p-ADIC UNIFORMIZATION OF SHIMURA CURVES: THE THEOREMS OF CEREDNIK AND DRINFELD

BOUTOT-CARAYOL

ABSTRACT. This is a translation of the *Asterisque* paper of Boutot and Carayol explaining Drinfeld's proof of the Cerednik- Drinfeld theorem. This p-adic uniformization result was originally obtained by Cerednik. Drinfeld then gave a more conceptual proof by interpreting the p-adic upper half plane as a moduli space for certain formal groups.

CONTENTS

Introduction

Let Δ denote an indefinite quaternion algebra over $\mathbf{Q}$. It corresponds to a projective system, indexed by compact open subgroups U of $\Delta(\mathbf{A}_f)^\times$, of *Shimura curves* S_U: they are algebraic curves (complete if Δ is a division algebra), defined over $\mathbf{Q}$, whose absolutely connected components are defined over cyclotomic extensions of $\mathbf{Q}$. The most well-known example of such a situation is the case when Δ is $M_2(\mathbf{Q}_p)$: the curves obtained are the usual modular curves. Their reduction modulo p has been well-studied, and the nature of this reduction depends on the exponent of p in the level, which is to say the component U_p at p of the subgroup U (supposing, for simplicity, that it decomposes into a product): in particular – when U is small enough to avoid problems of non-representability – our curve has good reduction at p if U_p is maximal, which is to say that p does not divide the level. One can consult [De-Ra] and [K-M] for the study of the bad reduction in the case when U_p is not maximal: this reduction can be described in terms of a moduli problem (where the famous Drinfeld bases play a role), which allows one to study the special fiber and the singularity obtained there. In the case of a general algebra Δ, at a place p where Δ is unramified, the sitation is formally similar to the case of modular curves: the Shimura curve has good reduction when U_p is maximal, and the case of bad reduction can be described in an analogous fashion to the modular case; moreover, everything can be generalised to the case of quaternion algebras over a totally real field ([Ca 1]).

The situation is quite different *at a place p where the algebra Δ is ramified*: in this case, supposing that U_p is maximal, Cerednik showed that the Shimura curve S_U admits a *p-adic uniformisation*, which is to say that $S_U \otimes \mathbf{Q}_p$ is the union of (twisted forms of) *Mumford quotients* of the "p-adic upper half-plane" by Schottky subgroups of $\mathrm{PGL}_2(\mathbf{Q}_p)$; such subgroups are obtained from the algebra $\overline{\Delta}$ obtained from Δ by interchanging local invariants at the places p and ∞ (that is, $\overline{\Delta}$ is definite and unramified at p). One can use this uniformisation to describe the special fiber at p, which is a quotient by a finite group of a graph of projective lines (whose general fiber is singular).

Cerednik obtained his result via an indirect method, where the principal was to consider a priori Mumford curves, and to compare them to Shimura curves on the one hand, and one the other to study the action of the fundamental group: this method, which was motivated by work of Ihara, is similar – unsurprisingly – to that used by Kazhdan to study conjugates of Shimura varieties.

Deligne and Kazhdan quickly remarked that the result of Cerednik pointed towards the existence of a universal family of formal groups over the rigid-analytic "half-plane" $\Omega_{\mathbf{Q}_p} = \mathbf{P}^1(\mathbf{C}_p) - \mathbf{P}^1(\mathbf{Q}_p)$; this is what Drinfeld's fundamental theorem proves: in a very precise manner, he proved that $\widehat{\Omega}_{\mathbf{Q}_p} \widehat{\otimes} \widehat{\mathbf{Z}_p}^{nr}$ – where $\widehat{\Omega}_{\mathbf{Q}_p}$ is a formal scheme over $\mathbf{Z}_p$ whose special fiber is $\Omega_{\mathbf{Q}_p}$ – parameterises a family of formal groups, of dimension 2 and height 4, along with an action of the maximal order of the quaternion field D of center $\mathbf{Q}_p$, and with a "rigidification". The local theorem of Drinfeld is also valid in higher dimensions (where $\Omega_{\mathbf{Q}_p}$ is replaced by $\mathbf{P}^{n-1}(\mathbf{C}_p)$ deprived of its rational hyperplanes: one obtains in this way a moduli space for formal groups of dimension n and height n^2, along with an action of the maximal order of the division algebra of invariant $1/n$), as well as for $\Omega_K = \mathbf{P}^1(\mathbf{C}_K) - \mathbf{P}^1(K)$ (where K is a non-archimedean local field) and its analogues in higher dimensions. The method used by Drinfeld for proving his theorem rests on the theory of Dieudonne-Cartier: it consists of an ingenious algebraic construction on the Dieudonne modules of the formal groups under consideration, and in this way one shows that their moduli is represented by the formal scheme $\widehat{\Omega}_{\mathbf{Q}_p}\widehat{\otimes}\widehat{\mathbf{Z}}_p^{nr}$, and in this way one obtains an isomorphism of functors.

One the local theorem has been proved, Drinfeld easily derives the original result of Cerednik: one sees in fact that S_U parameterises a family of abelian varieties classified by $\widehat{\Omega}_{\mathbf{Q}_p}\widehat{\otimes}\widehat{\mathbf{Z}_p}^{nr}$. The theorem of Drinfeld thus reveals a profound structure underlying the result of Cerednik. Moreover, the local theorem allows one to define a natural projective system of etale coverings Σ_n of $\Omega_{\mathbf{Q}_p}\otimes\mathbf{Q}_p^{nr}$: these coverings, which are a little mysterious, allow one to uniformise the curves S_U at a place p where Δ is ramified and where U_p is not maximal, something which was not possible using the methods of Cerednik. One can also apply the local theorem in other cases and obtains p-adic uniformisation results: for example, one can uniformise Shimura curves defined over totally real fields (which was already treated by Cerednik); the specialists in the subject know in principle how to do this, but it has not, to our knowledge, been written down. Moreover, Rapoport and Zink have, using the local theorem in higher dimensions, uniformised Shimura varieties associated to certain unitary groups. Finally we mention that Drinfeld knows how to uniformise "Elliptic Modules II" by the coverings Σ_n.

Drinfeld proved his theorem in a very brief article ([Dr 2]), which is very dense and difficult to digest; our goal here is to expand on his method, and prove the theorem in detail. The present work is divided into three distinct chapters. The first treats the non-archimedean half-plane, and its different perspectives (rigid-analytic or formal), as well as the different moduli problems that it represents. The second chapter constitutes the heart of the work: the local theorem is described and proved. Essential points regarind the method Drinfeld used in his proof were explained to us by Thomas Zink, who lectured to us a number of times on the subject in Strasbourg. Our debt in this regard is difficult to estimate: we wish to extend here our thanks, and we hope that our work is satisfactory. After a lot of hesitation, we made the, perhaps questionable, choice to restrict our treatment of the theorem to the case of the "half-plane", which is to say the case of dimension 1 (over an arbitrary p-adic field, however); this choice allows us to keep certain diagrams small, and discuss different cases more explicitely (however the necessary ideas to prove the theorem in higher dimensions are essentially similar). The third and final chapter treats the global situation: we describe, comment upon and finally prove the theorem of Cerednik (in the case of the base field $\mathbf{Q}$).

I. THE NONARCHIMEDEAN HALF PLANE

Let K be a non-archimedean local field and $\mathbf{C}$ the completion of the algebraic closure of K. The non-archimedean "half-plane" Ω over K is defined set-theoretically as $\Omega = \mathbf{P}^1(\mathbf{C}) - \mathbf{P}^1(K)$.

In section 1 we recall the construction of the tree I associated with $\mathrm{PGL}(2,K)$ [Se] and its geometric realisation $I_{\mathbf{R}}$. Then we define a map $\lambda\colon \Omega \to I_{\mathbf{R}}$ which allows us to describe the rigid analytic structure of Ω, in the sense of Tate [Ta 1], as the inverse image under λ of edges of the tree. This description was given by Drinfeld [Dr 1] and has been described in detail (in arbitrary dimensions) by Deligne and Husemoller [De-Hu]. For basics on rigid analytic geometry, one can consult [B-G-R] or [Fr-VdP].

In section 3 we define a formal model, in the sense of Raynaud [Ra 1], of the rigid analytic space Ω. It is a formal scheme $\widehat{\Omega}$ over the ring of integers of K, which we define by glueing formal schemes $\widehat{\Omega}_{[s,s']}$ corresponding to edges $[s,s']$ of the tree

I. This model was first introduced by Mumford [Mu 2] in his article on the non-archimedean analogue of Schottky uniformisation of Riemann surfaces.

Section 4 gives Deligne's (unpublished) functorial description of the formal scheme $\widehat{\Omega}_{[s,s']}$ in terms of edges adjacent to vertices s and s' of I. By glueing constructible sheaves along these edges, one obtains (in section 5) the functorial description of $\widehat{\Omega}$ used by Drinfeld [Dr 2]. One can find another treatment of this material in the recent article of Teitelbaum [Te].

We end this chapter in section 6 by describing the action of $\mathrm{PGL}(2, K)$ on the formal scheme $\widehat{\Omega}$ and on the corresponding functor.

1. **The building of** $\mathrm{PGL}_2(K)$.

1.0. Let K be a local non-archimedean field, $\mathcal{O}$ the ring of integers of K and π a uniformiser of $\mathcal{O}$. Let $k = \mathcal{O}/\pi\mathcal{O}$ be the residue field, p its characteristic and q its order. We write $\mathbf{C}$ for the completion of the algebraic closure of K and $|\cdot|$ for the norm on $\mathbf{C}$ normalised so that $|\pi| = q^{-1}$. The valuation v is given by: $v(x) = \log_q |x|$.

1.1. A lattice M of K^2 is an $\mathcal{O}$-submodule which is free of rank 2. Two lattices M and M' are homothetic if there exists $\lambda \in K^\times$ such that $M' = \lambda M$. We write S for the collection of homothety classes of lattices and $[M]$ for the class of the lattice M.

The building of $\mathrm{PGL}(2, K)$ is the graph I with vertices given by S, and where $s = [M]$ is joined to s' by an edge if and only if there exists a representative lattice M' for s' such that $\pi M \subsetneq M' \subsetneq M$. Then I is a tree such that each vertex has $q + 1$ adjacent edges: the edges leaving a vertex $s = [M]$ are in bijection with the lines in $M/\pi M$, in other words with $\mathbf{P}^1(k)$.

1.2. The points of the geometric realisation $I_{\mathbf{R}}$ of I are identified with proportionality classes of norms on the K-vectorspace K^2 (cf. [G-Iw]):

a) To a vertex $s = [M]$ corresponds the class of the norm $|\cdot|_M$ such that the corresponding unit ball is M. If (e_1, e_2) is a basis for M and if $v = a_1e_1 + a_2e_2$, one has

$$|v|_M = \sup\{|a_1|, |a_2|\}.$$

b) If s and s' are two adjacent edges and if $s = [M]$ and $s' = [M']$, with $\pi M \subset M' \subset M$, there exists a basis (e_1, e_2) of M such that $(e_1, \pi e_2)$ is a basis for M'. For $v = a_1e_1 + a_2e_2$, one has

$$|v|_M = \sup\{|a_1|, |a_2|\},$$
$$|v|_{M'} = \sup\{|a_1|, q|a_2|\}.$$

To a point $x = (1-t)s + ts'$, with $0 < t < 1$, on the edge between s and s', corresponds the class of the norm $|\cdot|_t$ defined by

$$|v|_t = \sup |a_1|, q^t |a_2|.$$

One has

$$M = \{v \in K^2 \mid |v|_t \leq \lambda\} \text{ for } q^t \leq \lambda < q,$$
$$M' = \{v \in K^2 \mid |v|_t \leq \lambda\} \text{ for } 1 \leq \lambda < q^2.$$

c) Conversely let $|\cdot|$ be a norm on K^2. For real $\lambda > 0$, the collection $M_\lambda = \{v \in K^2 \mid |v| \leq \lambda\}$ is a lattice in K^2. One has $M_{\lambda'} \subset M_\lambda$ if $\lambda' \leq \lambda$ and $M_{q^{-1}\lambda} = \pi M_\lambda$, thus $[M_\lambda]$ takes at most two values in S as λ varies.

If $[M_\lambda] = s$ is constant, then $|\cdot|$ corresponds to s.

Otherwise $[M_\lambda]$ equals s or s' for two adjacent vertices of I. After possibly replacing $|\cdot|$ by a proportional norm, one has $[M_\lambda] = s$ for $q^t \leq \lambda < q$ and $[M_\lambda] = s'$ for $1 \leq \lambda < q^t$, with $0 < t < 1$. Then $|\cdot|$ corresponds with the point $x = (1-t)s + ts'$ of the edge joining s and s'.

2. **The rigid analytic space Ω.**

2.1. We write $\Omega = \mathbf{P}^1(\mathbf{C}) - \mathbf{P}^1(K)$. If one identifies $\mathbf{P}^1(\mathbf{C})$ with the collection of $\mathbf{C}^\times$-homothety classes of nonzero K-linear maps of K^2 into $\mathbf{C}$, then $\mathbf{P}^1(K)$ corresponds to those maps with K-rank equal to one. Thus Ω is identified with the collection of $\mathbf{C}^\times$-homothety classes of K-linear injective maps of K^2 into $\mathbf{C}$.

2.2. By composing a K-linear injective map $z\colon K^2 \to \mathbf{C}$ with the norm on $\mathbf{C}$, one obtains a norm $|\cdot|_z$ on K^2:

$$|v|_z = |z(v)|\,, \text{ for } v \in K^2.$$

This defines a map $\lambda\colon \Omega \to I_{\mathbf{R}}$

$$\lambda(\text{class of } z) = \text{class of } |\cdot|_z\,.$$

One can verify that the image of λ is $I_{\mathbf{Q}}$.

2.3. Let $s = [M]$ and $s' = [M']$ be two adjacent vertices of I and (e_1, e_2) a basis adapted to the pair, so that $(e_1, \pi e_2)$ is a basis of M'. Identify Ω with $\mathbf{C} - K$ and choose a representative z for a point of Ω such that $z(e_2) = 1$ and $z(e_1) = \zeta \in \mathbf{C} - K$. Then one has

$$\begin{aligned}
\lambda^{-1}(s) =& \{\zeta \in \mathbf{C} \mid |\zeta| \leq 1\} - \bigcup_{a \in \mathcal{O}/\pi\mathcal{O}} \{\zeta \in C \mid |\zeta - a| < 1\},\\
\lambda^{-1}(x) =& \{\zeta \in \mathbf{C} \mid |\zeta| = q^{-t}\} \text{ if } x = (1-t)s + ts',\ 0 < t < 1,\\
\lambda^{-1}(s') =& \{\zeta \in \mathbf{C} \mid |\zeta| \leq q^{-1}\} - \bigcup_{b \in \pi\mathcal{O}/\pi^2\mathcal{O}} \{\zeta \in C \mid |\zeta - b| < q^{-1}\},\\
\lambda^{-1}([s, s']) =& \{\zeta \in \mathbf{C} \mid |\zeta| \leq 1\} - \bigcup_{a \in (\mathcal{O} - \pi\mathcal{O})/\pi\mathcal{O}} \{\zeta \in C \mid |\zeta - a| < 1\}\\
& - \bigcup_{b \in \pi\mathcal{O}/\pi^2\mathcal{O}} \{\zeta \in C \mid |\zeta - b| < q^{-1}\}.
\end{aligned}$$

In other words $\lambda^{-1}(s)$ [resp. $\lambda^{-1}(s')$] is the closed disc of radius 1 [resp. q^{-1}] centered on 0, but with the q open discs of radius 1 [resp. q^{-1}] centered on the K-rational points of the disc removed, while $\lambda^{-1}(]s, s'[)$ is the open anulus of interior radius q^{-1} and exterior radius 1 centered on 0.

Proof. The norms $|\cdot|_z = |\zeta a_1 + a_2|$ and $|\cdot|_t = \sup\{|a_1|, q^t |a_2|\}$ are proportional on K^2 if and only if $|\cdot|_z = q^{-t} |\cdot|_t$. This equality is satisfied if and only if it is so for $a_1 = 1$, which is to say if

$$(*) \qquad |\zeta + a_2| = \sup q^{-t}, |a_2| \text{ for } a_2 \in K.$$

If $0 < t < 1$, one has $|a_2| \neq q^{-t}$ for all $a_2 \in K$, so that $(*)$ is equivalent with $|\zeta| = q^{-t}$.

On the other hand, if $t = 0$, $(*)$ is equivalent with $|\zeta| = 1$ and $|\zeta + a_2| = 1$ for all $a_2 \in K$ such that $|a_2| = 1$, or $|\zeta| \leq 1$ and $|\zeta - a| \geq 1$ for $a \in \mathcal{O}$.

Finally if $t = 1$, $(*)$ is equivalent with $|\zeta| = q^{-1}$ and $|\zeta + a_2| = q^{-1}$ for all $a_2 \in K$ such that $|a_2| = q^{-1}$, or $|\zeta| \leq q^{-1}$ and $|\zeta - b| \geq q^{-1}$ for $b \in \pi\mathcal{O}$. □

Remark. Missing a picture of a standard affinoid.

2.4. The collections $\lambda^{-1}(s)$, $\lambda^{-1}(s')$ and $\lambda^{-1}([s,s'])$ possess natural structures as rigid analytic spaces defined over K; they are connected affinoid subsets of $\mathbf{P}^1_K$, that is, they are each a complement of a finite number of open discs in $\mathbf{P}^1_K$. Moreover $\lambda^{-1}(s)$ and $\lambda^{-1}(s')$ are opens of $\lambda^{-1}([s,s'])$.

More generally, if T is a finite subtree of I, $\lambda^{-1}(T)$ is a connected affinoid subset of $\mathbf{P}^1_K$; it is obtained by glueing the affinoids $\lambda^{-1}([s,s'])$ along the $\lambda^{-1}(s)$ for $[s,s']$ edges of T and s interior vertices of T.

Thus $\Omega = \bigcup \lambda^{-1}(T)$, for all finite subtrees T of I, has a natural structure as a rigid analtyic space defined over K; it is a connected analytic subspace of $\mathbf{P}^1_K$.

2.5. In this way one can imagine Ω as a tubular neighbourhood of $I_{\mathbf{R}}$.

Remark. Missing nice picture of $\lambda^{-1}([s,s'])$ as tubular neighbourhood of $I_{\mathbf{R}}$.

3. The formal scheme $\widehat{\Omega}$.

3.1. If M is a lattice in K^2, the generic fiber of the projective line $\mathbf{P}(M)$ over $\mathcal{O}$ is canonically identified with $\mathbf{P}^1_K$. Moreover, if M_1 is a homothetic lattice, the homothety between M and M_1 defines a unique $\mathcal{O}$-isomorphism between $\mathbf{P}(M)$ and $\mathbf{P}(M_1)$ inducing the identity on the generic fibers. We may thus write $\mathbf{P}_s$, where $s = [M]$ is the vertex of I corresponding to M, for the projective line $\mathbf{P}(M)$ over $\mathcal{O}$ along with the identification of its generic fiber with $\mathbf{P}^1_K$. The points $\lambda^{-1}(s)$ are exactly the points of $\mathbf{P}^1_K(\mathbf{C})$ which do not specialize to k-rational points of the special fiber of $\mathbf{P}_s$. We write Ω_s for the open subscheme of $\mathbf{P}_s$ with is the complement of the rational points of the special fiber and $\widehat{\Omega}_s$ for the formal completion of Ω_s along its special fiber. The canonical bijections $\mathbf{P}^1_K(\mathbf{C}) = \mathbf{P}_s(\mathcal{O}_{\mathbf{C}}) = \widehat{\mathbf{P}}_s(\mathcal{O}_{\mathbf{C}})$ induce a bijection $\lambda^{-1}(s) = \widehat{\Omega}_s(\mathcal{O}_{\mathbf{C}})$; more precisely, the rigid analytic space $\lambda^{-1}(s)$ is the generic fiber (in the sense of Raynaud) of the formal scheme $\widehat{\Omega}_s$. As an affine formal scheme, this means simply that the Tate algebra corresponding to $\lambda^{-1}(s)$ is $\Gamma(\widehat{\Omega}_s) \otimes_{\mathcal{O}} K$.

3.2. A vertex s' of I adjacent to s defines a k-rational point of the special fiber of $\mathbf{P}_s$: if $s = [M]$ and $s' = [M']$ with $\pi M \subset M' \subset M$, this point is defined by the map $M \to M/M' \cong k$. The $\mathcal{O}$-scheme $\mathbf{P}_{[s,s']}$ obtained by blowing up $\mathbf{P}_s$ at this point is equal to the blowup of $\mathbf{P}_{s'}$, at the point defined by s. Its generic fiber, which is the same as that of $\mathbf{P}_s$ and $\mathbf{P}_{s'}$, is canonically identified with $\mathbf{P}^1_K$.

We write $\Omega_{[s,s']}$ for the open subscheme of $\mathbf{P}_{[s,s']}$ with is the complement of the rational points of the special fiber, except for the singular point, and we write $\widehat{\Omega}_{[s,s']}$ for the formal completion of $\Omega_{[s,s']}$ along its special fiber. Identifying the generic fiber of $\mathbf{P}_{[s,s']}$ with $\mathbf{P}^1_K$ induces a bijection $\lambda^{-1}([s,s']) = \widehat{\Omega}_{[s,s']}(\mathcal{O}_{\mathbf{C}})$; in particular, the points of the annulus $\lambda^{-1}(]s,s'[)$ are those with specialise to the singular point of the special fiber of $\mathbf{P}_{[s,s']}$. In fact, retaining the notation of (2.3), the natural coordinates along the two components of the special fiber are ζ and ζ/π (modulo the maximal ideal), over one component the singular point corresponds to 0 after reduction of ζ, and on the other component ζ/π gives a coordinate for the reduction about ∞.

Remark. Picture of components and coordinates omitted.

Here we can say more precisely that the rigid analytic space $\lambda^{-1}([s,s'])$ is the generic fiber of the formal scheme $\widehat{\Omega}_{[s,s']}$. Moreover the canonical maps inducing the open immersions of $\widehat{\Omega}_s$ and $\widehat{\Omega}_{s'}$ in $\widehat{\Omega}_{[s,s']}$ corresponds on the generic fibers (or over the points with values in $\mathcal{O}_{\mathbf{C}}$) with the inclusions of $\lambda^{-1}(s)$ and $\lambda^{-1}(s')$ in $\lambda^{-1}([s,s'])$.

3.3. More generally, if T is a finite subtree of I, one obtains, by glueing the formal schemes $\widehat{\Omega}_{[s,s']}$ along the $\widehat{\Omega}_s$ for $[s,s']$ an edge of T and s an interior vertex of T (according to the incidence relations of T), a formal scheme $\widehat{\Omega}_T$ whose generic fiber is canonically identified with $\lambda^{-1}(T)$.

If $T \subset T'$, the open immersion $\widehat{\Omega}_T \subset \widehat{\Omega}_{T'}$ induce the inclusions $\lambda^{-1}(T) \subset \lambda^{-1}(T')$ on the generic fibers. One constructs in this way a formal scheme $\widehat{\Omega} = \bigcup_T \widehat{\Omega}_T$ whose generic fiber is Ω; in particular, $\widehat{\Omega}(\mathcal{O}_{\mathbf{C}}) = \Omega$. The special fiber of $\widehat{\Omega}$ is a tree of projective lines over k intersecting at k-rational points, which is dual to the tree I.

4. **The functor $\widehat{\Omega}$ of Deligne.** We describe, following Deligne, the functors on the category **Compl** of $\mathcal{O}$-algebras that are separated and complete for the π-adic topology which are represented by the formal schemes $\widehat{\Omega}_s$ and $\widehat{\Omega}_{[s,s']}$.

Definition 4.1. We write F_s for the functor which, to $R \in$ **Compl**, associates the collection of isomorphism classes of pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is a free R-module of rank 1 and $\alpha \colon M \to \mathcal{L}$ is a homomorphism of $\mathcal{O}$-modules satisfying the condition:

$$(*) \begin{cases} \text{for all } x \in \operatorname{Spec}(R/\pi R), \text{ the map} \\ \alpha(x) \colon M/\pi M \to \mathcal{L} \otimes_R k(x) \text{ is injective.} \end{cases}$$

Proposition 4.2. *The functor F_s is representable by the formal scheme $\widehat{\Omega}_s$.*

Proof. The condition on α implies that $\alpha(u)$ is a generator of $\mathcal{L}$ for all $u \in M - \pi M$, and in particular the map $\alpha \otimes \mathrm{id}_R \colon M \otimes_{\mathcal{O}} R \to \mathcal{L}$ is surjective. Thus F_s is a subfunctor of $\widehat{\mathbf{P}}_s$, the formal projective line over $\mathcal{O}$ defined by $s = [M]$.

