)$. Geometrically:



Definition (modulus). The **modulus** of a complex number $z = a + bi$, denoted by $|z|$, is defined by

$$|z| = \sqrt{a^2 + b^2}$$

The modulus of z is also called the **absolute value** of z.

Division of Complex Numbers (II)

Theorem (Relation between $\bar{z}$ and $|z|$). For any complex number z,

$$z\bar{z} = |z|^2$$

Division of complex numbers: We will define division as the inverse of multiplication. Thus, if $z_2 \neq 0$, then the definition of $z = z_1/z_2$ should be such that

$$z_1 = z_2 z \tag{1}$$

Theorem. If $z_2 \neq 0$, then Eq. (1) has a unique solution, which is

$$z = \frac{1}{|z_2|^2} z_1 \bar{z}_2$$

The above theorem allows us to write the division of the complex numbers z_1 and z_2 as

$$\boxed{z = \frac{z_1}{z_2} = \frac{1}{|z_2|^2} z_1 \bar{z}_2}$$

Theorem (Properties of the conjugate). For any complex numbers z_1, z_2 and z_3:

(a) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (c) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ (f) $\bar{\bar{z}} = z$

(b) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (d) $\overline{z_1/z_2} = \bar{z}_1/\bar{z}_2$