Polar Form of a Complex Number (I)

The **polar form** of a complex number $z = x + iy$ is given by

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

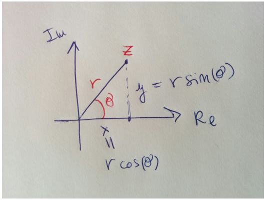
where θ is the angle from the real axes to the vector z and r is the modulus of z, i.e. $r = |z|$. Note that $x = r\cos\theta$ and $y = r\sin\theta$ (see figure below).

The angle θ is called the **argument of** z and is denoted by

$$\theta = \arg z$$

Note $\theta = \arg z$ is not uniquely determined (adding/substracting multiples of 2π from θ produces same argument). The **principal argument of** z is the argument of z that satisfies $-\pi < \theta < \pi$ and is denoted by

$$\theta = \operatorname{Arg} z$$



Polar Form of a Complex Number (II)

Multiplication and division (using polar form). Let

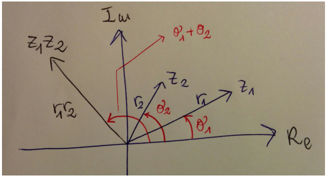
$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Then,

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$



DeMoivre's Formula. If n is a any integer and $z = r(\cos\theta + i\sin\theta)$ (and $z \neq 0$), then

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

In the special case where $r = 1$, we have

$$\boxed{(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)}$$

Polar Form of a Complex Number (III)

Application of DeMoivre's Formula (finding roots of z). If n is a positive integer, then n-th root of $z = r(\cos\theta + i\sin\theta)$ is given by

$$z^{1/n} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right], \quad k = 0, 1, 2, \ldots, n-1$$

Euler's Formula. In more detailed studies of complex numbers, complex exponents are defined, and it is shown that

$$\boxed{(\cos\theta + i\sin\theta) = e^{i\theta}}$$

where e is the irrational number $e \approx 2.71828\ldots$ (Euler number). Note: the expression above can be derived by expanding e^x around 0, substituting $x = i\theta$ and rearranging terms.

From Euler's formula it follows that the polar form $z = r(\cos\theta + i\sin\theta)$ can be expressed as

$$z = re^{i\theta}$$

and $\bar{z} = r(\cos\theta - i\sin\theta)$ as

$$\bar{z} = re^{-i\theta}$$

Definition (ordered n-tuple & n-space). If n is a positive integer, then an **ordered n-tuple** is a sequence of n real numbers $(v_1, v_2, \ldots, v_n)$. The set of all ordered n-tuples is called **n-space** and is denoted by R^n.

Definition (equal vectors). Vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ in R^n are said to be **equivalent** (also called **equal**) if

$$v_1 = w_1 \quad v_2 = w_2 \quad \ldots \quad v_n = w_n$$

We indicate this by writing $\mathbf{v} = \mathbf{w}$.

Definition (operations). If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are vectors in R^n, and if k is any scalar, then we define

(i) $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$
(ii) $k\mathbf{v} = (kv_1, kv_2, \ldots, kv_n)$
(iii) $-\mathbf{v} = (-v_1, -v_2, \ldots, -v_n)$
(vi) $\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1, w_2 - v_2, \ldots, w_n - v_n)$

Vectors in 2-space, 3-space, and n-space (II)

Theorem (properties of operations). If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are vectors in R^n, and if k and m are scalars, then:

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
(b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
(c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
(d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
(e) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
(f) $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
(g) $k(m\mathbf{u}) = (km)\mathbf{u}$
(h) $\mathbf{1u} = \mathbf{u}$

Theorem (additional properties). If $\mathbf{v}$ is a vector in R^n and k an scalar, then:

(a) $0\mathbf{v} = \mathbf{0}$ (b) $k\mathbf{0} = \mathbf{0}$ (c) $(-1)\mathbf{v} = -\mathbf{v}$

Definition (linear combination). If $\mathbf{w}$ is a vector in R^n, then $\mathbf{w}$ is said to be a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in R^n if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r \tag{1}$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the **coefficients** of the linear combination. If $r = 1$, formula (1) becomes $\mathbf{w} = k_1\mathbf{v}_1$, so that a linear combination of a single vector is just a scalar multiple of that vector.

Norm, Dot Product, and Distance in R^n (I)

Definition (norm). If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the **norm** (or **length** or **magnitude**) of $\mathbf{v}$ is denoted by $||\mathbf{v}||$, and is defined by the formula

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Theorem (facts about vectors in R^n). If $\mathbf{v}$ is a vector in R^n, and if k is any scalar, then:

$$\text{(a) } ||\mathbf{v}|| \geq 0 \qquad \text{(b) } ||\mathbf{v}|| = 0 \iff \mathbf{v} = \mathbf{0} \qquad \text{(c) } ||k\mathbf{v}|| = |k|||\mathbf{v}||$$

Unit vector: A vector of norm 1 is called a **unit vector**. If $\mathbf{v}$ is a nonzero vector, then

$$\mathbf{u} = \frac{1}{||\mathbf{v}||}\mathbf{v}$$

defines the unit vector that has the same direction as $\mathbf{v}$. Obtaining the unit vector $\mathbf{u}$ of $\mathbf{v}$ is called **normalizing** $\mathbf{v}$.