To describe this subfunctor, choose a basis (e_1, e_2) for M, which determines points $\{0, 1, \infty\}$ of $\widehat{\mathbf{P}}_s$. The pair $(\mathcal{L}, \alpha)$ is determined up to isomorphism by the relation $\alpha(e_1)/\alpha(e_2) = \zeta \in R$. Thus F_s is identified with a subfunctor of the formal affine line $\widehat{P}_s - \{\infty\}$.

The condition on α can be expressed in terms of the image $\overline{\zeta}$ of ζ in $R/\pi R$: for all $\alpha \in k$, $\overline{\zeta} - a$ does not vanish at any point of $\operatorname{Spec}(R/\pi R)$, or in other words, $\overline{\zeta} - a$ is invertible in $R/\pi R$. Thus F_s is the subfunctor of $\widehat{\mathbf{P}}_s - \{\infty\}$ which represents the open which is the complement of the k-rational points of the special fiber, which is to say $\widehat{\Omega}_s$. □

Definition 4.3. We write $F_{[s,s']}$ for the functor which, to $R \in$ **Compl**, associates the collection of isomorphism classes of commutative diagrams:

$$\begin{tikzcd} \pi M \arrow[r, hook] \arrow[d, "\alpha/\pi"] & M' \arrow[r, hook] \arrow[d, "\alpha'"] & M \arrow[d, "\alpha"] \\ \mathcal{L} \arrow[r, "c"] & \mathcal{L}' \arrow[r, "c'"] & \mathcal{L} \end{tikzcd}$$

where $\mathcal{L}$ and $\mathcal{L}'$ are free R-modules of rank 1, α and α' are homomorphisms of $\mathcal{O}$-modules, c and c' are homomorphisms of R-modules, satisfying the condition:

$$(*) \begin{cases} \text{for all } x \in \operatorname{Spec}(R/\pi R), \text{ one has} \\ \ker(\alpha(x) \colon M/\pi M \to \mathcal{L} \otimes_R k(x)) \subset M'/\pi M, \\ \ker(\alpha'(x) \colon M'/\pi M' \to \mathcal{L}' \otimes_r k(x)) \subset \pi M/\pi M'. \end{cases}$$

Proposition 4.4. *The functor $F_{[s,s']}$ is represented by the formal scheme.*

Proof. Let (e_1, e_2) be a basis for M such that $(e_1, \pi e_2)$ is a basis for M'. The condition (*) implies that $\alpha(e_2)$ generates $\mathcal{L}$ and $\alpha'(e_2)$ generates $\mathcal{L}'$. Identify $\mathcal{L}$ with R by putting $\alpha(e_2) = 1$ and $\mathcal{L}'$ with R by putting $\alpha'(e_1) = 1$. Let $\zeta = \alpha(e_1)$ and $\eta = \alpha'(\pi e_2)$. The commutativity of the diagram implies that $c = \eta$, $c' = \zeta$ and $\zeta\eta = \pi$.

Thus the choice of (e_1, e_2) allows us to identify $F_{[s,s']}$ with a subfunctor of the formal scheme $\mathrm{Spf}(\mathcal{O}\{\zeta, \eta\}/(\zeta\eta - \pi))$. The same choice identifies $\mathrm{Spf}(\mathcal{O}\{\zeta, \eta\}/(\zeta\eta - \pi))$ with the open subscheme of $\widehat{\mathbf{P}}_{[s,s']}$ which is the complement of the points at infinity $\overline{\zeta} = \infty$ and $\overline{\eta} = infty$ of the two components of the special fiber.

The condition (*) can be expressed in terms of the images $\overline{\zeta}$ and $\overline{\eta}$ of ζ and η in $R/\pi R$: for all $a \in k - \{0\}$, $\overline{\zeta} - a$ and $\overline{\eta} - a$ are invertible in $R/\pi R$. Thus $F_{[s,s']}$ is the subfunctor of $\widehat{\mathbf{P}}_{[s,s']} - (\{\overline{\zeta} = \infty\} \cup \{\overline{\eta} = \infty\})$ which represents the open which is the complement of the k-rational points of the two components of the special fiber $\mathrm{Spec}(k[\overline{\zeta}, \overline{\eta}]/(\overline{\zeta}\overline{\eta}))$ with the exception of the singular point $\overline{\zeta} = \overline{\eta} = 0$, which is to say $\widehat{\Omega}_{[s,s']}$. □

4.5. The open immersion $\widehat{\Omega}_s \hookrightarrow \widehat{\Omega}_{[s,s']}$ is, with the identifications made above, the restriction of the open immersion $\mathrm{Spf}(\mathcal{O}\{\zeta, \zeta^{-1}\}) \hookrightarrow \mathrm{Spf}(\mathcal{O}\{\zeta, \eta\}/(\zeta\eta - \pi))$. The arrow $F_s \to F_{[s,s']}$ defined by functorially associating to each arrow $\alpha\colon M \to \mathcal{L}$ the commutative diagram:

$$\begin{tikzcd} \pi M \arrow[r, hook] \arrow[d] & M' \arrow[r, hook] \arrow[d] & M \arrow[d, "\alpha"] \\ \mathcal{L} \arrow[r, "\pi"] & \mathcal{L} \arrow[r, equal, "\mathrm{id}"] & \mathcal{L} \end{tikzcd}$$

identifies F_s with the subfunctor of $F_{[s,s']}$ consisting of the diagrams above where c' is invertible.

Similarly the open immersion $\widehat{\Omega}_{s'} \hookrightarrow \widehat{\Omega}_{[s,s']}$ is the restriction of the immersion $\mathrm{Spf}(\mathcal{O}\{\eta, \eta^{-1}\}) \hookrightarrow \mathrm{Spf}(\mathcal{O}\{\zeta, \eta\}/(\zeta\eta - \pi))$. By functorially associating to each arrow $\alpha'\colon M' \to \mathcal{L}'$ the commutative diagram:

$$\begin{tikzcd} \pi M \arrow[r, hook] \arrow[d] & M' \arrow[r, hook] \arrow[d] & M \arrow[d, "\alpha"] \\ \mathcal{L}' \arrow[r, equal, "\mathrm{id}"] & \mathcal{L}' \arrow[r, "\pi"] & \mathcal{L}' \end{tikzcd}$$

on identifies $F_{s'}$ with the subfunctor of $F_{[s,s']}$ consisting of the diagrams where c is invertible.

5. **The functor $\widehat{\Omega}$ of Drinfeld.** From the functors F_s and $F_{[s,s']}$ defined above, one obtains a "modular" description of the open affines $\widehat{\Omega}_s$ and $\widehat{\Omega}_{[s,s']}$ which make up the formal scheme $\widehat{\Omega}$. A variant due to Drinfeld, which we will now describe, allows one to describe $\widehat{\Omega}$ directly in terms of a unique functor F defined on the category **Nilp** of $\mathcal{O}$-algebras where π is nilpotent.

If B is an $\mathcal{O}$-algebra, we write $B[\Pi]$ for the quotient of the algebra of polynimals $B[X]$ by the ideal generated by $X^2 - \pi$: it's thus a free B-module of rank 2, generated by 1 element Π (the image of X), which satisfies $\Pi^2 = \pi$. The algebra $B[\Pi]$ carries a $\mathbf{Z}/2\mathbf{Z}$-grading, such that the elements of B are of degree 0 and Π is of degree 1.

Definition 5.1. Let $B \in$ **Nilp** and $S = \mathrm{Spec}\, B$. Let $F(B)$ – or sometimes $F(S)$ – denote the collection of isomorphism classes of quadruples (η, T, u, r) consisting of the following:

(i) η is a constructible sheaf of flat $\mathbf{Z}/2\mathbf{Z}$-graded $\mathcal{O}[\Pi]$-modules, on S with the Zariski topology.

(ii) T is a sheaf of $\mathbf{Z}/2\mathbf{Z}$-graded $\mathcal{O}_S[\Pi]$-modules such that the homogeneous components T_0 and T_1 are invertible sheaves on S.

(iii) u is an $\mathcal{O}[\Pi]$-linear homomorphism of degree 0 of η in T, such that $u \otimes_{\mathcal{O}} \mathcal{O}_S \colon \eta \otimes_{\mathcal{O}} \mathcal{O}_S \to T$ is injective.

(iv) r is a K-linear isomorphism of the constant sheaf $\underline{K}^2$ with the sheaf $\eta_0 \otimes_{\mathcal{O}} K$.

These data are required to satisfy the following conditions:

[C1] Write $S_i \subset S$ for the zero locus of the morphism $\Pi\colon T_i \to T_{i+1}$ $(i = 0, 1)$; the restriction $\eta_i|S_i$ is a constant sheaf with stalk isomorphic to $\mathcal{O}^2$.

[C2] For all geometric points x of S, write $T(x) = T \otimes_B k(x)$; the map $\eta_x/\Pi\eta_x \to T(x)/\Pi T(x)$ induced by u is injective.

[C3] For $i = 0$, 1, $(\bigwedge^2 \eta_i)|S_i = \pi^{-1}(\bigwedge^2(\Pi^i r\underline{\mathcal{O}}^2))|S_i$.

We conclude this definition with some remarks:

(a) It is clear that the above definition of $F(S)$ makes sense for S which are not necessarily affine, but also for all $\mathcal{O}$-schemes S such that the image of π is nilpotent (i.e. an $(\mathcal{O}/\pi^n\mathcal{O})$-scheme).

(b) From the flatness of the $\mathcal{O}[\Pi]$-module η, and from the existence of r, one deduces that the homogeneous components η_0 and η_1 are flat sheaves of $\mathcal{O}$-modules such that each stalk is free of rank 2. The action of Π defines injective maps

$$\cdots \eta_0 \xrightarrow{\Pi} \eta_1 \xrightarrow{\Pi} \eta_0 \cdots$$

which compose to $\Pi^2 = \pi$.

(c) Giving a triple (η, T, u) amounts to giving a commutative diagram which is periodic of period 2:

$$\begin{tikzcd}
{} \arrow[r] & \eta_0 \arrow[r] \arrow[d, "u_0"] & \eta_1 \arrow[r] \arrow[d, "u_1"] & \eta_0 \arrow[r] \arrow[d, "u_0"] & {} \\
{} \arrow[r] & T_0 \arrow[r] & T_1 \arrow[r] & T_0 \arrow[r] & {}
\end{tikzcd}$$

(d) Using r, one can define sub-sheaves N_0 and N_1 of the constant sheaf $\underline{K}^2_S$:

$$N_0 = r^{-1}\eta_0$$
$$N_1 = r^{-1}(\Pi \otimes \mathbf{Q})^{-1}(\eta_1) = \pi^{-1}r^{-1}(\Pi(\eta_1)).$$

These are sub-sheaves of $\mathcal{O}$-modules of maximal rank (the "edges") which are isomorphic respectively to η_0 and η_1. At each geometric point x of S, one has the inclusions:

$$N_{0,x} \subset N_{1,x} \subset \pi^{-1}N_{0,x} \subset K^2,$$

and thus a simple (vertex and edge) of the tree. More precisely, one sees via condition [C2] that:

– If $\Pi|T_0(x)$ is invertible, then $N_{0,x} = N_{1,x}$. Moreover the normalization condition [C3] gives us: $\bigwedge^2 N_{1,x} = \pi^{-1}\bigwedge^2(\mathcal{O}^2)$.

– If $\Pi|T_1(x)$ is invertible, then $N_{1,x} = \pi^{-1}N_{0,x}$. From [C3] one obtains in this case: $\bigwedge^2 N_{0,x} = \bigwedge^2(\mathcal{O}^2)$.

– The case where $\Pi|T_0(x)$ and $\Pi|T_1(x)$ are both zero remains. One obtainss (using the surjectivity of $u_i \otimes_{\mathcal{O}} \mathcal{O}_S$) a branch $N_{0,x} \subsetneq N_{1,x} \subsetneq \pi^{-1}N_{0,x}$, with:

$$\bigwedge^2 N_{0,x} = \bigwedge^2(\mathcal{O}^2) \text{ and } \bigwedge^2 N_{1,x} = \pi^{-1}\bigwedge^2(\mathcal{O}^2).$$

5.2. Our goal is now to prove the following:

Proposition 5.3. *The functor F is representable by the formal scheme $\widehat{\Omega}$.*

For this, we will define for each vertex s (resp. for each edge $[s, s']$) a morphism of functors $F_s \to F$ (resp. $F_{[s,s']} \to F$), in a manner compatible with patching, which is to say compatible with the open immersions $F_s \hookrightarrow F_{[s,s']}$ and $F_{s'} \hookrightarrow F_{[s,s']}$. In this way we define a morphism of functors $\widehat{\Omega} \to F$, and we will show later that it is an isomorphism.

We begin by remarking that each vertex s of the tree is represented by a lattice $M \subset K^2$, so that we hae $\bigwedge^2 M = \bigwedge^2(\mathcal{O}^2)$, or $\bigwedge^2 M = \pi^{-1} \bigwedge^2(\mathcal{O}^2)$; such a representative M is unique, and the two possibilities are mutually exclusive: they define a partition on the collection of vertices, into even vertices (represented by M with $\bigwedge^2 M = \bigwedge^2(\mathcal{O}^2)$) and odd vertices (admitting a representative M such that $\bigwedge^2 M = \pi^{-1} \bigwedge^2(\mathcal{O}^2)$)). Note that a neighbouring vertex of an even vertex (resp. odd) is odd (resp. even). We suppose in what follows that the representatives M of vertices are chosen so that $\bigwedge^2 M = \bigwedge^2(\mathcal{O}^2)$ or $\bigwedge^2 M = \pi^{-1} \bigwedge^2(\mathcal{O}^2)$. Similarly, we always orient the edges $[s, s']$ so that s is odd and s' is even: that is we have representatives M and M' satisfying: $\bigwedge^2 M = \pi^{-1} \bigwedge^2(\mathcal{O}^2)$, $\bigwedge^2 M' = \bigwedge^2(\mathcal{O}^2)$ and $\pi M \subset M' \subset M$.

5.4. It is easiest to begin by defining $F_s \to F$. We distinguish two cases, according to whether s is even or odd:

I.5.3.1 – *Definition of $F_s \to F$ for $s = [M]$ odd* [$\bigwedge^2 M = \pi^{-1} \bigwedge^2(\mathcal{O}^2)$].

Giving a point of $F_s(B)$ corresponds with giving an invertible sheaf $\mathcal{L}$ on $S = \operatorname{Spec} B$ and an $\mathcal{O}$-linear morphism $\alpha\colon M \to \mathcal{L}$, such that the map $\alpha(x)\colon M/\pi M \to \mathcal{L} \otimes_B k(x)$ is injective for all points x of S.

To such a point corresponds the point of $F(B)$ defined by the following diagram:

$$\begin{tikzcd}
\eta_0 = \underline{M} \arrow[r, "\Pi=\mathrm{id}"] \arrow[d, "u_0=\alpha"'] & \eta_1 = \underline{M} \arrow[r, "\Pi=\pi"] \arrow[d, "u_1=\alpha"'] & \eta_0 = \underline{M} \arrow[d, "u_0=\alpha"'] \\
T_0 = \mathcal{L} \arrow[r, "\Pi=\mathrm{id}"] & T_1 = \mathcal{L} \arrow[r, "\Pi=\pi"] & T_0 = \mathcal{L}
\end{tikzcd}$$

and by the isomorphism $r\colon \underline{K}^2 \xrightarrow{\sim} \underline{M} \otimes K$ which corresponds to the inclusion $M \hookrightarrow K^2$. It is easy to see that all of the definitions of (5.1) are satisfied.

I.5.3.2 – *Definition of $F_{s'} \to F$ for $s' = [M']$ even* [$\bigwedge^2 M' = \bigwedge^2(\mathcal{O}^2)$].

To a point of $F_{s'}(B)$, represented by $\alpha'\colon M' \to \mathcal{L}'$, one associates the point of $F(B)$ defined by the diagram:

$$\begin{tikzcd}
\eta_0 = \underline{M'} \arrow[r, "\Pi=\pi"] \arrow[d, "u_0=\alpha'"'] & \eta_1 = \underline{M'} \arrow[r, "\Pi=\mathrm{id}"] \arrow[d, "u_1=\alpha'"'] & \eta_0 = \underline{M'} \arrow[d, "u_0=\alpha'"'] \\
T_0 = \mathcal{L}' \arrow[r, "\Pi=\pi"] & T_1 = \mathcal{L}' \arrow[r, "\Pi=\mathrm{id}"] & T_0 = \mathcal{L}'
\end{tikzcd}$$

and by the isomorphism $r\colon \underline{K}^2 \xrightarrow{\sim} \underline{M'} \otimes K$ which corresponds with the inclusion $M' \hookrightarrow K^2$.

5.5. *The case of an edge.* It remains to define the maps $F_{[s,s']} \to F$ for an edge $[s, s']$. Suppose that the orientation and the representatives M and M' are chosen as in

(5.2). Giving a point of $F_{[s,s']}(R)$ corresponds with giving an isomorphism class of commutative diagrams satisfying condition (*) of (4.3):

$$\begin{tikzcd}
\pi M \arrow[r, hook] \arrow[d, "\alpha/\pi"'] & M' \arrow[r, hook] \arrow[d, "\alpha'"'] & M \arrow[d, "\alpha"'] \\
\mathcal{L} \arrow[r, "c"] & \mathcal{L}' \arrow[r, "c'"] & \mathcal{L}
\end{tikzcd}$$

One sees that $S = \operatorname{Spec} R$ is the union of two closed subschemes S_0 and S_1, where S_0 (resp. S_1) is the locus of points where c (resp. c') vanishes. Write $\mathcal{U} \subset S_1$ (resp. $\mathcal{U}' \subset S_0$) for the open where c' (resp. c) is invertible.

On $\mathcal{U}$, the points of $F_{[s,s']}$ under our consideration comes from points of F_s defined by $\alpha\colon M \to \mathcal{L}$. The construction (5.3.1) yields the following point of $F(\mathcal{U})$:

$$\begin{tikzcd}
\underline{M} \arrow[r, "\mathrm{id}"] \arrow[d, "\alpha"] & \underline{M} \arrow[r, "\pi"] \arrow[d, "\alpha"] & \underline{M} \arrow[d, "\alpha"] \\
\mathcal{L} \arrow[r, "\mathrm{id}"'] & \mathcal{L} \arrow[r, "\pi"'] & \mathcal{L}
\end{tikzcd}$$

and where r is defined by the inclusion $M \hookrightarrow K^2$.

Or, what amounts to the same thing by using the isomorphism $c'\colon \mathcal{L}' \xrightarrow{\sim} \mathcal{L}$, the point defined by the diagram:

$$\begin{tikzcd}
\underline{M} \arrow[r, "\mathrm{id}"] \arrow[d, "(c')^{-1}\alpha"] & \underline{M} \arrow[r, "\pi"] \arrow[d, "\alpha"] & \underline{M} \arrow[d, "((c')^{-1}\alpha"] \\
\mathcal{L}' \arrow[r, "\sim", "c'"'] & \mathcal{L} \arrow[r, "c"'] & \mathcal{L}'
\end{tikzcd} \qquad (*)$$

Note that $(c')^{-1}\alpha$ is an extension to M of the arrow $\alpha'\colon M' \to \mathcal{L}'$ (it is not defined above $\mathcal{U}$).

Similarly, over $\mathcal{U}'$, the point under consideration corresponds to points of $F_{s'}$ defined by $\alpha'\colon M' \to \mathcal{L}'$. It is associated with the point of $F(\mathcal{U}')$ defined by:

$$\begin{tikzcd}
\underline{M'} \arrow[r, "\pi"] \arrow[d, "\alpha'"] & \underline{M'} \arrow[r, "\mathrm{id}"] \arrow[d, "\alpha'"] & \underline{M'} \arrow[d, "\alpha'"] \\
\mathcal{L}' \arrow[r, "\pi"'] & \mathcal{L}' \arrow[r, "\mathrm{id}"'] & \mathcal{L}'
\end{tikzcd}$$

(with r defined by the inclusion $M' \hookrightarrow K^2$).

Again, using $c\colon \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$, this is the same as the diagram:

$$\begin{tikzcd}
\underline{M'} \arrow[r, "\pi"] \arrow[d, "\alpha'"] & \underline{M'} \arrow[r, "\mathrm{id}"] \arrow[d, "c^{-1}\alpha'"] & \underline{M'} \arrow[d, "\alpha'"] \\
\mathcal{L}' \arrow[r, "c'"'] & \mathcal{L} \arrow[r, "\sim", "c"'] & \mathcal{L}'
\end{tikzcd} \qquad (**)$$

where $c^{-1}\alpha'$ extends the arrow $\alpha/\pi\colon \pi M \to \mathcal{L}$ to M'.

Finally, the complement of the union $\mathcal{U} \cup \mathcal{U}'$ is equal to the intersection $J = S_0 \cap S_1$. One verifies immediately that one can define a point of $F(J)$ via the diagram:

$$\begin{tikzcd}
\underline{M'} \arrow[r, hook] \arrow[d, "\alpha'"] & \underline{M} \arrow[r, "\pi"] \arrow[d, "\alpha"] & \underline{M'} \arrow[d, "\alpha'"] \\
\mathcal{L}' \arrow[r, "c'"'] & \mathcal{L} \arrow[r, "c"'] & \mathcal{L}'
\end{tikzcd} \qquad (***)$$

with r corresponding again with the inclusion $M' \hookrightarrow K^2$.

We thus have three points of F with valued respectively in the schemes $\mathcal{U}$, $\mathcal{U}'$ and J, which are described by the three diagram (*), (**) and (***) above. We must now explain how they glue to yield a single point (η, T, u, r) of $F(S)$.

The most simple part is to define T: we take $T_0 = \mathcal{L}'$ and $T_1 = \mathcal{L}$ on all of S, where the morphism Π is given by c' and c.

Turn next to the definition of the sheaf η: consider a sheaf on $S_0 \amalg S_1$ which is constant with value M on S_1 and of value M' on S_0. Its direct image under the morphism $S_0 \amalg S_1 \to S$, which we write ϕ, is such that the restriction to J is constant of value $M \oplus M'$. We define the sheaf η_0 (resp. η_1) as the subsheaf of sections of ϕ which, above J, take values in the submodule M', via the inclusion into $M \oplus M'$ given by $m' \mapsto (m', m')$ (resp. in the submodule M, given by the map $m \mapsto (m, \pi m)$). The morphisms $\Pi \colon \eta_0 \to \eta_1$ and $\eta_1 \colon \eta_0$ are obtained by restriction from the endomorphisms of ϕ given respectively by:

$$\left\{ \begin{array}{c} M \xrightarrow{\mathrm{id}} M \\ M' \xrightarrow{\pi} M' \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{c} M \xrightarrow{\pi} M \\ M' \xrightarrow{\mathrm{id}} M' \end{array} \right.$$

Alternatively, the definition of η is summarized by the following diagram:

$$\eta_0 \xrightarrow{\Pi} \eta_1 \xrightarrow{\Pi} \eta_0$$

$$\begin{tikzcd}
\text{restriction to } \mathcal{U}\ : & M \arrow[r, hook, "\mathrm{id}"] & M \arrow[r, hook, "\pi"] & M \\
\text{restriction to } J\ : \arrow[u, dash, squiggly, "S_1"] \arrow[d, dash, squiggly, "S_0"'] & M' \arrow[r, hook] \arrow[u] \arrow[d, "\mathrm{id}"] & M \arrow[r, hook, "\pi"] \arrow[u, "\mathrm{id}"] \arrow[d, "\pi"] & M' \arrow[u] \arrow[d, "\mathrm{id}"] \\
\text{restriction to } \mathcal{U}'\ : & M' \arrow[r, hook, "\pi"] & M' \arrow[r, hook, "\mathrm{id}"] & M'
\end{tikzcd}$$

Note in particular that $\eta_0 | S_0$ is constant of value M' and that $\eta_1 | S_1$ is constant of value M.

