Ex: [choose your row/column wisely]

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 \\ 3 & 2 \end{pmatrix}$$

$$\det(A) = a_{11}C_{11} + \cancelto{0}{a_{21}} C_{21} + \cancelto{0}{a_{31}} C_{31}$$

$$= a_{11}C_{11} =$$

$$= 1 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} =$$

$$= -1$$

Ex: [det. of triangular matrices]

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(A) = 1 \cdot 1 \cdot 1 = 1$$

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\det(B) = 2 \cdot 2 \cdot 2 = 8$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{pmatrix} \qquad \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix}$$

• Ex:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|cc} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 3 & 2 & 0 & 3 \end{array}\right.$$

$$\det(A) = 2 + 0 + 0 - (0 + 3 + 0)$$

$$= 2 - 3 = -1$$

Ex: $[\det(A) = \det(A^T)]$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \rightarrow A^T = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 2 & 2 \\ 0 & 3 & 1 \end{pmatrix}$$

$$\left.\begin{aligned} \det(A) &= 0 \\ \det(A^T) &= 0 \end{aligned}\right\} \quad \det(A) = \det(A^T)$$

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \rightarrow \det(A) = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$\underline{\underline{EX}}$: $[\det(B) = -\det(A)]$

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{pmatrix} \xrightarrow{\text{INTERCHANGE } A1 \,\&\, A2} B = \begin{pmatrix} 0 & \boxed{1 \;\; 1} \\ 1 & 0 \;\; 2 \\ 0 & \boxed{3 \;\; 2} \end{pmatrix}$$

$\hookrightarrow \det(A) = -1$ //

$\det(B) = b_{21}\, C_{21} =$

$= 1 \underbrace{(-1)^{2+1}}_{\downarrow} \underbrace{\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}}_{= -1} =$ 2 - 3

$= -1 \; (2 - 3) =$

$= 1$ //

<u>Ex</u>:
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Add $-2A1$ to $A2$

$$\to B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\to$ Using cof. expansion in row 2 $\to \det(B) = 0$

$$\det(A) = \det(B) = 0$$

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Ex : [Evaluating det. by row reduction]

i) $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow[\text{to } A2]{\text{Add } -A1} B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$\xrightarrow[\text{to } B3]{\text{Add } B2} C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

$\det(C) = 1 \cdot (-1) \cdot 2 = -2$

$\det(C) = \det(B) = -2$

$\det(B) = \det(A) = -2$

ii) $D = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \longrightarrow$

$\xrightarrow{\text{Multiplying } D2 \text{ by } 1/2} A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$\to B \to C$

$\det(C) = \det(B) = \det(A) = -2$

$\to \det(A) = \frac{1}{2} \det(D)$

$\Rightarrow \det(D) = 2 \cdot (-2) =$

$= -4$

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$\underline{\underline{Ex}}$: $\left[\det(k\underbrace{A}_{= A}) = k^n \det(A) \right]$

$$B = k \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det(B) = \det(kA) =$$

$$= \begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} =$$

$$= k^2 ad - k^2 bc$$

$$= k^2 \underbrace{(ad - bc)}_{= \det(A)}$$

Ex: $\left[\det(C) = \det(A) + \det(B)\right]$

$$\det\left[\overbrace{\begin{pmatrix} 1 & 2 \\ 3+1 & 2+1 \end{pmatrix}}^{\equiv C}\right]_{-5} =$$

$$= \det\left[\overbrace{\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}}^{\equiv A}\right]_{-4} + \det\left[\overbrace{\begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}}^{\equiv B}\right]_{-1} = \underline{\underline{-5}}$$

Ex: [$\det(EB) = \det(E) \cdot \det(B)$]

$$\underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}}_{\substack{\parallel \\ E}} \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{\substack{\parallel \\ B}} = \underbrace{\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}}_{\substack{\parallel \\ EB}}$$

$$\det(EB) = 2$$

$$\det(E) \cdot \det(B) = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= (2-0)(1-0)$$

$$= 2$$

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PROOF ($\Rightarrow$): (A INVERTIBLE $\Rightarrow$ $\Rightarrow \det(A) \neq 0$)

RREF of A:

$$R = E_1 E_2 \cdots E_r A$$

Equiv. theorem: A invertible $\Leftrightarrow$ $R = I_n$

Then, using the previous theorem

$$\det(R) = \det(I_n) = 1$$

$$= \underbrace{\det(E_1)\det(E_2)}_{\neq 0} \cdots \det(A)$$

$\Rightarrow \det(A) \neq 0$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\det(A) = 1 \cdot 2 - 1 \cdot 2 =$$

<u>Ex</u>: $[\det(AB) = \det(A) \cdot \det(B)]$

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} \rightarrow \det(AB) =$$
$$= 0 - 3 = -3$$

$$\det(A)\det(B) = \begin{array}{c} 2+1 \\ \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \end{array} \cdot \begin{array}{c} 1-2 \\ \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \end{array} = -3$$