Standard unit vectors are the unit vectors in the positive directions of the coordinate axes. In R^n the standard unit vectors are expressed as

$$\mathbf{e}_1 = (1, 0, \ldots, 0) \quad \mathbf{e}_1 = (0, 1, \ldots, 0) \quad \ldots \quad \mathbf{e}_n = (0, 0, \ldots, 1)$$

Every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n can be expressed as a linear combination of the standard unit vectors

$$\mathbf{v} = (v_1, v_2, \ldots, v_n) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$$

Norm, Dot Product, and Distance in R^n (II)

Definition (distance). If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in R^n, then we denote the distance between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u}, \mathbf{v})$ and define it to be

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

Definition (dot product in R^2, R^3). If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vecotrs in R^2 and R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the **dot product** (or **Euclidian inner product**) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| \, ||\mathbf{v}|| \cos\theta$$

If $\mathbf{u} = 0$ or $\mathbf{v} = 0$, then $\mathbf{u} \cdot \mathbf{v} = 0$.

Definition (dot product - component form). If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the **dot product** (or **Euclidian inner product**) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

Comment: In the particular case $\mathbf{u} = \mathbf{v}$

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

Theorem (properties of dot product). If $\mathbf{u}$, $\mathbf{v}$ are vectors in R^n, and if k is a scalar, then:

(a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$	Symmetry property
(b) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$	Distributive property
(c) $k(\mathbf{u}\cdot\mathbf{v}) = (k\mathbf{u})\cdot\mathbf{v}$	Homogeneity property
(d) $\mathbf{v}\cdot\mathbf{v}\geq 0$, and $\mathbf{v}\cdot\mathbf{v}=0 \iff \mathbf{v}=\mathbf{0}$	Positivity property

Theorem (additional properties). If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in R^n, and if k is a scalar, then:

(a) $\mathbf{0}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{0} = 0$

(b) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$

(c) $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{w}$

(d) $(\mathbf{u}-\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w}-\mathbf{v}\cdot\mathbf{w}$

(e) $k(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(k\mathbf{v})$

Angles in R^n: We can extend the notion of angle to R^n by means of the dot product definition in R^2 and R^3

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\,||\mathbf{v}||}\right)$$

This expression is defined if the argument satisfies

$$-1 \leq \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\,||\mathbf{v}||} \leq 1$$

Norm, Dot Product, and Distance in R^n (IV)

Theorem (Cauchy-Schwarz Inequality). If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then

$$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}||\, ||\mathbf{v}||$$

or in terms of components

$$|v_1 u_1 + \ldots + u_n v_n| \leq \sqrt{u_1^2 + \ldots + u_n^2}\, \sqrt{v_1^2 + \ldots + v_n^2}$$

Theorem (Triangle inequalities). If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in R^n, then
(a) $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ (triangle inequiality for vectors)
(b) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ (triangle inequality for distances)

Theorem (Parallelogram Equation for Vectors). If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then

$$||\mathbf{u} + \mathbf{v}||^2 + ||\mathbf{u} - \mathbf{v}||^2 = 2(||\mathbf{u}||^2 + ||\mathbf{v}||^2)$$

Theorem (Rel. between dot product and norm). If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}||\mathbf{u} + \mathbf{v}||^2 - \frac{1}{4}||\mathbf{u} - \mathbf{v}||^2$$

Table 1

Form	Dot Product	Example	
u a column matrix and **v** a column matrix	$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{u}$	$\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$	$\mathbf{u}^T\mathbf{v} = [1 \quad -3 \quad 5]\begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$ $\mathbf{v}^T\mathbf{u} = [5 \quad 4 \quad 0]\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$
u a row matrix and **v** a column matrix	$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\mathbf{v} = \mathbf{v}^T\mathbf{u}^T$	$\mathbf{u} = [1 \quad -3 \quad 5]$ $\mathbf{v} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$	$\mathbf{u}\mathbf{v} = [1 \quad -3 \quad 5]\begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$ $\mathbf{v}^T\mathbf{u}^T = [5 \quad 4 \quad 0]\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$
u a column matrix and **v** a row matrix	$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\mathbf{u} = \mathbf{u}^T\mathbf{v}^T$	$\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$ $\mathbf{v} = [5 \quad 4 \quad 0]$	$\mathbf{v}\mathbf{u} = [5 \quad 4 \quad 0]\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$ $\mathbf{u}^T\mathbf{v}^T = [1 \quad -3 \quad 5]\begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$
u a row matrix and **v** a row matrix	$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\mathbf{v}^T = \mathbf{v}\mathbf{u}^T$	$\mathbf{u} = [1 \quad -3 \quad 5]$ $\mathbf{v} = [5 \quad 4 \quad 0]$	$\mathbf{u}\mathbf{v}^T = [1 \quad -3 \quad 5]\begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$ $\mathbf{v}\mathbf{u}^T = [5 \quad 4 \quad 0]\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$

Dot product as matrix multiplication: If A is an $n \times n$ matrix and $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ matrices, then it follows that

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}$$

$$\mathbf{u} \cdot A\mathbf{v} = A^T\mathbf{u} \cdot \mathbf{v}$$

Dot product view of matrix multiplication: If the row vectors of A are $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_3$ and the column vectors of B are $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_3$, then the product AB can be expressed as

$$AB = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{c}_1 & \mathbf{r}_1 \cdot \mathbf{c}_2 & \ldots & \mathbf{r}_1 \cdot \mathbf{c}_n \\ \mathbf{r}_2 \cdot \mathbf{c}_1 & \mathbf{r}_2 \cdot \mathbf{c}_2 & \ldots & \mathbf{r}_2 \cdot \mathbf{c}_n \\ \vdots & & & \vdots \\ \mathbf{r}_n \cdot \mathbf{c}_1 & \mathbf{r}_n \cdot \mathbf{c}_2 & \ldots & \mathbf{r}_n \cdot \mathbf{c}_n \end{bmatrix}$$