The isomorphism $r \colon \underline{K}^2 \xrightarrow{\sim} \eta_0 \otimes_{\mathcal{O}} K$ is defined by the (compatible) inclusions of M and M' into K^2.

Finally, the morphism u_0 is given by α' on $S_0 = \mathcal{U}' \cup J$ and by $(c')^{-1}\alpha$ on $\mathcal{U}$. Similarly, u_1 is equal to α on $S_1 = \mathcal{U} \cup J$ and with $c^{-1}\alpha'$ on $\mathcal{U}'$.

It is clear that we have thus constructed a point (η, T, u, r) of $F(S)$, and have thus defined the morphism $F_{[s,s']} \to F$ that we were after. Moreover, our construction shows that these morphisms glue the morphisms $F_s \to F$ and $F_{s'} \to F$ defined in (5.3).

We thus obtain a morphism of functors: $\widehat{\Omega} \to F$.

5.6. We must still show that the morphism above is an *isomorphism* of funcotrs: in other words, for an arbitrary point of $F(R)$ there is a unique corresponding point of $\widehat{\Omega}(R)$.

Thus fix a point $(\eta, T, u, r) \in F(R)$. We associate to each edge $[s, s']$ of the tree, which we suppose is represented by two lattices M and M' such that $\pi M \subsetneq M' \subsetneq M$ with the conventions of (5.2), an (open) subscheme $S_{[s,s']}$ of $S = \operatorname{Spec} R$ defined in the following way: it's the collection of points x which satisfy, with the notations of remark (d) of (5.1), the inclusions:

$$M' \subset N_{0,x} \quad \text{and} \quad M \subset N_{1,x}.$$

If one prefers to distinguish the separate cases, then this amounts to:

$$\begin{cases} \text{case where } \Pi | T_0(x) \text{ is invertible:} & N_{0,x} = N_{1,x} = M \\ \text{case where } \Pi | T_1(x) \text{ is invertible:} & N_{0,x} = \pi N_{1,x} = M' \\ \text{case where neither is invertible:} & N_{0,x} = M',\ N_{1,x} = M. \end{cases}$$

Proof that $S_{[s,s']}$ is a Zariski open subscheme of S: It suffices to verify that the intersections $S_{[s,s']} \cap S_0$ and $S_{[s,s']} \cap S_1$ are Zariski opens, in S_0 and S_1 respectively. The intersection $S_{[s,s']} \cap S_0$ is the locus of points of S_0 which satisfy:

$$N_{0,x} = M' \text{ and } N_{1,x} = M \text{ or } \pi^{-1} M'.$$

As the sheaf η_0 is constant on S_0, N_0 is locally constant. The first condition above thus defines a subset $\mathcal{A} \subset S_0$ which is both open and closed. On the open subset of $\mathcal{A}$ where $\Pi | T_1$ is invertible, one has automatically $N_{1,x} = \pi^{-1} M'$. In other words, the complement of$S_{[s,s']} \cap S_0$ in $\mathcal{A}$ is contained in $S_0 \cap S_1$. This complement is defined in $S_0 \cap S_1$ by the condition $N_{1,x} \neq M$, and is thus closed (as well as open) since the sheaf $N_{1,x}$ is locally constant on S_1.

It follows that $S_0 \cap S_{[s,s']}$ is open in S_0; one shows similarly that $S_1 \cap S_{[s,s']}$ is open in S_1. Thus $S_{[s,s']}$ is open in S.

Using as always remark (d) of (5.1), one sees that *the $S_{[s,s']}$ cover S*. If $[s, s']$ and $[s, s'']$ are two distinct edges which intersect in s, then $S_{[s,s']} \cap S_{[s,s']}$ is the open S_s, which is the locus of points satisfying $N_{0,x} = N_{1,x} = M$; note that S_s can also be defined in $S_{[s,s']}$ or $S_{[s,s'']}$ by the condition: $\Pi | T_0(x)$ is invertible. Moreover, for two edges $[s, s']$ and $[s'', s']$ with intersection s', the intersection $S_{[s,s']} \cap S_{[s'',s']}$ is the open $S_{s'}$ "made up of" the points x which satisfy $N_{0,x} = \pi N_{1,x} = M'$, and which is also defined in $S_{[s,s']}$ or $S_{[s'',s']}$ by the condition: $\Pi | T_1(x)$ is invertible.

The following diagram defines a point of the functor $F_{[s,s']}$ with values in $S_{[s,s']}$:

$$\begin{tikzcd}
\pi M \arrow[r, hook] \arrow[d] & M' \arrow[r, hook] \arrow[d] & M \arrow[d] \\
\pi N_1 \arrow[r, hook] \arrow[d, "l"] & N_0 \arrow[r, hook] \arrow[d, "r"', "l"] & N_1 \arrow[d, "l"] \\
\eta_1 \arrow[r, "\Pi"] \arrow[d, "u_1"'] & \eta_0 \arrow[r, "\Pi"] \arrow[d, "u_0"'] & \eta_1 \arrow[d, "u_1"'] \\
T_1 \arrow[r, "\Pi"] & T_0 \arrow[r, "\Pi"] & T_1
\end{tikzcd}$$

It is clear that the points thus obtained glue: on S_s for example, one obtains the points of the functor F_s defined by the composition

$$M = N - 1 \xrightarrow{\sim} \eta_1 \to T_1.$$

By glueing one thus obtains a point of $\widehat{\Omega}(R)$, and it is easy to show that it maps to the initial point (η, T, u, r). One also verifies easily that this is the only point which maps to (η, T, u, r).

6. Action of the group $\mathrm{PGL}_2(K)$.

6.1. The group $\mathrm{GL}_2(K)$ acts naturally, through its quotient $\mathrm{PGL}_2(K)$, on the tree I: an element $g \in \mathrm{GL}_2(K)$ transforms the vertex $[M]$ (resp. the edge $[[M],[M']]$) to the vertex $[gM]$ (resp. to the edge $[[gM],[gM']]$).

One also has an action of the same group $\mathrm{PGL}_2(K)$ on the set $\Omega = \mathbf{P}^1(\mathbf{C}) - \mathbf{P}^1(K)$. It is clear that the map $\lambda\colon \Omega \to I_{\mathbf{R}}$ is equivariant for this action, and it thus follows that this group acts by automorphisms of the rigid analytic space Ω: the action permutes the different affinoid opens defined in §2. One sees without difficulty that the constructions of §§$1-4$ are equivariant, and thus this action yields an action on the formal scheme $\widehat{\Omega}$. This last action admits a description in terms of Deligne's functors: for $s = [M]$ and $gs = [gM]$, one has a morphism of functors:

$$g\colon F_s \to F_{gs},$$

given by $g \cdot (\mathcal{L}, \alpha) = (\mathcal{L}, \beta)$ where β denotes the composition

$$gM \xrightarrow{g^{-1}} M \xrightarrow{\alpha} \mathcal{L}.$$

Moreover, for $s' = [M']$ such that $[s, s']$ is an edge, the morphism $g : F_{[s,s']} \to F_{[gs,gs']}$ is given by:

$$g \cdot (\mathcal{L}, \mathcal{L}', c, c', \alpha, \alpha') = (\mathcal{L}, \mathcal{L}', c, c', \alpha \circ g^{-1}, \alpha' \circ g^{-1}).$$

It is a little more difficult to describe the action of $\mathrm{PGL}_2(K)$ on $\widehat{\Omega}$ in terms of the functor F of Drinfeld. One such description is furnished by the following proposition:

Proposition 6.2. *The action of an element $g \in \mathrm{GL}_2(K)$ on the functor F is given by the following formula:*

$$g \cdot (\eta, T, u, r) = (\eta[n], T[n], u[n], \Pi^n \circ r \circ g^{-1}),$$

where n denotes the valuation of $\det g$, and $[n]$ the shift by $n \pmod 2$ of the grading of (η, T, u).

Note that the shift ensures that the normalisation condition [C3] of definition (5.1) is satisfied for the image. In fact, we write r_1 for the composition:

$$r_1\colon \underline{K}^2 \xrightarrow{g^{-1}} \underline{K}^2 \xrightarrow{r} \eta_0 \otimes_{\mathcal{O}} K \xrightarrow{\Pi^n} \eta[n]_0 \otimes \mathcal{O}K.$$

The zero locus of the morphism $\Pi\colon T[n]_i \to T[n]_{i+1}$ is $S_{i'}$, with $i' \equiv i+n \pmod 2$, above which one has:

$$\bigwedge^2 \eta[n]_i = \bigwedge \eta_{i'} = \pi^{-1}\left(\bigwedge^2(\Pi^{i'} r\underline{\mathcal{O}}^2)\right) =$$
$$= \pi^{-i-n}\bigwedge^2(\Pi^{i+n} r\underline{\mathcal{O}}^2) =$$
$$= \pi^{-i}\bigwedge^2(\Pi^{i+n} r g^{-1}\underline{\mathcal{O}}^2) =$$
$$= \pi^{-i}\bigwedge^2(\Pi^{i} r_1\underline{\mathcal{O}}^2).$$

It is clear that the formula of the proposition defines a morphism of F into itself. To show that it is the one we want, we verify for example that, for $s = [M]$ and $g \in \mathrm{GL}_2(K)$, the following diagram is commutative:

$$\begin{tikzcd} F_s(B) \arrow[r] \arrow[d, "g"] & F(B) \arrow[d, "g"] \\ F_{gs}(B) \arrow[r] & F(B) \end{tikzcd}$$

(checking this is sufficient, since the union of the images of the F_s is dense and open).

For $(\mathcal{L},\alpha) \in F_s(B)$, one obtains: $g\cdot(\mathcal{L},\alpha) = (\mathcal{L}, \alpha\circ g^{-1})$. One can express the two cases (5.3.1) and (5.3.2) via a similar formula: the image of $(\mathcal{L},\alpha)$ in $F(B)$ is given by the following diagram:

$$\begin{tikzcd} \eta_e = \underline{M} \arrow[r, "\Pi=\pi"] \arrow[d, "\alpha"] & \eta_{e+1} = \underline{M} \arrow[r, "\Pi=\mathrm{id}"] \arrow[d, "\alpha"] & \eta_e = \underline{M} \arrow[d, "\alpha"] \\ T_e = \mathcal{L} \arrow[r, "\Pi=\pi"] & T_{e+1} = \mathcal{L} \arrow[r, "\Pi=\mathrm{id}"] & T_e = \mathcal{L} \end{tikzcd}$$

where $e = \log_q[\bigwedge^2 M \ : \ \bigwedge^2 \mathcal{O}^2]$ denotes the exponent of the (virtual) index of $\mathcal{O}^2$ in M (we do not assume here that M is normalised for $e \in \{0,1\}$). The “rigidification” r is the composite of the morphism $\underline{K}^2 \xrightarrow{\sim} \eta_e \otimes K$ associated with the inclusion $M \subset K^2$, with the morphism $\Pi^{-e}\colon \eta_e \otimes K \xrightarrow{\sim} \eta_0 \otimes K$.

The image of $(\mathcal{L}, \alpha\circ g^{-1})$ corresponds with the diagram:

$$\begin{tikzcd} \eta'_{e-n} = g\underline{M} \arrow[r, "\Pi=\pi"] \arrow[d, "\alpha\circ g^{-1}"] & \eta'_{e-n+1} = g\underline{M} \arrow[r, "\Pi=\mathrm{id}"] \arrow[d, "\alpha\circ g^{-1}"] & \eta'_{e-n} = g\underline{M} \arrow[d, "\alpha\circ g^{-1}"] \\ T'_{e-n} = \mathcal{L} \arrow[r, "\Pi=\pi"] & T'_{e-n+1} = \mathcal{L} \arrow[r, "\Pi=\mathrm{id}"] & T'_{e-n} = \mathcal{L} \end{tikzcd}$$

and with $r' = \underline{K}^2 \xrightarrow{\sim} \eta'_0 \otimes K$ obtained by composing $\underline{K}^2 \cong \eta'_{e-n}$ (associated with the inclusion $gM \hookrightarrow K^2$) with Π^{-e+n}.

One sees, via the isomorphism $g\colon \underline{M} \xrightarrow{\sim} g\underline{M}$, the point (η', T', u', r') obtained from (η, T, u, r) is the one predicted by the proposition (6.2).

II. DRINFELD’S THEOREM

We assume henceforth that the local field K is of *characteristic zero*.

In this chapter we consider certain p-divisible formal groups with an action of the ring of integers $\mathcal{O}_D$ of a quaternion algebra D over K: the *special formal $\mathcal{O}_D$-modules of height* 4 (defined in §2.2). Over an algebraic closure $\overline{k}$ of k, these form a single isogeny class; choose one and denote it Φ. Drinfeld's theorem (given precisely in §2.8) states that the formal scheme $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{\mathrm{nr}}$ parameterizes those special formal $\mathcal{O}_D$-modules X of height 4 with a quasi-isogeny $\rho\colon \Phi \to X$ of height 0.

In other words, the functor $\overline{G}$ on the category $\overline{\textbf{Nilp}}$ of $\mathcal{O}^{nr}$-algebras B where π is nilpotent in B, which to B associates the collection $\overline{G}(B)$ of isomorphism classes of pairs (X, ρ), where the isomorphisms are taken over B, is isomorphic to the functor $\overline{H}$ represented by $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{\mathrm{nr}}$. The functor $\overline{H}$ is the restriction to $\overline{\textbf{Nilp}}$ of the functor F classifying quadruples (η, T, u, r) defined in §1.5. In order to construct an isomorphism $\xi\colon \overline{G} \to \overline{H}$, it therefore suffices to associate to each pair (X, ρ) a quadruplet (η, T, u, r). The difficulty is to find the constructible sheaf η and the map $u\colon \eta \to T$, where $T = \mathrm{Lie}(X)$.

Rather than work with the p-divisible group X itself, we work instead with its Cartier (-Dieudonne) module M. This amounts to the same thing, thanks to the theory of Cartier (which is recalled briefly in §2.1), whose main advantage here is that it is valid for all base algebras B. The action of $\mathcal{O}_D$ on X yields an action on M, as described in §2.2, and with $T = \mathrm{Lie}(X)$, a $\mathbf{Z}/2\mathbf{Z}$-grading and an operator Π of degree 1 such that $\Pi^2 = \pi$. The usual operators F and V are also of degree 1, and the identification of T with M/VM is compatible with the grading and the action of Π.

We say that an index $i \in \mathbf{Z}/2\mathbf{Z}$ is *critical* if Π is zero on T_i, in other words if $\Pi M_i \subseteq VM_i$. Over $\overline{k}$ there always exists at least one critical index i; for this index, M_i and the operator $V^{-1}\Pi$ are a "unit crystal" and the invariants $M_i^{V^{-1}\Pi}$ are a free $\mathcal{O}$-module of rank 1. Putting $\eta_i = M_i^{V^{-1}\pi}$, we establish a natural bijection between $\overline{G}(\overline{k})$ and $\overline{H}(\overline{k})$ in §2.5.

Drinfeld's genius was to extend this bijection to all algebras B in $\overline{\textbf{Nilp}}$. We give his ingenius construction of a triple (η, T, u) over any base B in §2.3. We explain the link between this construction of η and the $M_i^{V^{-1}\Pi}$ in §2.4. Once we have defined the proper filtrations, we show in §2.6 that η is a constructible sheaf in the π-adic sense. The introduction of a rigidification, the quasi-isogeny $\rho\colon \Phi \to X$, allows us to recover the isomorphism $r\colon \underline{K}^2 \to \eta \otimes_{\mathcal{O}} K$ in §2.7 and to show that η is strictly constructible. The morphism of functors $\xi\colon \overline{G} \to \overline{H}$ is then well-defined.

After this it remains to compare the deformation theories (§2.10) and to show that $\overline{\xi}$ induces a bijection on tangent spaces (§2.11) of the geometric points of $\overline{G}$ and $\overline{H}$. A final argument of relative representability (§2.12) allows us to conclude that $\overline{\xi}$ is an isomorphism! Also, we describe in §2.9 the action of $\mathrm{GL}_2(K)$ and $D^\times$ on everything in sight.

To close the chapter we construct in §2.13, with the aid of torsion points of the universal special formal $\mathcal{O}_D$-modules of height 4 over $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{\mathrm{nr}}$, a projective system of etale coverings Σ_n of the rigid analytic space $\Omega \otimes_K \widehat{K}^{nr}$ whose Galois group is the profinite completion $\widehat{\mathcal{O}}_D^\times$.

1. **Cartier theory for formal $\mathcal{O}$-modules.** We briefly recall Cartier theory for formal $\mathcal{O}$-modules. For the most familiar case of formal groups with $\mathcal{O} = \mathbf{Z}_p$, the reader can consult M. Lazard [La] or Th. Zink [Zi 3]; this is the case required for the theorem of Cerednik. The general case is treated by M. Hazewinkel [Ha].

1.1. There exists a unique functor $W_{\mathcal{O}}$ from the category of commutative $\mathcal{O}$-algebras to itself such that, for every $\mathcal{O}$-algebra B, one has $W_{\mathcal{O}}(B) = B^{\mathbf{N}}$ and, for all $n \geq 0$, the map $w_n \colon W_{\mathcal{O}}(B) \to B$ defined by

$$w_n(a_0, a_1, \dots) = a_0^{q^n} + \pi a_1^{q^{n-1}} + \dots + \pi^n a_n$$

is a homomorphism of $\mathcal{O}$-algebras.

There is an $\mathcal{O}$-linear endomorphism τ of this functor defined by

$$\tau(a_0, a_1, \dots) = (0, a_0, a_1, \dots)$$

and an endomorphism of $\mathcal{O}$-algebras σ such that

$$w_n \sigma = w_{n+1}, \quad \text{for all } n \geq 0.$$ [1]

These endomorphisms satisfy the relations

$$\begin{aligned} \sigma\tau &= \pi \\ {}^{\tau}x.y &= {}^{\tau}(x.{}^{\sigma}y), \quad x, y \in W_{\mathcal{O}}(B). \end{aligned}$$

For each $a \in B$, we write $[a] = (a, 0, 0, \dots)$. We have:

$$\begin{aligned} [ab] &= [a] \cdot [b] \\ {}^{\sigma}[a] &= [a^q]. \end{aligned}$$

If B is a k-algebra, we have:

$$\begin{aligned} {}^{\sigma}(a_0, a_1, a_2, \dots) &= (a_0^q, a_1^q, a_2^q, \dots) \\ \tau\sigma = \sigma\tau &= \pi. \end{aligned}$$

When $\mathcal{O} = \mathbf{Z}_p$, the functor $W_{\mathcal{O}}$ is the functor W of Witt vectors.

1.2. For each $\mathcal{O}$-algebra B, consider the noncommutative $\mathcal{O}$- algebra $W_{\mathcal{O}}(B)[F, V]$ where F and V satisfy the relations:

$$\begin{aligned} Fx &= {}^{\sigma}xF \\ xV &= V{}^{\sigma}x \\ VxF &= {}^{\tau}x \\ FV &= \pi. \end{aligned}$$

The *Cartier ring* $E_{\mathcal{O}}(B)$ is the completion of the algebra above for the topology defined by the right ideals generated by the V^n, called the *V-adic topology*.

Every element of $E_{\mathcal{O}}(B)$ can be written in a unique way as

$$\sum_{m,n \geq 0} V^m [a_{m,n}] F^n, \quad a_{m,n} \in B,$$

subject to the condition: for each m, one has $a_{m,n} = 0$ for n large enough. The map:

$$(a_0, a_1, \dots) \mapsto \sum_{n \geq 0} V^n [a_n] F^n$$

is an $\mathcal{O}$-algebra homomorphism which identifies $W_{\mathcal{O}}(B)$ with its image in $E_{\mathcal{O}}(B)$. It follows that every element of $E_{\mathcal{O}}(B)$ can be written in a unique way as

$$\sum_{m>0} V^m x_m + x_0 + \sum_{n>0} y_n F^n, \quad x_m, y_n \in W_{\mathcal{O}}(B),$$

[1] I think this should be a congruence mod π^{n+1}, but this is what is written in Boutot-Carayol.

subject to the condition: $y_n \to 0$ in the τ-adic topology [2] as $n \to \infty$.

1.3. For $a \in \mathcal{O}$, we take care to distinguish the element $a = a.1$ giving the $\mathcal{O}$-algebra structure to $W_{\mathcal{O}}(B)$ and $E_{\mathcal{O}}(B)$ from the multiplicative representative $[a]$ of the element $a.1$ of B. For every $n \geq 0$, we have $w_n(a) = a$ and $w_n([a]) = a^{q^n}$. In particular

$$w_n(\pi - [\pi]) = \pi(1 - \pi^{q^n - 1}).$$

There exists a unit ε of $W_{\mathcal{O}}(B)$, contained in $W_{\mathcal{O}}(\mathcal{O})$, whose phantom components are given by

$$w_n(\varepsilon) = 1 - \pi^{q^{n+1}-1},$$

such that

$$\pi - [\pi] = {}^{\tau}\varepsilon = V\varepsilon F.$$

1.4. A *formal $\mathcal{O}$-module* over an $\mathcal{O}$-algebra B is a smooth formal group X over B with an action of $\mathcal{O}$, which is to say a ring homomorphism $i\colon \mathcal{O} \to \mathrm{End}(X)$, such that the action induced in the tangent space $\mathrm{Lie}(X)$ coincides with that provided by the B-module structure of $\mathrm{Lie}(X)$.

A *Cartier $\mathcal{O}$-module* over B is by definition a left $E_{\mathcal{O}}(B)$-module such that

(i) M/VM is a free B-module of finite rank,
(ii) V is injective on M,
(iii) M is separated and complete for the V-adic filtration.

Such a module is often described as *reduced* in the literature. The fundamental result of the theory of Dieudonne-Cartier is the following ([Zi 3] 4.23; [Ha] 26.3):

Theorem. *The category of formal $\mathcal{O}$-modules over B is equivalent with the category of Cartier $\mathcal{O}$-modules over B. Moreover, if M is the Cartier $\mathcal{O}$-module associated to the formal $\mathcal{O}$-module X, we have $M/VM = \mathrm{Lie}(X)$.*

If B' is a B-algebra, the Cartier module of the formal $\mathcal{O}$-module $X_{B'}$ obtained by base-change is

$$M' = E_{\mathcal{O}}(B')\widehat{\otimes}_{E_{\mathcal{O}}(B)}M,$$

the completion of $E_{\mathcal{O}}(B') \otimes_{E_{\mathcal{O}}(B)} M$ for the V-adic topology.

1.5. Let M be a Cartier $\mathcal{O}$-module over B. We say that elements $\gamma_1, \ldots, \gamma_d$ of M form a *V-basis* of M if their images $\overline{\gamma}_1, \ldots, \overline{\gamma}_d$ mod V form a basis for the free B-module M/VM. Every element of M can be described in a unique way as

$$\sum_{m \geq 0} \sum_{i=1}^{d} V^m [c_{m,i}] \gamma_i$$

with $c_{m,i} \in B$.