Ex: [adjoint A] $C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} = -2$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \to \boxed{\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & \cdot & \\ C_{31} & & \ddots \end{bmatrix}} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = 0$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

↳ do the same for the rest

$$\operatorname{adj}(A) = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} //$$

<u>**Ex**</u>: $\left[A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) \right]$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \operatorname{adj}(A) = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(A) = 1 \cdot 1 \cdot 1 = \underline{\underline{1}}$$

$$A^{-1} = \frac{1}{1} \cdot \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

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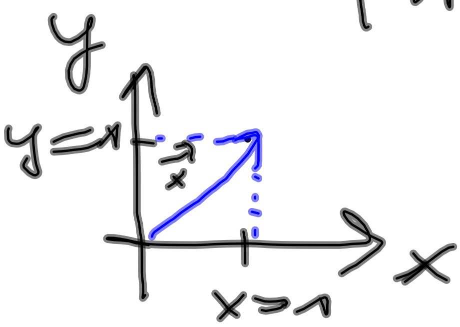
GEOMETRICAL INTERPRETATION

$$\vec{x} \xrightarrow{\text{Apply } A} \lambda \vec{x}$$

<u>Ex</u>: 2D - Case

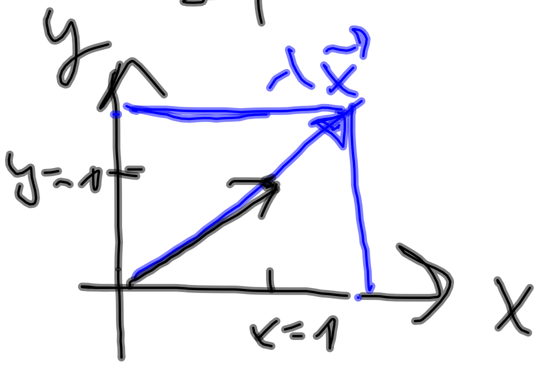
$$\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{A\vec{x}} \lambda \vec{x} = \begin{pmatrix} \lambda \\ \lambda \end{pmatrix}$$



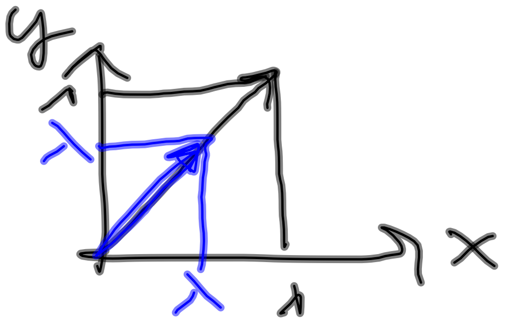
$\xrightarrow{A\vec{x}}$

<u>case (i)</u>

If $\lambda > 1$

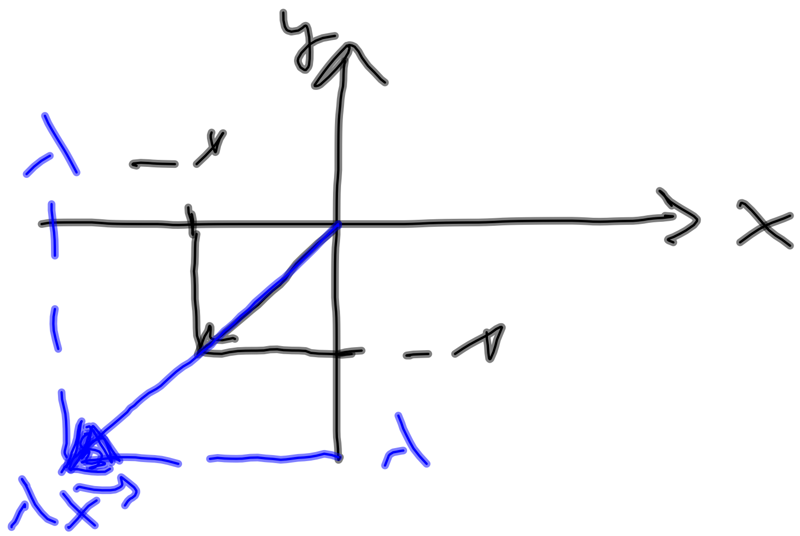


Case (ii): IF $0 < \lambda < 1$



Case (iii): IF $-1 < \lambda < 0$

Case (iv): IF $\lambda < -1$



→ $\lambda\vec{x}$ has the same direction and equal or different magnitude than $\vec{x}$.

related to the value of λ

$\underline{\underline{Ex}}$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det\left[\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] =$$

$$= \det\left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]$$

$$= \det\left[\begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix}\right]$$

$$= \begin{vmatrix} (\lambda - a) & -b \\ -c & (\lambda - d) \end{vmatrix} =$$

$$= \boxed{(\lambda - a)(\lambda - d) - bc = 0}$$

CHARACTERI. EQUATION OF A

2ND ORDER POLYNOMIAL
IN λ

2 ROOTS

These roots will
be the eigenvalues
of A.