In particular, the choice of the γ_i's determines a family $c_{m,i,j}$ ($m \in \mathbf{N}$; $i, j \in \{1, \ldots, d\}$) of elements of B such that

$$F(\gamma_j) = \sum_{m \geq 0} \sum_{i=1}^{d} V^m [c_{m,i,j}] \gamma_i, \quad j = 1, \ldots, d.$$

[2]Same as the V-adic topology

Conversely, given a family $c_{m,i,j}$ of elements $W_{\mathcal{O}}(B)$, there exists a Cartier $\mathcal{O}$-module M, unique up to isomorphism, and a V-basis $\gamma_1, \ldots, \gamma_d \in M$ satisfying the relations

$$(*) \quad F(\gamma_i) = \sum_{m \geq 0} \sum_{i=1}^{d} V^m c_{m,i,j} \gamma_i \quad j = 1, \ldots, d.$$

This module has a presentation

$$0 \to E_{\mathcal{O}}(B)^d \xrightarrow{\psi} E_{\mathcal{O}}(B)^d \xrightarrow{\phi} M \to 0$$

where, denoting by (ε_i) the canonical basis for $E_{\mathcal{O}}(B)^d$, the maps ϕ and ψ are the $E_{\mathcal{O}}(B)$-linear maps defined by

$$\phi(\varepsilon_i) = \gamma_i$$

$$\psi(\varepsilon_j) = F(\varepsilon_j) - \sum_{m \geq 0} \sum_{i=1}^{d} V^m c_{m,i,j} \varepsilon_i.$$

1.6. It is more convenient for what follows to use a modified version of the relations (*). The choice of a V-basis γ_i of M determines a family $d_{m,i,j}$ ($m \in \mathbf{N}^\times$; $i, j \in \{1, \ldots, d\}$) of elements of B such that

$$\pi\gamma_j = [\pi]\gamma_j + \sum_{m \geq 1} \sum_{i=1}^{d} V^m [d_{m,i,j}] \gamma_i, \quad j = 1, \ldots, d.$$

Conversely, given a family $d_{m,i,j}$ of elements of $W_{\mathcal{O}}(B)$, there exists a Cartier $\mathcal{O}$-module M, unique up to isomorphism, and a V-basis $\gamma_1, \ldots, \gamma_d$ of M satisfying the relations

$$(**) \quad \pi\gamma_j = [\pi]\gamma_j + \sum_{m \geq 1} \sum_{i=1}^{d} V^m d_{m,i,j} \gamma_i, \quad j = 1, \ldots, d.$$

Since $\pi - [\pi] = V\varepsilon F$, where ε is a unit in $W_{\mathcal{O}}(B)$, and since V is injective on M, we have

$$\begin{aligned} F\gamma_j &= \varepsilon^{-1} V^{-1} \left(\sum_{m \geq 1} \sum_{i=1}^{d} V^m d_{m,i,j} \gamma_i \right) \\ &= \sum_{m \geq 1} \sum_{i=1}^{d} V^{m-1} \, {}^{\sigma^{m-1}} \varepsilon^{-1} d_{m,i,j} \gamma_i \end{aligned}$$

which is the relation (*) with $c_{m,i,j} = {}^{\sigma^m} \varepsilon^{-1} d_{m+1,i,j}$.

2. Cartier theory for formal $\mathcal{O}_D$-modules.

2.1. Let D be a quaternion algebra over K and $\mathcal{O}_D$ its ring of integers. Let K' be a quadratic unramified extension of K contained in D, let $\mathcal{O}'$ be the ring of integers in K' and let σ denote the nontrivial Galois automorphism of K'/K. Let Π denote an element of $\mathcal{O}_D$ such that $\Pi^2 = \pi$ and $\Pi a = {}^{\sigma}a\Pi$ for all $a \in K'$.

A *formal $\mathcal{O}_D$-module* over an $\mathcal{O}$-algebra B is a formal $\mathcal{O}$-module X over B with an action $i \colon \mathcal{O}_D \to \mathrm{End}(X)$ extending the action of $\mathcal{O}$. A formal $\mathcal{O}_D$-module is said to be *special* if the action of $\mathcal{O}'$ makes $\mathrm{Lie}(X)$ a free $B \otimes_{\mathcal{O}} \mathcal{O}'$-module of rank one.

If B is an $\mathcal{O}'$-algebra and X is a formal $\mathcal{O}_D$-module over B, the B-module $\mathrm{Lie}(X)$ is $\mathbf{Z}/2\mathbf{Z}$-graded by the action of $\mathcal{O}'$:

$$(\mathrm{Lie}(X))_0 = \{x \in \mathrm{Lie}(X) \mid i(a)m = am \text{ for all } a \in \mathcal{O}'\}$$
$$(\mathrm{Lie}(X))_1 = \{x \in \mathrm{Lie}(X) \mid i(a)m = {}^{\sigma}am \text{ for all } a \in \mathcal{O}'\}.$$

Then X is special if each graded component of $\mathrm{Lie}(X)$ is a free B-module of rank one.

2.2. Give the Cartier ring $E_{\mathcal{O}}(B)$ the $\mathbf{Z}/2\mathbf{Z}$-grading defined by

$$\deg V = \deg F = 1,$$
$$\deg[b] = 0 \quad \text{for all } b \in B.$$

Note that the subring $W_{\mathcal{O}}(B)$ of $E_{\mathcal{O}}(B)$ is contained in the homogeneous component of degree 0; indeed, any element in $W_{\mathcal{O}}(B)$ can be written as $\sum_{n\geq 0} V^n[a_n]F^n$ for $a_n \in B$.

A $\mathbf{Z}/2\mathbf{Z}$-graded Cartier $\mathcal{O}$-module $M = M_0 \oplus M_1$ with an $E_{\mathcal{O}}(B)$-linear endomorphism Π of degree 1, such that $\Pi^2 = \pi$, is called a *graded Cartier $\mathcal{O}[\Pi]$-module*. As above, both M_0 and M_1 are automatically $W_{\mathcal{O}}(B)$-submodules of M.

We say that M is *special* if M_0/VM_1 and M_1/VM_0 are free B-modules of rank one.

Theorem 2.3. *If B is an $\mathcal{O}'$-algebra, the category of formal $\mathcal{O}_D$-modules is equivalent with the category of graded Cartier $\mathcal{O}[\Pi]$-modules over B. Moreover, a formal $\mathcal{O}_D$-module is special if and only if the corresponding Cartier $\mathcal{O}[\Pi]$-module is special.*

Proof. (cf. T. Zink [Zi 2], Satz 2.2). Note first that, if B is an $\mathcal{O}'$-algebra, the $\mathcal{O}$-algebra structure of $W_{\mathcal{O}}(B)$ and $E_{\mathcal{O}}(B)$ extend to a structure of $\mathcal{O}'$-algebra. If fact, $\mathcal{O}'$ is generated over $\mathcal{O}$ by the (q^2-1) roots of unity; for $\zeta \in \mathcal{O}'$ such that $\zeta^{q^2-1} = 1$ we have the multiplicative representative $[\zeta]$ in $W_{\mathcal{O}}(B)$ for the image of ζ in B and the mapping $\zeta \mapsto [\zeta]$ is an isomorphism between the groups of (q^2-1)th roots of unity in B and $W_{\mathcal{O}}(B)$; there thus exists a unique homomorphism of $\mathcal{O}$-algebras $j\colon \mathcal{O}' \to W_{\mathcal{O}}(B)$ such that $j(\zeta) = [\zeta]$.

We write σ for both the conjugation homomorphism of $\mathcal{O}'$ over $\mathcal{O}$ and for the Frobenius endomorphism of $W_{\mathcal{O}}(B)$. The homomorphism j is compatible with σ, since ${}^{\sigma}\zeta = \zeta^q$ in $\mathcal{O}'$ and ${}^{\sigma}[\zeta] = [\zeta^q]$ in $W_{\mathcal{O}}(B)$, so that $j({}^{\sigma}a) = {}^{\sigma}j(a)$ for all $a \in \mathcal{O}'$.

Thus every $E_{\mathcal{O}}(B)$-module, in particular every Cartier $\mathcal{O}$-module M over B, carries two natural $\mathcal{O}'$-module structures via j and $j\sigma$. We write simply am and ${}^{\sigma}am$, $a \in \mathcal{O}'$, $m \in M$, for these structures.

By (1.4), the category of formal $\mathcal{O}_D$-modules over B is equivalent with the category of Cartier $\mathcal{O}$-modules M over B with an action $i\colon \mathcal{O}_D \to \mathrm{End}(M)$ extending the action of $\mathcal{O}$. In particular such Cartier modules also have an action of $\mathcal{O}'$ via i, and one has a decomposition of the $\mathcal{O}$-module M as $M = M_0 \oplus M_1$, where

$$M_0 = \{m \in M \mid i(a)m = am,\ a \in \mathcal{O}'\},$$
$$M_1 = \{m \in M \mid i(a)m = {}^{\sigma}am,\ a \in \mathcal{O}'\}.$$

The operators V, F, $[b]$ are $\mathcal{O}$-module homomorphisms of degree 1, 1 and 0, respectively, since

$$\begin{aligned} aV &= V{}^{\sigma}a \\ Fa &= {}^{\sigma}aF \qquad a \in \mathcal{O}', b \in B. \\ a[b] &= [b]a \end{aligned}$$

Conversely, giving a $\mathbf{Z}/2\mathbf{Z}$-grading to an $\mathcal{O}$-module M, such that $\deg V = \deg F = 1$ and $\deg[b] = 0$ for all $b \in B$ is equivalent to giving an action of $\mathcal{O}'$ to M compatible with the $\mathcal{O}$-action.

Giving an action of $\mathcal{O}_D$ amounts to giving an action of Π; we write Π for the endomorphism of the $E_{\mathcal{O}}(B)$-module M defined by $i(\Pi)$. Since $\Pi^2 = \pi$ in $\mathcal{O}_D$, the endomorphism Π of M satisfies $\Pi^2 = \pi$, where π is defined by the $\mathcal{O}$-algebra structure of $E_{\mathcal{O}}(B)$. It's an operator of degree 1 since $\Pi(a) = {}^{\sigma}a\Pi$ for all $a \in \mathcal{O}'$.

Finally, the grading on the Lie algebra of a formal $\mathcal{O}_D$-module X and on its Cartier module M are both defined by the action of $\mathcal{O}'$, and these are compatible:

$$(\mathrm{Lie}(X))_0 = M_0/VM_1,$$
$$(\mathrm{Lie}(X))_1 = M_1/VM_0.$$

This shows that M is special if and only if X is special. □

2.4. Let M be a graded special Cartier $\mathcal{O}[\Pi]$-module over B. Let (γ_0, γ_1) be a homogeneous V-basis ($\gamma_0 \in M_0$, $\gamma_1 \in M_1$) for M. Every element of M can be written in a unique manner as

$$x = \sum_{m\geq 0}(V^m[c_{m,0}]\gamma_0 + V^m[c_{m,1}]\gamma_1), \quad c_{m,i} \in B.$$

Since V is of degree 1 and $[c_{m,i}]$ of degree 0, the decomposition of x into homogeneous components $x_0 \in M_0$ and $x_1 \in M_1$ is given by

$$x_0 = [c_{0,0}]\gamma_0 + \sum_{m>0} V^m[c_{m,\overline{m}}]\gamma_{\overline{m}}$$
$$x_1 = [c_{0,1}]\gamma_1 + \sum_{m>0} V^m[c_{m,\overline{m+1}}]\gamma_{\overline{m+1}}$$

where $\overline{m}$ is the class of m in $\mathbf{Z}/2\mathbf{Z} = \{0, 1\}$.

In particular, the choice of a homogeneous V-basis (γ_0, γ_1) determines elements $a_{m,i}$ ($m \in \mathbf{N}$, $i = 0, 1$) of B such that

$$\Pi\gamma_0 = [a_{0,0}]\gamma_1 + \sum_{m>0} V^m[a_{m,0}]\gamma_{\overline{m+1}},$$
$$\Pi\gamma_1 = [a_{0,1}]\gamma_0 + \sum_{m>0} V^m[a_{m,1}]\gamma_{\overline{m}}.$$

We deduce that

$$\Pi^2 \equiv [a_{0,0}.a_{0,1}] \pmod{VM}.$$

Since $\Pi^2 = \pi$ and $\pi \equiv [\pi] \pmod{VM}$ we have

$$a_{0,0}.a_{0,1} = \pi.$$

Conversely,

Proposition 2.5. *Let B be an $\mathcal{O}'$-algebra. Given elements $a_{m,i}$ ($m \in \mathbf{N}$, $i = 0, 1$) of B such that $a_{0,0}.a_{0,1} = \pi$, there exists a graded special Cartier $\mathcal{O}[\Pi]$-module M over B, unique up to isomorphism, and a homogeneous V- basis (γ_0, γ_1) of M satisfying the* *relations*

$$\Pi\gamma_0 = [a_{0,0}]\gamma_1 + \sum_{m>0} V^m[a_{m,0}]\gamma_{\overline{m+1}},$$
$$\Pi\gamma_1 = [a_{0,1}]\gamma_0 + \sum_{m>0} V^m[a_{m,1}]\gamma_{\overline{m}}.$$

Proof. Knowledge of the formulas giving the action of Π allows one to determine those giving the action of $\Pi^2 = \pi$ (since Π is an endomorphism of $E_{\mathcal{O}}(B)$-modules, which commutes with V and with $[a_{m,i}]$). Since $a_{0,0}.a_{0,1} = \pi$, one finds elements $d_{m,i}$ ($m \in \mathbf{N}^\times$, $i = 0, 1$) of B such that

$$\pi\gamma_i = [\pi]\gamma_i + \sum_{m>0} V^m[d_{m,i}]\gamma_{\overline{m+i}}, \quad i = 0, 1.$$

By (1.6), there exists a Cartier $\mathcal{O}$-module M over B, unique up to isomorphism, and a V-basis (γ_0, γ_1) of M such that these relations are satisfied. Put

$$M_i = \left\{ \sum_{m\geq 0} V^m x_m \gamma_{\overline{m+i}} \mid x_m \in W_{\mathcal{O}}(B) \right\}, \quad i = 0, 1.$$

Thus M_0 and M_1 are $W_{\mathcal{O}}(B)$-submodules of M such that $M = M_0 \oplus M_1$. By construction the operators Π, V and $[b]$ ($b \in B$) are of degree 1, 1 and 0, respectively. Hence $\pi - [\pi] \colon V \to VM$ is of degree 0 and, since V is injective on M and $F = \varepsilon^{-1}V^{-1}(\pi - [\pi])$ where ε is a unit in $W_{\mathcal{O}}(B)$ (1.3), F is of degree 1. Thus M is a graded Cartier $\mathcal{O}[\Pi]$-module. It is special since M_0/VM_1 is free with basis γ_0 and M_1/VM_0 is free with basis γ_1. □

2.6. Let B' be a B-algebra and $M' = E_{\mathcal{O}}(B')\widehat{\otimes}_{E_{\mathcal{O}}(B)}M$ the Cartier module over B' obtained from M by change of base. Then M' is a graded Cartier $\mathcal{O}[\Pi]$-module over B'; the image (γ'_0, γ'_1) in M' of (γ_0, γ_1) is a homogeneous V-basis of M' satisfying the relations

$$\Pi\gamma'_i = [a'_{0,i}]\gamma'_{i+1} + \sum_{m>0} V^m[a'_{m,i}]\gamma'_{\overline{m+1+i}}, \quad i = 0, 1,$$

where the $a'_{m,i}$ are the images of the elements $a_{m,i}$ of B inside B'.

3. **Construction of** (η_M, T_M, u_M)**.**

3.1. For each $\mathcal{O}'$-algebra B, we consider the noncommutative $\mathcal{O}'$-algebra $W_{\mathcal{O}}(B)[V, \Pi]$ where Π and V satisfy the relations:

$$\begin{aligned} \Pi V &= V\Pi \\ \Pi x &= x\Pi \\ xV &= V^\sigma x \quad x \in W_{\mathcal{O}}(B) \\ \Pi^2 &= \pi. \end{aligned}$$

We write $E'_{\mathcal{O}}(B)$ for the completion of this algebra for the topology defined by the right ideal generated by the V^ns. An element of $E'_{\mathcal{O}}(B)$ can be writen in a unique way as

$$\sum_{m\geq 0} V^m x_m + \sum_{m\geq 0} V^m x'_m \Pi, \quad x_m, x'_m \in W_{\mathcal{O}}(B).$$

We endow $E'_{\mathcal{O}}(B)$ with the $\mathbf{Z}/2\mathbf{Z}$-grading defined by

$$\deg x = 0, \quad x \in W_{\mathcal{O}}(B)$$
$$\deg V = \deg \Pi = 1.$$

Every graded Cartier $\mathcal{O}[\Pi]$-module over B is in particular a graded $E'_{\mathcal{O}}(B)$-module (by forgetting the action of F).

3.2. If M is a $W_{\mathcal{O}}(B)$-module, we write W^σ for the $W_{\mathcal{O}}(B)$-module obtained by restriction of scalars via $\sigma\colon W_{\mathcal{O}}(B) \to W_{\mathcal{O}}(B)$. If M is an $E'_{\mathcal{O}}(B)$-module, then M^σ is also an $E'_{\mathcal{O}}(B)$-module and V defines an $E'_{\mathcal{O}}(B)$-linear homomorphism of M^σ into M. If $M = M_0 \oplus M_1$ is graded, then $M^\sigma = M_0^\sigma \oplus M_1^\sigma$ is as well and $V\colon M^\sigma \to M$ is of degree 1.

For every $E'_{\mathcal{O}}(B)$-module M, we define an $E'_{\mathcal{O}}(B)$-module $N(M)$ via the exact sequence

$$M^\sigma \xrightarrow{\alpha_M} M \oplus M^\sigma \xrightarrow{\beta_M} N(M) \to 0$$
$$\alpha_M(m) = (Vm, -\Pi m).$$

If M is graded, then $N(M)$ is also graded with

$$N(M)_i = \beta_M(M_i \oplus M_i^\sigma), \quad i = 0, 1.$$

Note that, since V is injective on M, the map α_M is injective.

This defines a covariant functor N from the category of $E'_{\mathcal{O}}(B)$-modules into itself which is right exact. Moreover, if the sequence of $E'_{\mathcal{O}}(B)$-modules

$$0 \to M' \to M \to M'' \to 0$$

is exact and if V is injective on M'', the sequence

$$0 \to N(M') \to N(M) \to N(M'') \to 0$$

is exact.

3.3. We write $\beta_M(m, m') = ((m, m'))$.

The canonical $E'_{\mathcal{O}}(B)$-linear map $M^\sigma \to N(M)$ defined by $m \mapsto ((0, m))$ is injective if V is injective on M.

The map

$$M \oplus M^\sigma \to M/VM$$
$$(m, m') \mapsto \overline{m} \pmod{VM}$$

defines a canonical surjection $N(M) \to M/VM$.

Finally the map

$$M \oplus M^\sigma \to M$$
$$(m, m') \mapsto \Pi m + Vm'$$

defines a canonical $E'_{\mathcal{O}}(B)$-linear map $\lambda_M\colon N(M) \to M$ (of degree 1).

Lemma 3.4. *If B is a K-algebra, then λ_M is bijective.*

Proof. If B is a K-algebra, the family of maps w_n defines an isomorphism of $\mathcal{O}-algebras$, $W_{\mathcal{O}}(B) \cong B^{\mathbf{N}}$[3]; in particular, π is invertible in $W_{\mathcal{O}}(B)$. Since $\Pi^2 = \pi$, it follows that Π is invertible in $E'_{\mathcal{O}}(B)$.

[3] Since B is characteristic 0 in this case.

Thus the map $M \to N(M)$ defined by $m \mapsto ((m,0))$ is bijective and hence so is λ_M, since $\lambda_M((m,0)) = \Pi m$. $\square$

Lemma 3.5. *Let B be an $\mathcal{O}'$-algebra without torsion and M a Cartier $\mathcal{O}'$-module over B. Let $B_K = B \otimes_{\mathcal{O}} K$ and $M_K = M \widehat{\otimes}_{E_{\mathcal{O}}(B)} E_{\mathcal{O}}(B_K)$. Then the canonical map $M \to M_K$ is injective and V is injective on M_K/M.*

Proof. Let γ_0, γ_1 denote a V-basis for M. If

$$x = \sum_{i,m} V^m[a_{m,i}]\gamma_i, \quad a_{m,i} \in B,$$

is an element of M, its image x_K in M_K can be written

$$x_K = \sum V^m[a_{m,i;K}]\gamma_{i,K}$$

where $\gamma_{i,K}$ is the V-basis for M_K obtained from γ_i and $a_{m,i;K}$ is the image of $a_{m,i}$ in B_K. If $x_K = 0$ we have $a_{m,i;K} = 0$ for all m, i; thus, since B is without torsion, $a_{m,i} = 0$ and $x = 0$.

Moreover if

$$x' = \sum_{i,m} V^m[a'_{m,i}]\gamma_{i,K}, \quad a'_{m,i} \in B_K$$

is an element of B_K, we have

$$Vx' = \sum_{i,m} V^{m+1}[a'_{m,i}]\gamma_{i,K}.$$

If $Vx' \in M$, we have $a'_{m,i} \in B$ for all m, i; thus $x' \in M$. $\square$

Lemma 3.6. *Let B be an $\mathcal{O}'$-algebra without torsion and M a Cartier $\mathcal{O}[\Pi]$-module over B. Then the map $\lambda_M \colon N(M) \to M$ is injective.*

Proof. By (3.5) we have an exact sequence of $E'_{\mathcal{O}}(B)$-modules

$$0 \to M \to M_K \to M_K/M \to 0$$

and V is injective on M_K/M. Therefore by (3.2), the canonical map $N(M) \to N(M_K)$ is injective. The commutative diagram

$$\begin{tikzcd} N(M) \arrow[r, "\lambda_M"] \arrow[d] & M \arrow[d] \\ N(M_K) \arrow[r, "\lambda_{M_K}"] & M_K \end{tikzcd}$$

and the injectivity of λ_{M_K} (3.4) shows that λ_M is injective. $\square$

Lemma 3.7. *Let $B \to B'$ be a surjection of $\mathcal{O}'$-algebras with kernel I. Let M be a Cartier $\mathcal{O}$-module over B and $M' = M \widehat{\otimes}_{E_{\mathcal{O}}(B)} E_{\mathcal{O}}(B')$. Let $\{\gamma_i\}$ be a V-basis for M and*

$$M_I = \left\{ \sum_{m,i} V^m[a_{m,i}]\gamma_i \mid a_{m,i} \in I \right\}.$$

Then we have an exact sequence of $E_{\mathcal{O}}(B)$-modules

$$0 \to M_I \to M \to M' \to 0$$

and an exact sequence of $E'_{\mathcal{O}}(B)$-modules

$$0 \to N(M_I) \to N(M) \to N(M') \to 0.$$

Proof. An element $\sum_{m,i} V^m[a_{m,i}]\gamma_i$ of M maps to zero in M' if and only if the $a_{m,i}$ map to zero in B', so that the first sequence is exact. Exactness of the second follows from (3.2). □

Proposition 3.8. *For all $\mathcal{O}'$-algebras B and all special graded Cartier $\mathcal{O}[\Pi]$-modules M over B, there is exactly one way to define a map $L_M\colon M \to N(M)$ such that*

(i) *if $B \to B'$ is a homomorphism of $\mathcal{O}'$-algebras and if $M' = M\widehat{\otimes}_{E_{\mathcal{O}}(B')}E_{\mathcal{O}}(B')$, the diagram*

$$\begin{array}{ccc} M & \xrightarrow{L_M} & N(M) \\ \downarrow & & \downarrow \\ M' & \xrightarrow{L_{M'}} & N(M') \end{array}$$

is commutative.

(ii) *we have $F = \lambda_M \circ L_M$.*

Proof. a) Suppose first that B is an $\mathcal{O}'$-algebra without torsion. Since λ_M is injective (3.6), it suffices to show that $F(M) \subset \lambda_M(N(M))$, where $\lambda_M(N(M)) = \Pi M + VM$ is an $E'_{\mathcal{O}}(B)$-submodule of M.

Let (γ_0, γ_1) be a homogeneous V-basis for M. Every element of M can be written $[a_0]\gamma_0 + [a_1]\gamma_1 \pmod{VM}$ with $a_i \in B$. As $FVM = \pi M = \Pi^2 M$ and $F[a_i]\gamma_i = [a_i^q]F\gamma_i$, it suffices to show that $F\gamma_i \in \Pi M + VM$ for $i = 0, 1$. We have $V\varepsilon F = \pi - [\pi]$, where ε is a unit in $W_{\mathcal{O}}(B)$ (1.3) and V is injective on M; it is therefore equivalent to show that

$$(\pi - [\pi])\gamma_i \in V\varepsilon(\Pi M + VM) = V(\Pi M + VM).$$

We have

$$\Pi\gamma_0 = [a_{0,0}]\gamma_1 + Vx_0$$
$$\Pi\gamma_1 = [a_{0,1}]\gamma_0 + Vx_1$$

where $a_{0,0}$ and $a_{0,1}$ are elements of B such that $a_{0,0}.a_{0,1} = \pi$ and $x_0 \in M_0$, $x_1 \in M_1$. Therefore

$$(\pi - [\pi])\gamma_0 = (\Pi^2 - [\pi])\gamma_0 = [a_{0,0}]Vx_1 + \Pi V x_0.$$

Since

$$\Pi V x_0 = V\Pi x_0$$
$$[a_{0,0}]Vx_1 = V[a_{0,0}][a_{0,0}^{q-1}]x_1 \in V[a_{0,0}]M_1,$$

but $[a_{0,0}]\gamma_1 = \Pi\gamma_0 - Vx_0$, we have $[a_{0,0}]M_1 \subset \Pi M + VM$. We have shown that $(\pi - [\pi])\gamma_0 \in V(\Pi M + VM)$.

We also have

$$(\pi - [\pi])\gamma_1 = [a_{0,1}]Vx_0 + \Pi V x_1$$

and we thus conclude that $[a_{0,1}]M_0 \subset \Pi M + VM$.

Note that L_M is additive and satisfies

$$L_M(ax) = {}^{\sigma}a L_M(x), \quad a \in W_{\mathcal{O}}(B), x \in M,$$
$$L_m(Vx) = ((\Pi x, 0)).$$

Thus

$$\lambda_M L_M(ax) = F(ax) = {}^{\sigma}aFx = {}^{\sigma}\lambda_M L_M(x) = \lambda_M({}^{\sigma}a L_M(x))$$
$$\lambda_M L_M(Vx) = FVx = \Pi^2 x = \Lambda_M((\Pi x, 0)).$$

b) Let I be an ideal of B, let $B' = B/I$ and $M' = M \widehat{\otimes}_{E_{\mathcal{O}}(B)} E_{\mathcal{O}}(B')$. To show the existence of $L_{M'}$ making the diagram in (i) commute, it is necessary and sufficient, by (3.7), to show that $L_M(M_I) \subset N(M_I)$. Every element of M_I can be written in the form

$$x = [c_0]\gamma_0 + [c_1]\gamma_1 + Vy$$

with c_0 and $c_1 \in I$, and $y \in M_I$. Thus one has

$$L_M(x) = [c_0^q]L_M(\gamma_0) + [c_1^q]L_M(\gamma_1) + ((\Pi y, 0)).$$

It is clear that each term of the right side of this formula maps to zero in $N(M')$, so that $L_M(x) \in N(M_I)$.

c) Let B now be an arbitrary $\mathcal{O}$-algebra and M a special graded Cartier $\mathcal{O}[\Pi]$-module over B. Let (γ_0, γ_1) be a homogeneous basis for M and $a_{m,i}$ the elements of B such that

$$\Pi\gamma_0 = [a_{0,0}]\gamma_1 + \sum_{m>0} V^m[a_{m,0}]\gamma_{m+1}$$
$$\Pi\gamma_1 = [a_{0,1}]\gamma_0 + \sum_{m>0} V^m[a_{m,1}]\gamma_m.$$

Let $\widetilde{B} = \mathcal{O}'[X_b; b \in B]/(X_{a_{0,0}}.X_{a_{0,1}} - \pi)$, where the X_b are independent variables indexed by B. Let $\widetilde{M}$ be the special graded Cartier $\mathcal{O}[\Pi]$-module over $\widetilde{B}$ with homogeneous basis $(\widetilde{\gamma}_0, \widetilde{\gamma}_1)$ satisfying

$$\Pi\widetilde{\gamma}_i = [X_{a_{0,i}}]\widetilde{\gamma}_{i+1} + \sum_{m>0} V^m[X_{a_{m,i}}]\widetilde{\gamma}_{m+i+1}.$$

Then $\widetilde{B}$ is an $\mathcal{O}'$-algebra without torsion while B is a quotient via $X_b \mapsto b$, and M is obtained from $\widetilde{M}$ by basechange. From (a) and (b), there exist unique maps $L_{\widetilde{M}} \colon \widetilde{M} \to N(\widetilde{M})$ and $L_M \colon M \to N(M)$ such that the diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{L_{\widetilde{M}}} & N(\widetilde{M}) \\ \downarrow & & \downarrow \\ M & \xrightarrow{L_M} & N(M) \end{array}$$

is commutative, and such that $\lambda_{\widetilde{M}} L_{\widetilde{M}} = F$ and $\lambda_M L_M = F$.

d) If $B \to B'$ is an $\mathcal{O}'$-algebra homomorphism and if $M' = M \widehat{\otimes}_{E_{\mathcal{O}}(B)} E_{\mathcal{O}}(B')$, the preceding construction furnishes a commutative diagram of $\mathcal{O}'$-algebras

$$\begin{array}{ccc} \widetilde{B} & \longrightarrow & B \\ \downarrow & & \downarrow \\ \widetilde{B}' & \longrightarrow & B' \end{array}$$

where $\widetilde{B}$ and $\widetilde{B}'$ are without torsion. The commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{L_M} & N(M) \\ \downarrow & & \downarrow \\ M' & \xrightarrow{L_{M'}} & N(M') \end{array}$$

is deduced from the corresponding diagram for $\widetilde{M}$ and $\widetilde{M}'$ by passing to the quotient. The commutativity of this analogous diagram follows by injectivity of $\lambda_{\widetilde{M}}$, and from the commutativity of the two diagrams:

$$\begin{array}{ccc} N(\widetilde{M}) & \xrightarrow{\lambda_{\widetilde{M}}} & \widetilde{M} \\ \downarrow & & \downarrow \\ N(\widetilde{M}) & \xrightarrow[\lambda_{\widetilde{M}}]{} & \widetilde{M}' \end{array} \qquad \begin{array}{ccc} \widetilde{M} & \xrightarrow{F} & \widetilde{M} \\ \downarrow & & \downarrow \\ \widetilde{M}' & \xrightarrow[F]{} & \widetilde{M}' \end{array}$$

This completes the proof of proposition (3.8). $\square$

Remark 3.9. The map $L_M\colon M \to N(M)$ so defined is additive and satisfies

$$\begin{aligned} L_M(ax) &= {}^{\sigma}aL_M(x) \\ L_M(Vx) &= ((\Pi x, 0)) \qquad a \in W_{\mathcal{O}}(B), x \in M. \end{aligned}$$

That this follows in the case where B is torsion free follows by the remark in (a); it holds generally by passage to the quotient.

An additive map $L\colon M \to N(M)$ satisfying

$$\begin{aligned} L([a]x) &= [a^q]L(x) \\ L(Vx) &= ((\Pi x, 0)) \qquad a \in B, x \in M, \end{aligned}$$

is completely determined by giving $L(\gamma_0)$ and $L(\gamma_1)$. In fact every element in M can be written in a unique way as

$$x = [a_0]\gamma_0 + [a_1]\gamma_1 + Vy,$$

with $a_0, a_1 \in B$ and $y \in M$. Then L is defined by

$$L(x) = [a_0^q]L(\gamma_0) + [a_1^q]L(\gamma_1) + ((\Pi x, 0)).$$

Moreover to have $\lambda_M L = F$ it suffices that $\lambda_M L(\gamma_i) = F\gamma_i$ for $i = 0, 1$. Indeed, we then have

$$\begin{aligned} \lambda_M L(x) &= [a_0^q]F\gamma_0 + [a_1^q]F\gamma_1 + \Pi^2 y \\ &= F([a_0]\gamma_0 + [a_1]\gamma_1 + Vy) = F(x). \end{aligned}$$

The essential point is to show that there exist elements y_i in $N(M)$ such that $\lambda_M(y_i) = F\gamma_i$. This may be verified in the "universal" case where B is without torsion and λ_M is injective, as in (a) above.

Remark 3.10. Note also that L_M commutes with Π. Indeed, it suffices again to check in the case where B is torsion-free and after composition with λ_M; since $\lambda_M L_M = F$ and both F and λ_M commute with Π, so does L_M. This also shows that L_M is of degree 0, since F and λ_M are both of degree 1.

Definition 3.11. The map $M \oplus M^{\sigma} \to N(M)$ given by $(x, x') \mapsto L_M(x) + ((x', 0))$ defines a map $\phi_M\colon N(M) \to N(M)$. Indeed, $(Vx, -\Pi x) \mapsto 0$ since $L_M(Vx) = ((\Pi x, 0))$.

The map ϕ_M is additive of degree 0, it commutes with Π, and satisfies

$$\phi_M(ay) = {}^{\sigma}a\phi_M(y), \quad a \in W_{\mathcal{O}}(B), y \in N(M).$$

In particular ϕ_M is $\mathcal{O}[\Pi]$-linear.

Proposition 3.12. *Let $B \to B'$ be a surjective map of $\mathcal{O}'$-algebras with kernel I such that $I^2 = 0$ and $\pi I = 0$. Let M be a special graded Cartier $\mathcal{O}[\Pi]$-module over B and $M' = M \widehat{\otimes}_{E_{\mathcal{O}}(B)} E_{\mathcal{O}}(B')$. Then the kernel of the map $N(M) \to N(M')$ is killed by ϕ_M^3.*

Proof. Let (γ_0, γ_1) be a homogeneous V-basis for M. Every element of $M_f = \ker(M \to M')$ can be written (3.7):

$$x = [a_0]\gamma_0 + [a_1]\gamma_1 + Vx', \quad a_i \in I, x' \in M_f.$$

Moreover we have $N(M_f) = \ker(N(M) \to N(M'))$. Thus

$$\begin{aligned}\phi_M((x,0)) &= L_M(x)\\ &= [a_0^q]L_M(\gamma_0) + [a_1^q]L_M(\gamma_1) + ((\Pi x', 0))\\ &= ((\Pi x', 0))\end{aligned}$$

since $a_0^q = a_1^q = 0$. By writing:

$$x' = [a_0']\gamma_0 + [a_1']\gamma_1 + Vx'', \quad a_i' \in I, x'' \in M_I,$$

it follows that:

$$\phi_M^2((x,0)) = ((\Pi^2 x'', 0)) = ((\pi x'', 0)) = 0,$$

since $\pi M_I = 0$. In fact since $x \in M_I$ we have:

$$x = \sum_{m,i} V^m [a_{m,i}]\gamma_i, \quad a_{m,i} \in I,$$

$$\pi x = \sum_{m,i} V^m \pi [a_{m,i}]\gamma_i.$$

But $\pi[a] = 0$ if $a \in I$, since:

$$\begin{aligned}\pi[a] &= ([\pi] + V\varepsilon F)[a]\\ &= [\pi a] + V\varepsilon[a^q]F.\end{aligned}$$

We conclude that $\phi_M((0,x)) = ((x,0))$. □

Definition 3.13. To each special graded Cartier $\mathcal{O}[\Pi]$-module M over B associate a graded $\mathcal{O}[\Pi]$-module η_M defined by

$$\eta_M = N(M)^{\phi} M = \{x \in N(M) \mid \phi_M(z) = z\}$$

and an $\mathcal{O}[\Pi]$-linear map of degree zero $u_M \colon \eta_M \to M/VM$, by composing the inclusion $\eta_M \hookrightarrow N(M)$ with the canonical map $N(M) \to M/VM$ of (3.3).

Moreover, if B' is a B-algebra and $M' = M \widehat{\otimes}_{E_{\mathcal{O}}(B)} E_{\mathcal{O}}(B')$, the canonical map $M \to M'$ induces an $\mathcal{O}[\Pi]$- linear map of degree zero $\eta_M \to \eta_{M'}$ such that the diagram

$$\begin{array}{ccc} \eta_M & \xrightarrow{u_M} & M/VM \\ \downarrow & & \downarrow \\ \eta_{M'} & \xrightarrow[u_{M'}]{} & M'/VM' \qquad (= (M/VM) \otimes_B B') \end{array}$$

is commutative.

Proposition 3.14. *If B' is a quotient of B by a nilpotent ideal killed by a power of π, then the canonical map $\eta_M \to \eta_{M'}$ is bijective.*

Proof. The proposition is true if $I^2 = 0$ and $\pi I = 0$, since $N(M) \to N(M')$ is surjective (3.7) and $1 - \phi_M$ is invertible on the kernel (3.12). The general case is treated by induction. □

4. **Calculation of the homogeneous components of η_M.**

4.1. In this subsection B is an $\mathcal{O}'$-algebra such that $\pi B = 0$ and $M = M_0 \oplus M_1$ is a graded Cartier $\mathcal{O}[\Pi]$-module over B. An index $i \in \mathbf{Z}/2\mathbf{Z}$ is said to be *critical* if the map $\Pi\colon M_i/VM_{i-1} \to M_{i+1}/VM_i$ is zero, or in other words, if $\Pi M_i \subset VM_i$.

If B is integral, at least one of the two indices 0 or 1 is critical. In fact M_i/VM_{i-1} is a free B-module of rank 1 ($i = 0, 1$) and the composition $\Pi \circ \Pi = \pi$ is zero.

Lemma 4.2. *If i is a critical index and $x \in M_i$, then we have $L_M x = ((V^{-1}\Pi x, 0))$. [Note that $V^{-1}\Pi x$ is well-defined since $\Pi M_i \subset VM_i$ and V is injective]*

Proof. We have already seen that if $x = Vx'$, then we have $L_M x = ((\Pi x', 0))$ and $\Pi x' = V^{-1}\Pi x$, since $\Pi V = V\Pi$. Moreover, for $a \in B$, we have $L_M[a]x = [a^q]L_M x$ and $V^{-1}\Pi[a]x = [a^q]V^{-1}\Pi x$. Since every element of M_i is of the form $[a]\gamma_i + Vx'$, it thus suffices to prove the lemma when $x = \gamma_i$.

Let, as in (3.8), $\widetilde{B}$ be a torsion-free $\mathcal{O}'$-algebra and let B be a quotient of $\widetilde{B}$ and $\widetilde{M}$ a lifting of M to a special graded Cartier $\mathcal{O}[\Pi]$-module over $\widetilde{B}$. Letting $\widetilde{\gamma}_i$ lift γ_i, we have

$$\Pi\widetilde{\gamma}_i = [\widetilde{a}_i]\widetilde{\gamma}_{i+1} + V\widetilde{x}_i$$

with $\widetilde{a}_i \in \widetilde{B}$ and $\widetilde{x}_i \in \widetilde{M}_i$. Since i is critical, the image a_i of $\widetilde{a}_i$ in B is zero and

$$\Pi\gamma_i = Vx_i$$

where x_i is the image of $\widetilde{x}_i$ in M_i.

This shows that $L_M\gamma_i = ((x_i, 0))$. By definition $L_M\gamma_i$ is the image of $L_{\widetilde{M}}\widetilde{\gamma}_i$. Recall the calculation of $L_{\widetilde{M}}\widetilde{\gamma}_i$ made in (3.8), supposing that $i = 0$ to simplify the notations: [4]

$$\begin{aligned} V\varepsilon F\widetilde{\gamma}_0 &= (\Pi^2 - [\pi])\widetilde{\gamma}_0 \\ &= [\widetilde{a}_0]V\widetilde{x}_1 + \Pi V\widetilde{x}_0, \quad \widetilde{x}_i \in \widetilde{M}_i. \end{aligned}$$

Moreover

$$[\widetilde{a}_0]V\widetilde{x}_1 = V[\widetilde{a}_0^q]\widetilde{x}_1$$

and

$$[\widetilde{a}_0]\widetilde{x}_1 \in \Pi\widetilde{M}_0 + V\widetilde{M}_0$$

so that there exist $\widetilde{u}_0, \widetilde{v}_0 \in \widetilde{M}_0$ such that

$$[\widetilde{a}_0]V\widetilde{x}_1 = V\Pi[\widetilde{a}_0]\widetilde{u}_0 + V^2[\widetilde{a}_0]\widetilde{v}_0.$$

Since $F = \lambda_{\widetilde{M}}L_{\widetilde{M}}$, it follows:

$$\lambda_{\widetilde{M}}L_{\widetilde{M}}\widetilde{\gamma}_0 = \Pi\left(\varepsilon^{-1}\widetilde{x}_0 + \varepsilon^{-1}[\widetilde{a}_0]\widetilde{u}_0\right) + V\left({}^{\sigma}\varepsilon^{-1}[\widetilde{a}_0]\widetilde{v}_0\right)$$

so that the injectivity of $\lambda_{\widetilde{M}}$ gives:

$$L_{\widetilde{M}}\widetilde{\gamma}_0 = \left(\left(\varepsilon^{-1}\widetilde{x}_0 + \varepsilon^{-1}[\widetilde{a}_0]\widetilde{u}_0,\ {}^{\sigma}\varepsilon^{-1}[\widetilde{a}_0]\widetilde{v}_0\right)\right)$$

The image of ε in $W_{\mathcal{O}}(B)$ is 1 since $\pi B = 0$; in fact ${}^{\tau}\varepsilon = \pi - [\pi] = \pi =^{\tau\sigma} 1$, so that $\varepsilon =^{\sigma} 1 = 1$. The image of $\widetilde{a}_0$ in B is zero. Thus:

$$L_M\gamma_0 = ((x_0, 0)).$$

□

[4] bad translation of this sentence, I think

Lemma 4.3. *If i is a critical index, then the map*

$$\pi\colon N(M)_i \to M_i/VM_{i-1} \oplus M_i^\sigma$$
$$((m,m')) \mapsto (\overline{m}, V^{-1}\Pi m + m'))$$

is a $W_{\mathcal{O}}(B)$-linear isomoprhism. [We write $\overline{m}$ for the class of m modulo V. Note that $V^{-1}\Pi m + m' = V^{-1}\lambda_M((m,m'))$].

Proof. We define the inverse map

$$\rho^{-1}\colon M_i/VM_{i-1} \oplus M_i^\sigma \to N(M)_i$$
$$(\overline{m}, m'') \mapsto ((m, -V^{-1}\Pi m + m'')).$$

The map ρ^{-1} is well-defined: in fact, for $m \in M_i$, the element $V^{-1}\Pi m$ is defined and the map $m \mapsto ((m, -V^{-1}\Pi m))$ of M_i in $N(M)_i$ is trivial on VM_{i-1} since $Vm_1 \mapsto ((VM_i, -\Pi m_1)) = 0$. The map is clearly $W_{\mathcal{O}}(B)$-linear and inverse to the one of the lemma, which concludes the proof. □

Lemma 4.4. *If i is a critical index, the endomorphism $\rho\phi_M\rho^{-1}$ of $M_i/VM_{i-1} \oplus M_i^\sigma$ induced by ϕ_M via the isomorphism ρ is given by*

$$\rho\phi_M\rho^{-1}(\overline{m}, m'') = (\overline{m}'', V^{-1}\Pi m'').$$

Proof. This follows immediately from the definitions of ϕ_M and ρ, and from lemma (4.2):

$$(0, m'') \overset{\rho^{-1}}{\mapsto} ((0, m'')) \overset{\phi_M}{\mapsto} ((m'', 0)) \overset{\rho}{\mapsto} (\overline{m}'', V^{-1}\Pi m'')$$
$$(\overline{m}, 0) \overset{\rho^{-1}}{\mapsto} ((m, -V^{-1}\Pi m)) \overset{\phi_M}{\mapsto} L_M m + ((-V^{-1}\Pi m, 0)) = 0.$$

□

We deduce that:

Proposition 4.5. *If i is a critical index for M, the map $V^{-1}\lambda_M$ induces an $\mathcal{O}$-linear isomorphism*

$$\eta_{M,i} = N(M)_i^{\phi_M} \xrightarrow{\cong} M_i^{V^{-1}\Pi}.$$

More precisely $\eta_{M,i} = \left\{((m,0)) \mid m \in M_i^{V^{-1}\Pi}\right\}$ and the restriction of $V^{-1}\lambda_M$ to $\eta_{M,i}$ is the map $((m,0)) \mapsto m$.

When there does not exist a critical index, one has the following result:

Lemma 4.6. *If the map $\Pi\colon M_j/VM_{j-1} \to M_{j+1}/VM_j$ is an isomorphism, the map*

$$\lambda_M\colon N(M)_j \to M_{j+1}$$
$$((m,m')) \mapsto \Pi m + Vm'$$

is a $W_{\mathcal{O}}(B)$-linear isomorphism.

Proof. By the hypothesis on Π, every element of M_{j+1} can be written $\Pi m + Vm'$, so that λ_M is surjective.

Now we show that λ_M is surjective. Let $m, m' \in M$ be such that $\Pi m + Vm' = 0$. Since $\Pi m \in VM_j$, there exists $m'' \in M_{j-1}$ such that $m = Vm''$. Moreover $\Pi m + Vm' = V(\Pi m'' + m') = 0$ and thus $m' = -\Pi m''$. So $((m,m')) = (Vm'', -\Pi m'')) = 0$. □

Lemma 4.7. *Under the hypotheses of (4.6), the endomorphism $\lambda_M\phi_M\lambda_M^{-1}$ of M_{j+1} induced by ϕ_M is equal to $V^{-1}\Pi$.*

Proof. We note that $j+1$ is necessarily critical, since $\Pi^2 = 0$ on M/VM. For $m, m' \in M_j$, we have:

$$\begin{aligned} \phi_M((m,m')) &= L_M m + ((m',0)) \\ \lambda_M \phi_M((m,m')) &= Fm + \Pi m' \\ \lambda_M((m,m')) &= \Pi m + V m' \\ (V^{-1}\Pi)\lambda_M((m,m')) &= V^{-1}\pi m + \Pi m' = Fm + \Pi m', \end{aligned}$$

which proves the assertion. □

Proposition 4.8. *Under the hypotheses of (4.6), the map λ_M induces an $\mathcal{O}$-linear isomorphism*

$$\eta_{M,j} = N(M)_j^{\phi_M} \xrightarrow{\cong} M_{j+1}^{V^{-1}\Pi}.$$

Moreover the diagram

$$\begin{tikzcd} M_{j+1}^{V^{-1}\Pi} \arrow[r, equal, "\mathrm{id}"] & M_{j+1}^{V^{-1}\Pi} \\ \eta_{M,j} \arrow[u, "\lambda_M", "\cong"'] \arrow[r, "\Pi"'] & \eta_{M,j+1} \arrow[u, "\cong", "V^{-1}\lambda_M"'] \end{tikzcd}$$

is commutative.

Proof. The isomorphism follows from the previous two lemmas. The commutativity of the diagram above follows from the commutativity of the diagram

$$\begin{tikzcd} M_{j+1} \arrow[r, "V^{-1}\Pi"] & M_{j+1} \\ N(M)_j \arrow[u, "\lambda_M"] \arrow[r, "\Pi"'] & N(M)_{j+1} \arrow[u, "V^{-1}\lambda_M"'] \end{tikzcd}$$

since $V^{-1}\Pi$ restricts to id on $M_{j+1}^{V^{-1}\Pi}$. □

5. **Special formal $\mathcal{O}_D$-modules over an algebraicaly closed field.** In this subsection we suppose that $B = L$ is an algebraically closed field of characteristic p. We write $\mathcal{W} = W_{\mathcal{O}}(L)$ and let $\mathcal{K}$ denote the field of fractions of $\mathcal{W}$. The ring $\mathcal{W}$ is a complete discrete valuation ring with uniformizer π and residue field L, equipped with an automorphism σ such that $\mathcal{W}^\sigma = \mathcal{O}$.

If X is a formal (smooth) $\mathcal{O}$-module over L, its Cartier module M is a free $\mathcal{W}$-module of finite rank. We call rank of M over $\mathcal{W}$ the *height* of X.

Proposition 5.1. *If X is a special formal $\mathcal{O}_D$-module, its height is a multiple of* 4.

Proof. For $M' \subset M''$ two free $\mathcal{W}$-modules, we write $[M''\colon M']$ for the length of the $\mathcal{W}$-module W''/W'. By hypothesis we have $[M_0\colon VM_1] = [M_1\colon VM_0] = 1$ and V is injective, so that M_0 and M_1 have the same rank r over $\mathcal{W}$ and $M = M_0 \oplus M_1$ is of rank $2r$. We have $\Pi^2 = \pi$, so that Π is injective and

$$\begin{aligned} r = [M_0\colon \pi M_0] &= [M_0\colon \Pi M_1] + [\Pi M_1\colon \Pi^2 M_0] \\ &= [M_0\colon \Pi M_1] + [M_1\colon \Pi M_0]. \end{aligned}$$

The inclusions

$$\begin{tikzcd} & VM_1 \arrow[dr, hook] & \\ \Pi VM_0 \arrow[ur, hook] \arrow[dr, hook] & & M_0 \\ & \Pi M_1 \arrow[ur, hook] & \end{tikzcd}$$

yield the equality

$$[M_0\colon VM_1] + [VM_1\colon \Pi VM_0] = [M_0\colon \Pi M_1] + [\Pi M_1\colon \Pi VM_0].$$

Since

$$[\Pi M_1\colon \Pi VM_0] = [M_1\colon VM_0] = [M_0\colon VM_1] = 1$$

and

$$[VM_1\colon \Pi VM_0] = [M_1\colon \Pi M_0]$$

we deduce that

$$[M_1\colon \Pi M_0] = [M_0\colon \Pi M_1],$$

and hence that r is even. $\square$

Proposition 5.2. *There exists a single isogeny class of special formal $\mathcal{O}_D$-modules of height 4.*

Proof. We know that the isogeny class of a formal $\mathcal{O}$-module X is determined by the isocrystal $(M\otimes_{\mathcal{W}}\mathcal{K}, V)$. This is in turn determined by the isocrystal $(M_0\otimes_{\mathcal{W}}\mathcal{K}, V\Pi^{-1})$, since $M_1\otimes_{\mathcal{W}}\mathcal{K}$ is identified with $M_0\otimes_{\mathcal{W}}\mathcal{K}$ by Π.

But $(M_0\otimes_{\mathcal{W}}\mathcal{K}, V\Pi^{-1})$ is a unit isocrystal: if i is critical, M_i is a lattice stable under $V\Pi^{-1}$ in $M_i\otimes_{\mathcal{W}}\mathcal{K}$ and $V\Pi^{-1}|_{M_i}$ is bijective. Since L is algebraically closed, such an isocrystal is unique up to isomorphism; there exists a basis e_1, e_2 of M_i over $\mathcal{W}$ such that $V\Pi^{-1}e_1 = e_1$ and $V\Pi^{-1}e_2 = e_2$. $\square$

Remark (5.2'). The formal $\mathcal{O}$-module X is isogenous to the sum of two formal $\mathcal{O}$-modules of dimension 1 and height 2. In other words, the isocrystal $(M\otimes_{\mathcal{W}}\mathcal{K}, V)$ is of dimension 2 and of slope $1/2$.

Proposition 5.3. *We have* $\operatorname{End}^0_D X \cong M_2(K)$*, where we write* $\operatorname{End}^0_D X = \operatorname{End}_{\mathcal{O}_D} X \otimes_{\mathbf{Z}} \mathbf{Q}$.

Proof. The correspondence $X \mapsto (M\otimes_{\mathcal{W}}\mathcal{K}, V) \mapsto (M_0\otimes_{\mathcal{W}}\mathcal{K}, V\Pi^{-1})$ induces isomorphisms

$$\operatorname{End}^0_D X = \operatorname{End}_D(M\otimes_{\mathcal{W}}\mathcal{K}, V) = \operatorname{End}_K(M_0\otimes_{\mathcal{W}}\mathcal{K}, V\Pi^{-1}).$$

Finally $\operatorname{End}_K(M_0\otimes_{\mathcal{W}}\mathcal{K}, V) \cong M_2(K)$, since $(M_0\otimes_{\mathcal{W}}\mathcal{K}, V\Pi^{-1})$ is a unit isocrystal of rank 2: a $\mathcal{K}$-linear endomorphism of $M_0\otimes_{\mathcal{W}}\mathcal{K}$ commutes with the σ^{-1}-linear map $V\Pi^{-1}$ if and only if its matrix in the basis e_1, e_2 has coefficients in $\mathcal{K}^\sigma = K$. $\square$

We suppose henceforth that X is a special formal $\mathcal{O}_D$-module of height 4. Then

$$[M_1\colon \Pi M_0] = [M_0\colon \Pi M_1] = 1$$

In particular, if i is critical, we have $\Pi M_i = VM_i$.

5.4. Recall that, for all $\mathcal{O}$-algebras B, we write $B[\Pi]$ for the commutative $\mathbf{Z}/2\mathbf{Z}$-graded algebra generated by B in degree 0 and an element Π of degree 1 such that $\Pi^2 = \pi$ (I.5).

We associate to X a triple (η, T, u) where

(i) η is a graded $\mathcal{O}[\Pi]$-module,

(ii) T is a graded $L[\Pi]$-module whose homogeneous components T_0 and T_1 are of dimension 1,

(iii) $u\colon \eta \to T$ is an $\mathcal{O}[\Pi]$-linear map of degree 0, with $\eta = \eta_M$, $T = M/VM$ and where u is defined by composing the inclusion $\eta_M \subset N(M)$ with the map $((m, m')) \mapsto \overline{m}$ of $N(M)$ into M/VM.

Proposition 5.5. *The homogeneous components η_0 and η_1 of η are free $\mathcal{O}$-modules of rank 2 and the map $\eta \otimes_{\mathcal{O}} L \to T$ is surjective.*

Proof. If i is critical, we have seen in (4.5) that η_i is identified with $M_i^{V^{-1}\Pi}$ and $u_i\colon \eta_i \to T_i$ with the composition $M_i^{V^{-1}\Pi} \hookrightarrow M_i \to M_i/VM_{i-1}$. The σ-linear endomorphism $V^{-1}\Pi\colon M_i \to M_i$ is bijective, that is, $(M_i, V^{-1}\Pi)$ is a unit crystal over L and L is algebraically closed. Therefore $M_i^{V^{-1}\Pi}$ is a free $\mathcal{O}$-module of rank 2 and the map $M_i^{V^{-1}\Pi} \otimes_{\mathcal{O}} L \to M_i/VM_{i-1}$ is surjective.

If j is not critical, we have a commutative diagram

$$\begin{tikzcd}
\eta_j \arrow[r, "\Pi"] \arrow[d, "u_j"'] & \eta_{j+1} \arrow[d, "u_{j+1}"] \\
T_j \arrow[r, "\Pi"] & T_{j+1}
\end{tikzcd}$$

as in (4.8), where the horizontal arrows induced by Π are isomorphisms, and $j+1$ is critical. $\square$

Proposition 5.6. *The map $\eta/\Pi\eta \to T/\Pi T$, induced by u, is injective.*

Proof. To show that $\eta_i/\Pi\eta_{i-1} \to T_i/\Pi T_{i-1}$ is injective, we distinguish three cases.

First case: i is critical and $i-1$ is not – We have $\Pi\eta_{i-1} = \eta_i$ and the assertion is clear.

Second case: i and $i-1$ are both critical – We have a commutative diagram

$$\begin{tikzcd}
\eta_{i-1} \arrow[r, "\Pi"] \arrow[d, "\sim"] & \eta_i \arrow[d, "\sim"] \\
M_{i-1}^{V^{-1}\Pi} \arrow[r, "\Pi"] \arrow[d] & M_i^{V^{-1}\Pi} \arrow[d] \\
T_{i-1} \arrow[r, "\Pi=0"] & T_i
\end{tikzcd}$$

If $x \in M_i^{V^{-1}\Pi}$ maps to zero in T_i, there exists $y \in M_{i-1}$ such that $x = Vy$. But, since $\Pi x = Vx$, we deduce that $\Pi y = Vy$, that is $y \in M_{i-1}^{V^{-1}\Pi}$ and $x = \Pi y$.

Third case: i is not critical but $i-1$ is critical – The commutative diagram

$$\begin{tikzcd}
\eta_{i-1} \arrow[r, "\Pi"] \arrow[d, "u_{i-1}"'] & \eta_i \arrow[r, "\Pi", "\sim"'] \arrow[d, "u_i"'] & \eta_{i-1} \arrow[d, "u_{i-1}"] \\
T_{i-1} \arrow[r, "\Pi=0"'] & T_i \arrow[r, "\sim", "\Pi"'] & T_{i-1}
\end{tikzcd}$$

shows that is suffices to prove that the map $\eta_{i-1}/\pi\eta_{i-1} \to T_{i-1}$ induced by u_{i-1} is injective. Moreover $u_{i-1}\colon \eta_{i-1} \to T_{i-1}$ is identified with $M_{i-1}^{V^{-1}\Pi} \to M_{i-1}/VM_i$.

If $x \in M_{i-1}^{V^{-1}\Pi}$ maps to zero in M_{i-1}/VM_i, there exists $y \in M_i$ such that $x = Vy$. From $\Pi x = Vx$, we deduce $\Pi y = Vy$; in particular the image $\overline{y}$ of y in M_i/VM_{i-1} is zero, because $\Pi\overline{y} = \overline{x} = 0$ and $\Pi\colon M_i/VM_{i-1} \to M_{i-1}/VM_i$ is an isomorphism. So there exists $z \in M_{i-1}$ such that $y = Vz$. But again since $\Pi y = Vy$, we deduce $\Pi z = Vz$; thus $z \in M_{i-1}^{V^{-1}\Pi}$ and $x = \pi z$. $\square$

Proposition 5.7. *The triple (η, T, u) determines X up to isomorphism.*

Proof. It suffices to show that M_0, M_1, Π and V are determined by (η, T, u).

If i is a critical index and σ the automorphism $\mathrm{id}\otimes\sigma$ of $\eta_i \otimes_{\mathcal{O}} \mathcal{W}$, the inclusion $\eta_i \subset M_i$ induces an isomorphism $(\eta_i \otimes_{\mathcal{O}} \mathcal{W}, \sigma) \cong (M_i, \Pi V^{-1})$. Moreover if

$$\mathcal{H} = \ker \left\{ \eta_i \otimes_{\mathcal{O}} \mathcal{W} \overset{u_i\otimes \mathrm{id}}{\to} T_i \right\},$$

the isomorphism above identifies $\mathcal{H}$ with VM_{i-1} and $\sigma(\mathcal{H})$ with ΠM_{i-1}. So the diagram

$$\begin{tikzcd} M_{i-1} \arrow[r, bend left, "\Pi"] \arrow[r, bend right, "V"'] & M_i \arrow[r, bend left, "\Pi"] \arrow[r, bend right, "V"'] & M_{i-1} \end{tikzcd}$$

is identified with the diagram

$$\begin{tikzcd} \sigma(\mathcal{H}) \arrow[r, bend left, "incl"] \arrow[r, bend right, "incl\circ\sigma^{-1}"'] & \eta_i \otimes_{\mathcal{O}} \mathcal{W} \arrow[r, bend left, "\pi"] \arrow[r, bend right, "\pi\circ\sigma^{-1}"'] & \sigma(\mathcal{H}). \end{tikzcd}$$

$\square$

Definition 5.8. A triple (η, T, u) is said to be *admissible* if it satisfies the conditions of (5.5) and (5.6). An index $i \in \{0, 1\}$ is said to be *critical* for (η, T, u) if $\Pi\colon T_i \to T_{i+1}$ is zero.

Lemma 5.9. *An admissible triple (η, T, u) is determined up to isomorphism by (η_i, T_i, u_i) for i critical.*

Proof. Write $H = \ker u_i$. We have $\pi\eta_i \subset H \subset \eta_i$ and $H \neq \eta_i$ by (5.5).

If $i-1$ is not critical, we have $\Pi T_{i-1} = T_i$ and thus $\Pi\eta_{i-1} = \eta_i$, by (5.6). Hence the diagram

$$\begin{tikzcd} \eta_i \arrow[r, "\Pi"] \arrow[d, "u_i"] & \eta_{i-1} \arrow[r, "\Pi"] \arrow[d, "u_{i-1}"] & \eta_i \arrow[d, "u_i"] \\ T_i \arrow[r, "0"] & T_{i-1} \arrow[r, "\Pi"] & T_i \end{tikzcd}$$

is identified with the diagram

$$\begin{tikzcd} \eta_i \arrow[r, "\pi"] \arrow[d, "u_i"] & \eta_{i-1} \arrow[r, "\mathrm{id}"] \arrow[d, "u_{i-1}"] & \eta_i \arrow[d, "u_i"] \\ T_i \arrow[r, "0"] & T_{i-1} \arrow[r, "\mathrm{id}"] & T_i \end{tikzcd}$$

Moreover the condition (5.6) implies in this case that $H = \pi\eta_i$.

If $i-1$ is critical, we have $\Pi\eta_{i-1} \neq \eta_i$; otherwise we would have $u_i = 0$, contradicting (5.5). Moreover, since i is critical, we have $\Pi\eta_i \neq \eta_{i-1}$, thus $\pi\eta_i \neq \Pi\eta_{i-1}$. Then, by (5.6), u_i and $u_{i-1}\Pi^{-1}$ induce isomorphisms

$$\eta_i/\Pi\eta_{i-1} \otimes_k L \xrightarrow{\sim} T_i \quad \text{and} \quad \Pi\eta_{i-1}/\pi\eta_i \otimes_k L \xrightarrow{\sim} T_{i-1}.$$

In this case we have $H = \Pi\eta_{i-1} \neq \pi\eta_i$ and the diagram

$$\begin{tikzcd} \eta_i \arrow[r, "\Pi"] \arrow[d, "u_i"'] & \eta_{i-1} \arrow[r, "\Pi"] \arrow[d, "u_{i-1}"'] & \eta_i \arrow[d, "u_i"'] \\ T_i \arrow[r, "0"] & T_{i-1} \arrow[r, "0"] & T_i \end{tikzcd}$$

is identified with the diagram

$$\begin{tikzcd} \eta_i \arrow[r, "\pi"] \arrow[d] & H \arrow[r, "incl"] \arrow[d] & \eta_i \arrow[d] \\ \eta_i/H \otimes_k L \arrow[r, "0"] & H/\pi\eta_i \otimes_k L \arrow[r, "0"] & \eta_i/H \otimes_k L \end{tikzcd}$$

where the vertical arrows are the canonical maps.

Note that we recognize whether $i-1$ is or is not critical depending on whether $H \neq \pi\eta_i$ or $H = \pi\eta_i$, respectively; the proof of the lemma is complete. $\square$

Proposition 5.10. *Every admissible triple (η, T, u) is isomorphic to a triple associated to a special formal $\mathcal{O}_D$-module of height 4.*

Proof. Let i be a critical index, σ the automorphism $\mathrm{id}\otimes\sigma$ of $\eta_i \otimes_{\mathcal{O}} \mathcal{W}$ and $\mathcal{H} = \ker\{u_i \otimes \mathrm{id}\colon\ \eta_i \otimes_{\mathcal{O}} \mathcal{W} \to T_i\}$. Since $\eta_i \otimes_{\mathcal{O}} L \to T_i$ is surjective, we have $\pi(\eta_i \otimes_{\mathcal{O}} \mathcal{W}) \subsetneq \mathcal{H} \subsetneq \eta_i \otimes_{\mathcal{O}} \mathcal{W}$; the same is true for $\sigma(\mathcal{H})$.

We define the diagram

$$\begin{tikzcd} M_{i-1} \arrow[r, shift left, "\Pi"] \arrow[r, shift right, "V"'] & M_i \arrow[r, shift left, "\Pi"] \arrow[r, shift right, "V"'] & M_{i-1} \end{tikzcd}$$

to be equal to the diagram

$$\begin{tikzcd} \sigma(\mathcal{H}) \arrow[r, shift left, "incl"] \arrow[r, shift right, "incl\circ\sigma^{-1}"'] & \eta_i \otimes_{\mathcal{O}} \mathcal{W} \arrow[r, shift left, "\pi"] \arrow[r, shift right, "\pi\circ\sigma^{-1}"'] & \sigma(\mathcal{H}). \end{tikzcd}$$

Thus V is σ^{-1}-linear, Π is linear, $\Pi^2 = \pi$ and

$$[M_i\colon VM_{i-1}] = [M_{i-1}\colon VM_i] = [M_i\colon \Pi M_{i-1}] = [M_{i-1}\colon \Pi M_i] = 1,$$

thus (M, Π, V) is a special formal Cartier $\mathcal{O}_D$-module of height 4.

The index i is critical for M, the homogeneous component of index i of the triple associated to M is

$$(M_i^{V\Pi^{-1}}, M_i/VM_{i-1}, can) \cong (\eta_i, \eta_i \otimes_{\mathcal{O}} \mathcal{W}/\mathcal{H}, can) \cong (\eta_i, T_i, u_i),$$

because this triple is isomorphic with (η, T, u). $\square$

We can summarize the preceding propositions in a single theorem:

Theorem 5.11. *Over an algebraically closed field of characteristic p, the correspondence $X \mapsto (\eta, T, u)$ gives an equivalence of categories between on the one hand the groupoid of special formal $\mathcal{O}_D$-modules of height 4 and their isomorphisms, with the groupoid of admissible triples and their isomorphisms.*

Lemma 5.12. *Let (η, T, u) be an admissible triple associated to a special formal Cartier $\mathcal{O}_D$-module M of height 4. Then, for $i \in \{0,1\}$, the isocrystal $(M_i \otimes_{\mathcal{W}} \mathcal{K}, V\Pi^{-1})$ is canonically isomorphic with $(\eta_i \otimes_{\mathcal{O}} \mathcal{K}, \sigma^{-1})$.*

Proof. For i critical, $(M_i, V\Pi^{-1})$ is a unit crystal, that is $V\Pi^{-1}$ is a bijective σ^{-1}-linear endomorphism of M_i, such that η_i is identified with $M_i^{V\Pi^{-1}}$ (4.5). It follows then from the structure theorem for unit crystals ([Zi 3], Satz 6.26) that $(M_i, V\Pi^{-1})$ is identified with $(\eta_i \otimes_{\mathcal{O}} \mathcal{W}, \sigma^{-1})$, and so a fortiori $(M_i \otimes_{\mathcal{W}} \mathcal{K}, V\Pi^{-1})$ is identified with $(\eta_i \otimes_{\mathcal{O}} \mathcal{K}, \sigma^{-1})$.

Also Π induces isomorphisms of isocrystals:

$$(M_i \otimes_{\mathcal{W}} \mathcal{K}, V\Pi^{-1}) \xrightarrow{\sim} (M_{i+1} \otimes_{\mathcal{W}} \mathcal{K}, V\Pi^{-1}),$$
$$(\eta_i \otimes_{\mathcal{O}} \mathcal{K}, \sigma^{-1}) \xrightarrow{\sim} (\eta_{i+1} \otimes_{\mathcal{O}} \mathcal{K}, \sigma^{-1}).$$

which are compatible with the preceding isomorphisms when 0 and 1 are both critical. □

Definition 5.13. Let X and X' be two special formal $\mathcal{O}_D$-modules of height 4 over L. We call a *quasi-isogeny* between X and X' an element of $\mathrm{Hom}_{\mathcal{O}_D}(X, X') \otimes_{\mathcal{O}} K$ which is invertible in $\mathrm{Hom}_{\mathcal{O}_D}(X', X) \otimes_{\mathcal{O}} K$. That is, if α is a quasi-isogeny of X in X', there exists $n \geq 0$ such that $\pi^n \alpha$ gives an $\mathcal{O}_D$-isogeny of X in X'. We say that α is of *height* zero if $h(\pi^n \alpha) = h(\pi^n)$.

Proposition 5.14. *Let (η, T, u) and (η', T', u') be admissible triples associated to X and X'. We have a canonical isomorphism:*

$$\mathit{QIsog}(X, X') \cong \mathit{Isom}_K(\eta_0, \otimes_{\mathcal{O}} K, \eta'_0 \otimes_{\mathcal{O}} K).$$

Proof. Let M and M' be the Cartier modules of X and X'. We have, following (2.2), a canonical isomorphism:

$$\mathrm{QIsog}(X, X') = \mathrm{Isom}((M \otimes_{\mathcal{W}} \mathcal{K}, V), (M' \otimes_{\mathcal{W}} \mathcal{K}, V)),$$

where the isomorphisms of the isocrystals on the right must be compatible with the grading and the action of Π; they are determined by their action on the homgeneous component of degree 0, more precisely, we have:

$$\mathrm{Isom}((M \otimes_{\mathcal{W}} \mathcal{K}, V), (M' \otimes_{\mathcal{W}} \mathcal{K}, V)) = \mathrm{Isom}((M_0 \otimes_{\mathcal{W}} \mathcal{K}, V\Pi^{-1}), (M'_0 \otimes_{\mathcal{W}} \mathcal{K}, V\Pi^{-1})).$$

Lemma (5.12) now finishes the proof. □

Proposition 5.15. *Suppose that 0 is critical for X. Then, in the preceding isomorphism, the quasi-isogenies of height 0 correspond to isomorphisms $r\colon \eta_0 \otimes_{\mathcal{O}} K \xrightarrow{\sim} \eta'_0 \otimes_{\mathcal{O}} K$ such that:*

$$[\eta'_0\colon r(\eta_0)] = 0 \text{ if } 0 \text{ is critical for } X',$$
$$[\eta'_1\colon \Pi r(\eta_0)] = 1 \text{ if } 1 \text{ is critical for } X';$$

that is, such that $\Lambda^2 \eta'_i = \pi^{-i} \Lambda^2 \Pi^i r(\eta_0)$ if $i \in \{0,1\}$ is critical for X'.

Proof. A quasi-isogeny is of height 0 if and only if the corresponding isomorphism $\alpha\colon M \otimes_{\mathcal{W}} \mathcal{K} \xrightarrow{\sim} M' \otimes_{\mathcal{W}} \mathcal{K}$ is such that $[M'\colon \alpha(\pi^n M)] = [M\colon \pi^n M]$ for n such that $\alpha(\pi^n M) \subset M'$, that is if $[M'\colon \alpha(M)] = 0$. Since $[M_1\colon \Pi M_0] = [M_0\colon \Pi M_1] = 1$, and the same for M', the conditions $[M'_0\colon \alpha(M_0)] = 0$ and $[M'_1\colon \Pi\alpha(M_0)] = 1$ are equivalent.

Since 0 is critical for X, M_0 is identified with $\eta_0 \otimes_{\mathcal{O}} \mathcal{W}$. If 0 is also critical for X', M'_0 is identified with $\eta'_0 \otimes_{\mathcal{O}} \mathcal{W}$ and $[M'_0\colon \alpha(M_0)] = [\eta'_0\colon r(\eta_0)]$. If 1 is critical for X', then M'_1 is identified with $\eta'_1 \otimes_{\mathcal{O}} \mathcal{W}$ and $[M'_1\colon \Pi\alpha(M_0)] = [\eta'_1\colon \Pi r(\eta_0)]$. □

5.16. Let $\overline{k}$ be an algebraic closure of $k = \mathcal{O}/\pi\mathcal{O}$. Choose a special formal $\mathcal{O}_D$-module Φ of height 4 over $\overline{k}$ such that 0 is critical for Φ (such a module is unique up to isogeny by (5.2)) and fix an isomorphism $\mathcal{O}^2 \cong \eta_{0,\Phi}$.

A special formal $\mathcal{O}_D$-module of height 4 over a field extension L of $\overline{k}$ is said to be *rigidified* if it is equipped with a quasi-isogeny $\rho\colon \Phi_L \to X$ of height 0.

An admissible triple (η, T, u) over L is said to be *rigidified* if it is equipped with an isomorphism $r\colon K^2 \xrightarrow{\sim} \eta_0 \otimes_{\mathcal{O}} K$ such that $[\eta_i\colon \Pi^i r(\mathcal{O}^2)] = i$ if $i \in \{0,1\}$ is critical for η, that is $\Lambda^2\eta_i = \pi^{-i}\Lambda^2\Pi^i r(\mathcal{O}^2)$ if i is critical.

Following what we have seen above, if (η, T, u) is associated to X, a rigidification ρ of X corresponds to a rigidification r of (η, T, u), and we have:

Theorem 5.17. *Let L be an algebraically closed field extension of $\overline{k}$. The correspondence $(X,\rho) \mapsto (\eta, T, u, r)$ gives a bijection between the collection of isomorphism classes of special formal $\mathcal{O}_D$-modules of height 4, rigidified over L, with the isomorphism classes of rigidified admissible triples over L.*

Recall (I.5.2) that this latter collection is identified with $\widehat{\Omega}(L)$.

6. Filtrations on $N(M)$ and η_M.

6.1. In this subsection, B is an $\mathcal{O}'$-algebra such that $\pi B = 0$ and M is a graded special Cartier $\mathcal{O}[\Pi]$-module over B. We moreover suppose that $i \in \{0,1\}$ is a critical index, that is $\Pi M_i \subset V M_i$. To ease notations we write simply $N = N(M)$, $\phi = \phi_M$ and $\eta = \eta_M$.

Consider the filtrations of N_i and N_{i-1} given by the $\mathcal{O}$-submodules

$$V^{2n}M_i = ((0, V^{2n}M_i)) \subset N_i$$
$$V^{2n-1}M_i = ((0, V^{2n-1}Mi)) \subset N_{i-1}$$

for $n > 0$. Let $N_{i,n} = N_i/V^{2n}M_i$ and $N_{i-1,n} = N_{i-1}/V^{2n-1}M_i$. For $j \in \{0,1\}$, we put $\varepsilon = 0$ if $j = i$ and $\varepsilon = 1$ if $j = i-1$, so that $N_{j,n} = N_j/V^{2n-\varepsilon}M_i$.

Lemma 6.2. *For all $r > 0$, we have $\phi(V^r M_i) \subset V^r M_i$.*

Proof. For $m \in M_i$, we have

$$\phi((0, V^r m)) = ((V^r m, 0)) = ((0, \Pi V^{r-1} m)),$$

but $\Pi m \in \Pi M_i \subset V M_i$, so that $\Pi V^{r-1} m = V^{r-1}\Pi m \in V^r M_i$. □

Thus ϕ induces an $\mathcal{O}$-linear endomorphism of $N_{j,n}$. In what follows, we write $\eta_{j,n} = \{z \in N_{j,n} \mid \phi(z) = z\}$.

Lemma 6.3. *We have $N_j = \varprojlim N_{j,n}$ and $\eta_j = \varprojlim \eta_{j,n}$.*

Proof. Consider the exact sequence of $\mathcal{O}$-modules defining N_j:

$$0 \to M_{j-1} \xrightarrow{\alpha} M_j \oplus M_j \to N_j \to 0 \qquad (6.3.1)$$

where $\alpha(m) = (Vm, -\Pi m)$. Since V is injective, we have $\alpha(M_{j-1}) \cap (0, V^{2n-\varepsilon}M_i) = \{0\}$; we thus have a projective system of exact sequences:

$$0 \to M_{j-1} \to M_j \oplus M_j/V^{2n-\varepsilon}M_i \to N_{j,n} \to 0,$$

and, by passing to the projective limit, an exact sequence:

$$0 \to M_{j-1} \to M_j \oplus \varprojlim M_j/V^{2n-\varepsilon}M_i \to \varprojlim N_{j,n} \to 0.$$

But M is complete for the V-adic topology, hence $M_j = \varprojlim M_j/V^{2n-\varepsilon}M_i$ and $N_j = \varprojlim N_{j,n}$.

The assertion for η_j is deduced by taking the kernel of $1-\phi$. □

6.4. In the remainder of this subsection we consider M_j, N_j, η_j, $N_{j,n}$ and $\eta_{j,n}$ as functors on the category of B-algebras: for such an algebra B', $M(B') = M\widehat{\otimes}_{E'_{\mathcal{O}}(B)}E'_{\mathcal{O}}(B')$ is a special graded Cartier $\mathcal{O}[\Pi]$-module over B' and $N_j(B')$, $\eta_j(B')$, $N_{j,n}(B')$, $\eta_{j,n}(B')$ are obtained from $M(B')$ by the preceding constructions; the transition maps are defined in the obvious way, given the fact that ϕ is compatible with basechange.

Lemma 6.5. *The functor $N_{j,n}$ is representable by a scheme in an affine $\mathcal{O}$-module over B such that the underlying scheme is the affine space of dimension $2n+1-\varepsilon$ over B.*[5]

Proof. Let (γ_0, γ_1) be a homogeneous V-basis for M and let $M_{j,(0)}$ be the subfunctor of M_j defined by

$$M_{j,(0)}(B') = \{[a]\gamma_j \mid a \in B'\}.$$

The exact sequence (6.3.1) defines a natural map:

$$M_{j,(0)} \times M_j/V^{2n-\varepsilon}M_i \to N_{j,n}.$$

This map is bijective.

Indeed, let m, $m' \in M_j$; we have $m = m_0 + Vm_1$ with $m_0 \in M_{j,(0)}$ and $m_1 \in M_{j-1}$, where $((m, m')) = ((m_0, m' + \Pi m_1))$ in N_j.

Moreover, let m_0 and $l_0 \in M_{j,(0)}$, m' and $l' \in M_j$, be such that $((m_0, m'))$ and $((l_0, l'))$ have the same image in $N_{j,n}$; by (6.3.1), there exists $m_1 \in M_{j-1}$ such that

$$\begin{aligned} m_0 + Vm_1 &= l_0 \\ m' - \Pi m_1 &\equiv l' \pmod{V^{2n-\varepsilon}M_i}. \end{aligned}$$

The first equality implies that necessarily $m_1 = 0$, and the assertion of bijectivity follows.

Thus the maps

$$\begin{aligned} B' &\to M_{j,(0)}(B') \\ a &\mapsto [a]\gamma_j \end{aligned}$$

and

$$\begin{aligned} (B')^{2n-\varepsilon} &\to M_j/V^{2n-\varepsilon}M_i(B') \\ (a_k) &\mapsto \sum_{0\le k<2n-\varepsilon} V^k[a_k]\gamma_{j-k} \end{aligned}$$

are functorial and bijective. We thus obtain a functorial bijection between $\mathbf{A}^{2n+1-\varepsilon}$ and $N_{j,n}$. □

Lemma 6.6. *The functor $\eta_{j,n}$ is representable by an affine scheme of $\mathcal{O}$-modules which is of finite presentation and étale over B.*[6]

Proof. Indeed $\eta_{j,n} = \ker(1-\phi)$ is the inverse image of the zero section of $N_{j,n}$ by $1-\phi$ and this section, which coincides with the zero section of $\mathbf{A}^{2n+1-\varepsilon}$ under the previous isomorphism, is a closed immersion of finite presentation. Hence $\eta_{j,n} \hookrightarrow N_{j,n}$ is also a closed immersion of finite presentation.

[5]not happy with this translation

[6]similar to above; what is un schema en O-modules affine?

To show that $\eta_{j,n}$ is étale over B it remains to show that, if $B' \to B''$ is a surjection of B-algebras defined by an ideal of square zero, the map $\eta_{j,n}(B') \to \eta_{j,n}(B'')$ is bijective. In the commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & V^{2n-\varepsilon}M_i(B') \arrow[r] \arrow[d, "\alpha"'] & N_j(B') \arrow[r] \arrow[d, "\beta"'] & N_{j,n}(B') \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & V^{2n-\varepsilon}M_i(B'') \arrow[r] & N_j(B'') \arrow[r] & N_{j,n}(B'') \arrow[r] & 0, \end{tikzcd}$$

α and β are surjective by (3.7), thus γ is surjective and ϕ is nilpotent on $\ker\gamma$, because it is on $\ker\beta$ by (3.12) and $\ker\gamma$ is a quotient of $\ker\beta$. □

In other words, $\eta_{j,n}$ is a constructible sheaf for the étale topology on $\mathrm{Spec}(B)$ and its formation is compatible with basechange. Write $S = \mathrm{Spec}(B)$ and let S_{i-1} be the closed subset of S where $i-1$ is critical. We have:

Proposition 6.7. (1) *$\eta_{i,n}$ is a smooth sheaf over S of free $\mathcal{O}/\pi^n$-modules of rank 2.* Smooth *here means locally constant for the étale topology.*

(2) *$\eta_{i-1,n}$ is constructible over S and $\Pi\colon \eta_{i-1,n} \to \eta_{i,n}$ is injective, moreover:*

(2a) *$\eta_{i-1,n}$ is smooth above $S - S_{i-1}$ and Π is an isomorphism above $S - S_{i-1}$.*

(2b) *η_{i-1}, n is smooth over S_{i-1} and $(\eta_{i,n}/\Pi\eta_{i-1,n})$ is smooth over S_{i-1} "en $\mathcal{O}/\pi$-vectoriels de rang 1"*[7].

Proof. Given what we have just seen, to prove that the sheaves in question are smooth, it suffices to prove that the size of their fibers above geometric points of S is constant. To prove the proposition we may thus suppose that $B = L$ is an algebraically closed field of characteristic p.

(1) By (4.3), we have an isomorphism

$$N_{i,n} \cong M_i/VM_{i-1} \oplus M_i/V^{2n}M_i$$

such that the endomorphism ϕ of $N_{i,n}$ corresponds with

$$(\overline{m}, \overline{m}'') \mapsto (\overline{m}'', V^{-1}\Pi\overline{m}'').$$

We thus have an isomorphism:

$$\eta_{i,n} \cong (M_i/V^{2n}M_i)^{V^{-1}\Pi}.$$

Identify $(M_i, V^{-1}\Pi)$ with $(\eta_i \otimes_{\mathcal{O}} W, \sigma)$; then V^2 is identified with $\pi.\sigma^{-2}$ and $V^{2n}M_i$ with $\pi^n\eta_i \otimes_{\mathcal{O}} W$; thus

$$\eta_{i,n} \cong \eta_i/\pi^n\eta_i \cong (\mathcal{O}/\pi^n)^2.$$

(2) The map $\Pi\colon N_{i-1} \to N_i$ is injective and $\Pi N_{i-1} \cap V^{2n}M_i = V^{2n}M_i = \Pi V^{2n-1}M_i$, thus Π is injective. This shows, in particular, that the map $\Pi\colon \eta_{i-1,n} \to \eta_{i,n}$ induced on the ϕ-invariants is injective.

(2a) If $i-1$ is not critical, we have by (4.6) an isomorphism:

$$N_{i-1,n} \cong M_i/V^{2n}M_i$$

such that ϕ corresponds with $V^{-1}\Pi$, thus an isomorphism

$$\eta_{i-1,n} \cong (M_i/V^{2n}M_i)^{V^{-1}\Pi} \cong \eta_{i,n}.$$

(2b) If $i-1$ is critical, we have by (1) an isomorphism

$$\eta_{i-1,n} \cong (M_{i-1}/V^{2n-1}M_i)^{V^{-1}\Pi}.$$

[7] what?

Moreover the diagram

$$M_{i-1} \overset{\Pi}{\underset{V}{\rightrightarrows}} M_i \overset{\Pi}{\underset{V}{\rightrightarrows}} M_{i-1}$$

is identified with the diagram

$$\eta_{i-1} \otimes_{\mathcal{O}} \mathcal{W} \overset{\Pi}{\underset{\Pi\circ\sigma^{-1}}{\rightrightarrows}} \eta_i \otimes_{\mathcal{O}} \mathcal{W} \overset{\Pi}{\underset{\Pi\circ\sigma^{-1}}{\rightrightarrows}} \eta_{i-1} \otimes_{\mathcal{O}} \mathcal{W} .$$

The inclusions $\Pi V^{2n-1} M_i = V^{2n} M_i \subset \Pi M_{i-1} \subset M_i$ are σ-invariant and by tensoring with $\mathcal{W}$ we deduce inclusions $\pi^n \eta_i \subset \Pi \eta_{i-1} \subset \eta_i$. Thus

$$\eta_{i-1,n} \cong \Pi \eta_{i-1} / \pi^n \eta_i;$$

moreover we have in this case $[\eta_i \colon \Pi \eta_{i-1}] = 1$, so that

$$\eta_{i,n} / \Pi \eta_{i-1,n} \cong \mathcal{O}/\pi.$$

□

Remark 6.8. The preceding calculations show moreover that, for $j \in \{0, 1\}$ and $m \geq n$, the canonical maps $\eta_{j,m} \otimes_{\mathcal{O}} \mathcal{O}/\pi^n \to \eta_{j,n}$ are isomorphisms. Thus the projective system of $\eta_{j,n}$'s defines a *π-adic sheaf* η_j. Proposition (6.7) may be rewritten as:

Proposition 6.9. (1) *η_i is a smooth π-adic sheaf of free $\mathcal{O}$-modules of rank 2.*

(2) *η_{i-1} is a constructible π-adic sheaf of free $\mathcal{O}$-modules of rank 2 and $\Pi \colon \eta_{i-1} \to \eta_i$ is injective. Moreover:*

(2a) *η_{i-1} is smooth over $S - S_{i-1}$ and Π is an isomorphism over $S - S_{i-1}$.*

(2b) *η_{i-1} is smooth over S_{i-1} and $(\eta_i / \Pi \eta_{i-1})$ is smooth "en O/pi-vectoriels de rang 1" over S_{i-1}.*[8]

7. **Rigidification.**

7.1. Let B be a $\bar{k}$-algebra and X a special formal $\mathcal{O}_D$-module of height 4 over B. A *rigidification* of X is a quasi-isogeny $\rho \colon \Phi_B \to X$ of height 0, where Φ_B is obtained by basechange from the formal $\mathcal{O}_D$-module Φ over $\bar{k}$ chosen in (5.16).

By definition, an *isogeny* $\alpha \colon \Phi_B \to X$ is a homomorphism of formal $\mathcal{O}_D$-modules such that the kernel is representable by a finite group scheme which is locally free over B. We say that α is of height h if $\ker \alpha$ is of degree q^h over B.

Over $\bar{k}$, multiplication by π is an isogney of height 4 from Φ into itself; by basechange, the same is true for multiplication by π from Φ_B into itself. In particular, Φ_B is π-divisible and the $\mathcal{O}$-module $\mathrm{Hom}_{\mathcal{O}_D}(\Phi_B, X)$ is torsion-free ([Zi 3], 5.31).

By definition, a *quasi-isogney* $\rho \colon \Phi_B \to X$ is an element of $\mathrm{Hom}_{\mathcal{O}_D}(\Phi_B, X) \otimes_{\mathcal{O}} K$ such that $\pi^n \rho$ is an isogeny for n a sufficiently large integer. Note that, by preceding work, $\pi^n \rho$ determines ρ without ambiguity. We say that ρ is of height zero if $\pi^n \rho$ is of height $4n$.

One can show that a homomorphism $\alpha \colon \Phi_B \to X$ is an isogeny if and only if there exists an integer m and a homomorphism $\beta \colon X \to \Phi_B$ such that $\beta \circ \alpha = \pi^m$ ([Zi 3], Satz 5.25). Therefore, and element ρ of $\mathrm{Hom}_{\mathcal{O}_D}(\Phi_B, X) \otimes_{\mathcal{O}} K$ is a quasi-isogeny if and only if it admits an inverse in $\mathrm{Hom}_{\mathcal{O}_D}(X, \Phi_B) \otimes_{\mathcal{O}} K$.

We also note the following result of T. Zink ([Zi 3], Satz 5.15, or [Zi 1]): if B is noetherian, a homomorphism $\alpha \colon \Phi_B \to X$ is an isogeny of height h if and only if for

[8] again!

all prime ideals $\mathfrak{p}$ of B, the homomorphism $\alpha_{\overline{k}(\mathfrak{p})}: \Phi_{\overline{k}(\mathfrak{p})} \to X_{\overline{k}(\mathfrak{p})}$, obtained from α by restriction of scalars to an algebraic closure $\overline{k}(\mathfrak{p})$ of the residue field $k(\mathfrak{p}) = B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$, is an isogeny of height h.

Write $S = \operatorname{Spec}(B)$ and, for $j \in \{0,1\}$, let S_j be the closed subscheme of S above which j is critical for X.

Suppose first that there exists an index $i \in \{0,1\}$ such that $S = S_i$ schematically, or in other words that i is critical for X, and consider the π-adic sheaf η_j associated to X in section 6.

Proposition 7.2. *Suppose $S = S_i$ and X is rigidified. Then:*

(1) *The π-adic sheaf η_i is constant on S.*
(2) *The restrictions to $S - S_{i-1}$ and to S_{i-1} of the constructible π-adic sheaf η_{i-1} are constant.*
(3) *To each rigidifaction ρ of X is associated an isomorphism of sheaves $r: \underline{K}^2 \xrightarrow{\sim} \eta_0 \otimes_{\mathcal{O}} K$ such that $[\eta_j|_{S_j}: \Pi^j r\mathcal{O}^2] = j$ for $j \in \{0,1\}$.*

Proof. Recall that 0 is a critical index for Φ and choose an identification of $\eta_{\Phi,0}$ with $\mathcal{O}^2$. By the following lemma, the isogeny $\pi^n\rho: \Phi_B \to X$ induces a homomorphism of π-adic sheaves $\pi^n r: \underline{\mathcal{O}}^2 \to \eta_0$, thus a homomorphism $r: \underline{K}^2 \to \eta_0 \otimes_{\mathcal{O}} K$.

Lemma 7.3. *Let $\alpha: X \to X'$ be a homomorphism of special formal $\mathcal{O}_D$-modules of height 4 over S. Suppose that $i \in \{0,1\}$ (resp. j) is critical for X (resp. X') over S. Let η (resp. η') denote the corresponding π-adic $\mathbf{Z}/2\mathbf{Z}$-graded sheaf associated to X (resp. X') along with the filtration V^rM_i (resp. $V^rM'_j$). Then α induces a natural homomorphism of π-adic sheaves of η_i into η'_j.*

Proof. The problem is that the construction of ϕ, and therefore η, is not functorial in X; however this lemma establishes a partial functoriality and also the complete functoriality of $\eta \otimes_{\mathcal{O}} K$.

For all $n > 0$, we have following (4.3) isomorphisms

$$N_{i,n} \cong M_iVM_{i-1} \oplus M_i/V^{2n}M_i$$
$$N'_{j,n} \cong M'_j/VM'_{j-1} \oplus M'_j/V^{2n}M'_j$$

such that the endomorphism ϕ of the left side above corresponds with the map

$$(\overline{m}, \overline{m}'') \mapsto (\overline{m}'', V^{-1}\Pi\overline{m}'')$$

on the right side.

Write $\varepsilon = |i-j|$. The natural map of M_i in M'_j induced by $\Pi^\varepsilon\alpha$ commutes with V and Π; it therefore defines a map of $N_{i,n}$ into $N'_{j,n}$ which commutes with ϕ and hence a map of $\eta_{i,n}$ into $\eta'_{j,n}$. As n varies, these maps are compatible and thus define a homomorphism of η_i into η'_j.

When $j \neq i$, we show that the homomorphism factors through the natural injection $\Pi: \eta'_i \to \eta'_j$. This claim can be verified on stalks, so that we may suppose S is the spectrum of an algebraically closed field. There are then two cases:

a) i is not critical for X'. Then Π induces an isomorphism of η'_i in η'_j, so that there is nothing to prove.
b) i is critical for X'. In this case we have isomorphisms

$$N'_{i,n} \cong M'_i/VM'_{i-1} \oplus M'_i/V^{2n-1}M'_j$$

such that the endomorphism ϕ of the left side corresponds with the previously described map of the right side. Therefore the map of $N_{i,n}$ into $N'_{j,n}$

induced by $\Pi\alpha$ factors through the map induced by α of $N_{i,n}$ into $N'_{j,n}$ (note that $\alpha(V^{2n}M_i) \subset V^{2n}M'_i \subset V^{2n-1}M'_j$) followed by the map induced by Π of $N'_{i,n}$ into $N'_{j,n}$. These maps commute with ϕ and define maps $\eta_{i,n} \to \eta'_{i,n}$ and $\eta'_{i,n} \to \eta'_{j,n}$, whose compositions gives maps $\eta_{i,n} \to \eta'_{j,n}$ which are induced by $\Pi\alpha$. One obtains the desired factorisation by taking a projective limit in n.

□

Return now to the proof of Proposition (7.2). For every geometric point $\overline{s}$ of S, the homomorphism $r_{\overline{s}}$ induced on the fiber over $\overline{s}$ is an isomorphism, which follows by our study of the situation in (5.14) over algebraically closed fields; therefore $r\colon \underline{K}^2 \to \eta_0 \otimes_{\mathcal{O}} K$ is an isomorphism. In particular the sheaf $\eta_0 \otimes_{\mathcal{O}} K$ is constant; the same is true for $\eta_1 \otimes_{\mathcal{O}} K$, since $\Pi\colon \eta_0 \to \eta_1$ induces an isomorphism $\eta_0 \otimes_{\mathcal{O}} K \xrightarrow{\sim} \eta_1 \otimes_{\mathcal{O}} K$, since this is true for the fibers.

The π-adic sheaf η_i is smooth over S and $\eta_i \otimes_{\mathcal{O}} K$ is constant, thus η_i is constant. In fact, to prove this we may suppose that S is connected; then η_i corresponds to a representation of the fundamental group $\Pi_1(S, \overline{s})$ in $\eta_{i,\overline{s}} \cong \mathcal{O}^2$ and $\eta_i \otimes_{\mathcal{O}} K$ to the respresentation in K^2 obtained by tensoring; since this last one is trivial, the representation in $\mathcal{O}^2$ is as well.

Thus, since the restrictions of the π-adic sheaf η_{i-1} to $S - S_{i-1}$ and S_{i-1} are smooth and since $\eta_{i-1} \otimes_{\mathcal{O}} K$ is constant, these restrictions are also constant.[9]

Finally the properties of the isomorphism $r\colon \underline{K}^2 \to \eta_0 \otimes_{\mathcal{O}} K$ relative to the constant sheaves $\eta_j|_{S_j}$ hold for the fibers above a geometric point, as we established in (5.15).

□

Remark 7.4. We note that the construction of the sheaves η_j and the isomorphism r associated to (X, ρ) commute with arbitrary basechange from B to a B-algebra B'. We have seen this in (6.6) with respect to η_j and the lemma proves the result for r.

We have implicitely used properties of fields in the proof when we reduced to the case of an algebraically closed field to calculate the fibers.

Consider now the general case (that is, no longer suppose $S = S_i$) and regard η as a functor of B-algebras (and no longer as a π-adic sheaf).

Proposition 7.5. *Suppose X is rigidified. Then for $j \in \{0, 1\}$:*

(1) *η_j is a constructible sheaf for the Zariski topology on S, of free $\mathcal{O}$-modules of rank 2.*
(2) *The restriction of η_j to S_j is a constant sheaf.*
(3) *To a rigidification ρ of X is associated an isomorphism of sheaves $r\colon \underline{K}^2 \xrightarrow{\sim} \eta_0 \otimes_{\mathcal{O}} K$ such that $[\eta_j|_{S_j} : \Pi^j r\mathcal{O}^2] = j$.*

Moreover the formation of (n, r) from (X, ρ) commutes with arbitrary basechange by B-algebras B'.

Proof. When one of the indices $i \in \{0, 1\}$ is critical over all of S, the analogous assertions for the π-adic sheaves η_j follow by (7.2), given the fact that $\eta_j(B') = \varprojlim \eta_{j,n}(B')$ for all B-algebras B' (6.3). In particular (7.5.2) follows by (7.2.1).

Thanks to the following lemma, the constructibility of η in the general case is deduced by reducing to the case where B is reduced, which does not change η (3.14), and then from the case when B is integral; for then one of the indices is critical over all of S, and one concludes by gluing irreducible components:

[9] Not sure about this one

Lemma 7.6. *Let I_1 and I_2 be two ideals of B such that $I_1 \cap I_2 = 0$. Let i_1, i_2 and i_{12} be closed immersions of the subschemes of S defined by I_1, I_2 and $I_1 + I_2$, respectively. Then we have an exact sequence*

$$0 \to \eta \to i_{1*}i_1^*\eta \oplus i_{2*}i_2^*\eta \to i_{12*}i_{12}^*\eta$$

of presheaves on S.

Proof. For every B-algebra B', let $B'_1 = B'/I_1B'$, $B'_2 = B'/I_2B'$ and $B'_{12} = B'/(I_1 + I_2)B'$. From the exact sequence:

$$\begin{aligned} 0 \to B' \to B'_1 \times B'_2 &\to B'_{12} \to 0 \\ (a, b) &\mapsto a - b \end{aligned}$$

we obtain an exact sequence of Cartier modules:

$$0 \to M_{B'} to M_{B'_1} \times M_{B'_2} \to M_{B'_{12}} \to 0$$

and an exact sequence of modified Cartier modules:

$$0 \to N(M_{B'}) \to N(M_{B'_1}) \times N(M_{B'_2}) \to N(M_{B'_{12}}) \to 0.$$

By taking the kernel of $1 - \phi$, we obtain an exact sequence:

$$0 \to \eta(B') \to \eta(B'_1) \times \eta(B'_2) \to \eta(B'_{12})$$

which proves the lemma. □

Finally note that the isomorphism $r\colon \underline{K}^2 \to \eta_0 \otimes_{\mathcal{O}} K$ associated to the rigidification ρ given by lemma (7.3) does not depend on the choice of $i \in \{0, 1\}$ when both indices 0 and 1 are critical; therefore the isomorphisms $r|_{S_0}$ and $r||_{S_1}$ can be glued to make an isomorphism r defined on all of S. □

8. **Drinfeld's theorem.** Recall that we have fixed an algebraic closure $\overline{k}$ of k, a special formal $\mathcal{O}_D$-module Φ of height 4 over $\overline{k}$ such that 0 is critical for Φ and an isomorphism $\mathcal{O}^2 \cong \eta_{\Phi,0}$. We write $\mathcal{O}^{nr}$ for the *strict henselization* (that is, maximal unramified extension) of $\mathcal{O}$ with residue field $\overline{k}$.

Definition 8.1. Let $\overline{\textbf{Nilp}}$ denote the category of $\mathcal{O}^{nr}$-algebras such that the image of π is nilpotent. We define a functor $\overline{G}$ on $\overline{\textbf{Nilp}}$ which associates to $B \in \mathrm{Ob}\,\overline{\textbf{Nilp}}$ the set $\overline{G}(B)$ of isomorphism classes of pairs (X, ρ) consisting of:

1) a special formal $\mathcal{O}_D$-module X of height 4 over B.
2) a quasi-isogeny $\rho\colon \Phi_{B/\pi B} \to X_{B/\pi B}$ of height zero.

However in this definition, it is convenient to take a more general definition of formal $\mathcal{O}_D$-module: we ask only that $\mathrm{Lie}(X)$ is a projective B-module. Locally for the Zariski topology on B, we recover the formal $\mathcal{O}_D$-modules defined in 2.2.

Drinfeld's fundamental result is the following:

Theorem 8.2. *The functor $\overline{G}$ is represented by the formal $\widehat{\mathcal{O}}^{nr}$-scheme $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{\mathrm{nr}}$.*

Definition 8.3. Let **Nilp** denote the category of $\mathcal{O}$-algebras such that the image of π is nilpotent. We define a functor G on **Nilp** which associates to $B \in \mathrm{Ob}\,\textbf{Nilp}$ the set $G(B)$ of pairs made up of

1) a k-homomorphism $\psi\colon \overline{k} \to B/\pi B$
2) an isomorphism class of pairs (X, ρ) consisting of:

(2.1) a special formal $\mathcal{O}_D$-module X of height 4 over B.

(2.2) a quasi-isogeny $\rho\colon \psi_*\Phi \to X_{B/\pi B}$ of height 0.

If $B \to B'$ is a morphism in **Nilp**, we define a map $G(B) \to G(B')$ in the obvious way by associating to ψ its composition with $B/\pi B \to B'/\pi B'$ and to the pair (X, ρ) the pair $(X_{B'}, \rho_{B'/\pi B'})$ obtained by extension of scalars from B to B'.

The functor G is none other than the functor obtained from $\overline{G}$ by restriction of scalars from $\mathcal{O}^{nr}$ to $\mathcal{O}$. Indeed, it is the same to give a k-homomorphism $\psi\colon \overline{k} \to B/\pi B$ or an $\mathcal{O}$-homomorphism $\widetilde{\psi}\colon \mathcal{O}^{nr} \to B$ making B an $\mathcal{O}^{nr}$-algebra $B_\psi \in \operatorname{Ob} \overline{\textbf{Nilp}}$, and an element of $G(B)$ corresponds to a couple consisting of $\widetilde{\psi}$ and an element of $\overline{G}(B_\psi)$.

From theorem (8.2), we thus obtain the following theorem by restriction of scalars:

Theorem 8.4. *The functor G is representable by the formal $\mathcal{O}$-scheme $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{\mathrm{nr}}$.*

Write $\overline{H}$ for the restriction to the category $\overline{\textbf{Nilp}}$ of the functor F on **Nilp** defined in (I.5.1). We saw in section (I.5.2) that F is representable by the formal $\mathcal{O}$-scheme $\widehat{\Omega}$; it follows that $\overline{H}$ is representable by the formal $\widehat{\mathcal{O}}^{nr}$-scheme $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{\mathrm{nr}}$.

For $B \in \operatorname{Ob} \overline{\textbf{Nilp}}$, we define a map $\overline{\xi}_B\colon \overline{G}(B) \to \overline{H}(B)$ which associates to a pair (X, ρ) the quadruple $(\eta_X, T_X, u_X, r_{(X,\rho)})$ where, if M is the Cartier module of X over B, we have:

1) $\eta_X = \eta_M$ viewed as a sheaf on $\operatorname{Spec}(B)$: if B' is a B-algebra and if $M' = M_{B'}$, we have $\eta_X(B') = \eta_{M'}$;
2) $T_X = \operatorname{Lie}(X) = M/VM$;
3) $U_X\colon \eta_X \to T_X$ is the sheaf homomorphism such that $u_X(B') = u_{M'}\colon \eta_{M'} \hookrightarrow N(M') \to M'/VM' = (M/VM) \otimes_B B'$ (cf. (3.13));
4) $r_{(X,\rho)}\colon \underline{K}^2 \xrightarrow{\sim} \eta_{X,0}$ is the isomorphism associated to the rigidification of X.

The propositions (7.5), (5.5) and (5.6) show that the quadruple $(\eta_X, T_X, u_X, r_{(X,\rho)})$ satisfies the conditions of definition (I.5.1) and thus yields a well-defined element of $\overline{H}(B) = F(B)$.

Moreover, if $B \to B'$ is a morphism in $\overline{\textbf{Nilp}}$, the diagram:

$$\begin{tikzcd} \widehat{G}(B) \arrow[r, "\overline{\xi}_B"] \arrow[d] & \overline{H}(B) \arrow[d] \\ \widehat{G}(B') \arrow[r, "\overline{\xi}_{B'}"'] & \overline{H}(B') \end{tikzcd}$$

is commutative. This follows from the fact that the construction of η_X and $r_{(X,\rho)}$ from (X, ρ) commutes with basechange (cf. (6.6) and (7.4)). Thus $\overline{\xi}_B$ defines a natural transformation $\overline{\xi}\colon \overline{G} \to \overline{H}$.

Theorem (8.2) follows from the precise statement:

Theorem 8.5. *The natural transformation $\overline{\xi}\colon \overline{G} \to \overline{H}$ is an isomorphism of functors.*

We will prove this theorem in subsections 10 through 12.

Let H denote the functor on **Nilp** obtained from $\overline{H}$ by restriction of scalars from $\mathcal{O}^{nr}$ to $\mathcal{O}$. For $B \in \operatorname{Ob} \textbf{Nilp}$, an element of $H(B)$ consists of a k-homomorphism $\psi\colon \overline{k} \to B/\pi B$ and an element of $\overline{H}(B_\psi) = F(B)$. It is clear that H is representable by the formal $\mathcal{O}$-scheme $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{\mathrm{nr}}$.

We define a natrual transformation $\xi\colon G \to H$ by restriction of scalars, which associates to a pair (ψ, a), where $a \in \overline{G}(B_\psi)$, the pair $(\psi, \overline{\xi}_{B_\psi}(a))$. Theorem (8.5) implies:

Theorem 8.6. *The natural transformation $\xi\colon G \to H$ is an isomorphism of functors.*

We take care to note that the map $\overline{\xi}_{B_\psi}\colon \overline{G}(B_\psi) \to \overline{H}(B_\psi) = F(B)$ depends on the $\mathcal{O}^{nr}$-algebra structure of B_ψ, and in particular the $\mathbf{Z}/2\mathbf{Z}$-grading of η_X and T_X associated to X depends on its structure as an $\mathcal{O}'$-algebra.[10]

9. Action of the groups $\mathrm{GL}_2(K)$ and D^*.

9.1. *Frobenius morphism.* We write $\mathrm{Fr}\colon \overline{k} \to \overline{k}$ for the Frobenius homomorphism $\mathrm{Fr}(x) = x^q$ and $\mathrm{Frob}\colon \mathrm{Fr}_*^{-1}\Phi \to \Phi$ for the Frobenius morphism. The latter is a $\overline{k}$-morphism of formal $\mathcal{O}_D$-modules from the formal $\mathcal{O}_D$-module $\mathrm{Fr}_*^{-1}\Phi$ obtained from Φ by extension of scalars via Fr^{-1} (sometimes written $\Phi^{q^{-1}}$) into Φ. It is an isogeny of height 2 (equal to the dimension of Φ).

If M_Φ is the graded Cartier $\mathcal{O}[\Pi]$-module of Φ over $\overline{k}$, then Fr_*^{-1} is identified with $M_\Phi^\sigma[1]$, which follows by restricting scalars of M_Φ via $\sigma\colon \mathcal{W}_\mathcal{O}(\overline{k}) \to \mathcal{W}_\mathcal{O}(\overline{k})$ and shifting the grading (since the action of $\mathcal{O}'$ via $\mathcal{O}_D$ is unchanged). The Frobenius morphism corresponds to the $\mathcal{W}_\mathcal{O}(\overline{k})$-linear homomorphism $V\colon M_\Phi^\sigma[1] \to M_\Phi$ of degree 0.

Since 0 is critical for Φ, 1 is critical for $\mathrm{Fr}_*^{-1}\Phi$ and the identification $M_{\mathrm{Fr}_*^{-1}\Phi} = M_\Phi^\sigma[1]$ induces an identification:

$$\eta_{\mathrm{Fr}_*^{-1}\Phi,1} = M^{V^{-1}\Pi}_{\mathrm{Fr}_*^{-1}\Phi,1} = M^{V^{-1}\Pi}_{\Phi,0} = \eta_{\Phi,0}$$

and thus an identification $\eta_{\mathrm{Fr}_*^{-1}\Phi} \otimes_\mathcal{O} K = (\eta_\Phi \otimes_\mathcal{O} K)[1]$. The Frobenius morphism thus corresponds to the K-linear isomorphism $\Pi\colon (\eta_\Phi \otimes_\mathcal{O} K)[1] \to \eta_\Phi \otimes_\mathcal{O} K$ of degree 0.

9.2. *Action of $\mathrm{GL}_2(K)$ on the functor G.* Let v denote the valuation of K normalized by $v(\pi) = 1$. Via the identification (5.14): $\mathrm{GL}_2(K) = \mathrm{GL}(\eta_{\Phi,0} \otimes_\mathcal{O} K) = (\mathrm{End}^0_{\mathcal{O}_D}\Phi)^*$, an element g of $\mathrm{GL}_2(K)$ defines a quasi-isogeny of Φ of height $2n$, where $v(\det g) = n$. Thus $g^{-1} \circ \mathrm{Frob}^n\colon \mathrm{Fr}_*^{-n}\Phi \to \Phi$ is a quasi-isogeny of height 0. We define an action of $\mathrm{GL}_2(K)$ on the functor G by putting, for $B \in \mathrm{Ob}\,\mathbf{Nilp}$ and $(\psi; X, \rho)$ representing an element of $G(B)$:

$$g \cdot (\psi; X, \rho) = (\psi \circ \mathrm{Fr}^{-n}; X, \rho \circ \psi_*(g^{-1} \circ \mathrm{Frob}^n)).$$

We write $\widetilde{\mathrm{Fr}}\colon \mathcal{O}^{nr} \to \mathcal{O}^{nr}$ for the lifting of the k-homomorphism $\mathrm{Fr}\colon \overline{k} \to \overline{k}$ to an $\mathcal{O}$-homomorphism.

Theorem 9.3. *The action of $\mathrm{GL}_2(K)$ on the functor G corresponds with the action on the formal scheme $\widehat{\Omega}\widehat{\otimes}_\mathcal{O}\widehat{\mathcal{O}}^{\mathrm{nr}}$ defined by the natural action of $\mathrm{PGL}_2(K)$ on $\widehat{\Omega}$ and the action $g \mapsto \widetilde{\mathrm{Fr}}^{-v(\det g)}$ on $\mathcal{O}^{nr}$.*

Proof. After (I.6.2), the latter action is described on elements of $H(B)$ by:

$$g \cdot (\psi; \eta, T, u, r) = (\psi \circ \mathrm{Fr}^{-n}; \eta[n], T[n], u[n], \Pi^n r g^{-1}).$$

It thus suffices to verify, if $\overline{\xi}_{B_\psi}(X, \rho) = (\eta, T, u, r)$, that we have:

$$\overline{\xi}_{B_{\psi\circ\mathrm{Fr}^{-n}}}(X, \rho \circ \psi_*(g^{-1} \circ \mathrm{Frob}^n)) = (\eta[n], T[n], u[n], \Pi^n r g^{-1}).$$

The shift in the grading of (η, T, u) associated to X depends on whether we are using the $\overline{k}$-algebra structure of B_ψ or $B_{\psi\circ\mathrm{Fr}^{-n}}$, which changes the action of $\mathcal{O}'$ on M_X via $W_\mathcal{O}(\overline{k})$ by σ^n, while the action of $\mathcal{O}'$ via $\mathcal{O}_D$ remains unchanged.[11]

---

[10]This should be cleaned up!

[11]Not too sure about this paragraph

To calculate the rigidification, we will write

$$g^{-1}\colon \eta_\Phi \otimes_{\mathcal{O}} K \xrightarrow{\sim} \eta_\Phi \otimes_{\mathcal{O}} K \quad \text{and} \quad r\colon \psi_*(\eta_\Phi \otimes_{\mathcal{O}} K) \xrightarrow{\sim} \eta \otimes_{\mathcal{O}} K$$

for the isomorphism defined, in light of the identification $K^2 = \eta_{\Phi,0} \otimes_{\mathcal{O}} K$, by g^{-1} and r on the components of degree 0 and extended by conjugation by Π to the components of degree 1, so that we have $g^{-1} \circ \Pi = \Pi \circ g^{-1}[1]$ and $r \circ \Pi = \Pi \circ r[1]$.

Since the quasi-isogeny

$$g^{-1} \circ \mathrm{Frob}^n\colon\ \mathrm{Fr}_*^{-n} \Phi \to \Phi$$

corresponds via $\overline{\xi}_{\overline{k}}$ with:

$$(\eta_\Phi \otimes_{\mathcal{O}} K)[n] \xrightarrow{\Pi^n} \eta_\Phi \otimes_{\mathcal{O}} K \xrightarrow{g^{-1}} \eta_\Phi \otimes_{\mathcal{O}} K,$$

the quasi-isogeny

$$\rho \circ \psi_*(g^{-1} \circ \mathrm{Frob}^n)\colon \psi_* \circ \mathrm{Fr}_*^{-n} \Phi \to X$$

correpsonds via $\overline{\xi}_{B_\psi}$ with:

$$\psi_*(\eta_\Phi \otimes_{\mathcal{O}} K)[n] \xrightarrow{\Pi^n} \psi_*(\eta_\Phi \otimes_{\mathcal{O}} K) \xrightarrow{g^{-1}} \psi_*(\eta_\Phi \otimes_{\mathcal{O}} K) \xrightarrow{r} \eta \otimes_{\mathcal{O}} K,$$

or, by switching Π^n with g^{-1} and r and shifting the grading by n, via $\overline{\xi}_{B_{\psi \circ \mathrm{Fr}^{-n}}}$ with:

$$\psi_*(\eta_\Phi \otimes_{\mathcal{O}} K) \xrightarrow{g^{-1}} \psi_*(\eta_\Phi \otimes_{\mathcal{O}} K) \xrightarrow{r} \eta \otimes_{\mathcal{O}} K \xrightarrow{\Pi^n} (\eta \otimes_{\mathcal{O}} K)[n].$$

The result is obtained from above by taking degree 0 components and composing with the fixed identification $K^2 = \eta_{\Phi,0} \otimes_{\mathcal{O}} K$. □

9.4. *Action of D^* on the functor G.* Let $N_{D/K}\colon D^* \to K^*$ denote the reduced norm: every element of D^* can be written as $g = \Pi^n g_0$ with $g_0 \in \mathcal{O}_D^*$ and $n = v(N_{D/K} g)$.

For $g \in D^*$, write ${}^gX$ for the formal $\mathcal{O}_D$-module which is equal to X as an $\mathcal{O}$-module, but where the action of $a \in \mathcal{O}_D$ on ${}^gX$ is equal to the action of $g^{-1}ag$ on X.

The action of $\mathcal{O}_D$ on Φ associates to g^{-1} an $\mathcal{O}_D$- equivariant quasi-isogeny $g^{-1}\colon \Phi \to {}^gX$ of height $-2n$, if $v(N_{D/K}g) = n$. Thus $g^{-1} \circ \mathrm{Frob}^n\colon\ \mathrm{Fr}_*^{-1} \Phi \to {}^g\Phi$ is an $\mathcal{O}_D$-equivariant quasi-isogeny of height 0. We define an action of D^* on the functor G by putting, for $B \in \mathrm{Ob}\,\mathbf{Nilp}$ and $(\psi; X, \rho)$ a representative of an element of $G(B)$:

$$g \cdot (\psi; X, \rho) = (\psi \circ \mathrm{Fr}^{-n};\ {}^gX, \rho \circ \psi_*(g^{-1} \circ \mathrm{Frob}^n)).$$

Theorem 9.5. *The action of D^* on the functor G corresponds with the action of D^* on the formal scheme $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{\mathrm{nr}}$ defined by $g \mapsto \widetilde{\mathrm{Fr}}^{-v(N_{D/K}g)}$ over $\mathcal{O}^{nr}$.*

Proof. Note first that $\mathcal{O}_D^*$ acts trivially since, if $g \in \mathcal{O}_D^*$, the map $g^{-1}\colon X \to {}^gX$ is an isomorphism of (X, ρ) onto $({}^gX, \rho \circ \psi_*(g^{-1}))$.

It remains to verify that the action of Π on G corresponds with $\widetilde{\mathrm{Fr}}^{-1}$ on $\mathcal{O}^{nr}$; in other words that, if $\overline{\xi}_{B_\psi}(X, \rho) = (\eta, T, u, r)$, we have:

$$\overline{\xi}_{B_{\psi \circ \mathrm{Fr}^{-1}}}({}^\Pi X, \rho \circ \psi_*(\Pi^{-1} \circ \mathrm{Frob})) = (\eta, T, u, r).$$

Let M_X and M_{Π_X} be the graded Cartier $\mathcal{O}[\Pi]$-modules over B_ψ associated to X and ${}^\Pi X$. As M_{Π_X} coincides with M_X as $W_{\mathcal{O}}(\overline{k})[V, \Pi]$-modules [12], but the action of $a \in \mathcal{O}'$ via $\mathcal{O}_D$ on M_{Π_X} is identified with the action of $\Pi^{-1}a\Pi = \sigma(a)$ on M_X, we have $M_{\Pi_X} = M_X[1]$. Thus the triple over B associated to ΠX is $(\eta, T, u)[1]$ if B is equipped

[12]should this be $\mathcal{W}$?

with the same $\overline{k}$-algebra structure as B_ψ, and (η, T, u) if B is equipped with the same $\overline{k}$-algebra structure as $B_{\psi\circ\mathrm{Fr}^{-1}}$.

The quasi-isogeny

$$\Pi^{-1}\circ\mathrm{Frob}\colon\ \mathrm{Fr}_*^{-1}\Phi\to\ {}^{\Pi}\Phi$$

correponds via $\overline{\xi}_{\overline{k}}$ with:

$$(\eta_\Phi\otimes_{\mathcal{O}}K)[1]\xrightarrow{\Pi}\eta_\Phi\otimes_{\mathcal{O}}K\xrightarrow{\Pi^{-1}}\eta_\Phi\otimes_{\mathcal{O}}K[1],$$

that is to say with $\mathrm{id}_{\eta_\Phi\otimes_{\mathcal{O}}K[1]}$. Therefore the quasi-isogeny

$$\rho\circ\psi_*(\Pi^{-1}\circ\mathrm{Frob})\colon\ \psi_*\,\mathrm{Fr}_*^{-1}\Phi\to\ {}^{\Pi}X$$

corresponds via $\overline{\xi}_{B_\psi}$ with

$$r[1]\colon\ \psi_*(\eta_\Phi\otimes_{\mathcal{O}}K)[1]\to\eta[1]$$

and via $\overline{\xi}_{B_{\psi\circ\mathrm{Fr}^{-1}}}$ with r. □

10. **Deformation theory.** Here we work in the category $\overline{\mathbf{Nilp}}$ of $\mathcal{O}^{nr}$-algebras such that the image of π is nilpotent. We call a surjective homomorphism $B'\to B$ with nilpotent kernel an *infinitesimal extension*.

Proposition 10.1. *Let $B'\to B$ be an infinitesimal extension. Let $B'_0=B'/\pi B'$ and $B_0=B/\pi B$. Let X' be a special formal $\mathcal{O}_D$-module of height 4 over B' and $X=X'_B$. Suppose X is equipped with a rigidification ρ, which is to say a quasi-isogeny $\rho\colon\Phi_{B_0}\to X_{B_0}$ of height 0. Then:*

(i) *X' is π-divisible (that is, $\pi\colon X'\to X'$ is an isogney).*
(ii) *ρ lifts in a unique way to a rigidification of X', which is to say a quasi-isogeny $\rho'\colon\Phi_{B'_0}\to B'_0$.*

Proof. (i) Let n be such that $\alpha=\pi^n\rho$ is an isogeny. Since Φ_{B_0} is π-divisible, $\alpha\circ\pi$ is also an isogeny. But $\alpha\circ\pi=\pi\circ\alpha$, thus $\pi\colon X_{B_0}\to X_{B_0}$ is also an isogeny ([Zi 3], 5.10). By deformation, the same is true for $\pi\colon X'\to X'$ ([Zi 3], 5.12).

(ii) By induction, we may suppose that $I=\ker(B'\to B)$ is of square zero. The isogeny $\alpha=\pi^n\rho\colon\Phi_{B_0}\to X_{B_0}$ does not always lift, but $\beta=\pi\alpha=\pi^{n+1}\rho$ does always lift ([Zi 3], 4.47) to an isogeny $\beta'\colon\Phi_{B'_0}\to X'_{B_0}$; thus ρ lifts to $\rho'=\beta/\pi^{n+1}$. Moreover this lifting is unique by the rigidity of p-divisible groups ([Zi 3], 5.30). □

We may thus ignore rigidification to study deformation theory. Moreover, there do not exist infinitesimal automorphisms, by rigidity.

Proposition 10.2. *Let $B'\to B$ and $B''\to B$ be two infinitesimal extensions. Then the canonical map*

$$\overline{G}(B'\times_B B'')\to\overline{G}(B')\times_{\overline{G}(B)}\overline{G}(B'')$$

is bijective.

Proof. One can prove this by making the obvious changes to the usual proof in the case of p-divisible groups ([Zi 3], 5.40). Let X' and X'' be deformations over B' and B'', respectively, of X over B. There exists a unique simultaneous deformation $\widetilde{X}$ of X' and X'' over $B'\times_B B''$; its graded Cartier $\mathcal{O}[\Pi]$-module is $M_{\widetilde{X}}=M_{X'}\times_{M_X}M_{X''}$. □

For $x\in\overline{G}(B)$ and $C\to B$ an infinitesimal extension, we write $\overline{G}_x(C)$ the the inverse image of x in $\overline{G}(C)$ and $\overline{H}_x(C)$ for the inverse image of $\overline{\xi}(x)$ in $\overline{H}(C)$.

Corollary 10.3. *Let $B' \to B$ an an infinitesimal extension with a kernel of square zero, and let $B[I]$ denote the B algebra $B \oplus I$ with $I^2 = 0$. Then:*

(i) *$\overline{G}_x(B[I])$ is an abelian group,*
(ii) *if $\overline{G}_x(B')$ is nonempty, it is a principal homogeneous $\overline{G}_X(B[I])$-set.*

These structures are canonical.

Proof. This is a consequence of a classical result, since $\overline{G}$ commutes with fibre products, which is usually stated for local artinian rings, but is true in the present context (cf. Schlessinger [Sc], Artin [A]). The group structure is defined by the homomorphism

$$B[I] \times_B B[I] \to B[I]$$
$$(b+i) \times_b (b+j) \mapsto b+i+j$$

which yields, in view of (10.2), a map:

$$\overline{G}_x(B[I]) \times \overline{G}_x(B[I]) \to \overline{G}_x(B[I]).$$

The isomorphism

$$B' \times_B B' \xrightarrow{\sim} B' \times_B B[I]$$
$$a \times_b c \mapsto a \times_b b + (c-a)$$

gives a bijection

$$\overline{G}_x(B') \times \overline{G}_x(B') \xrightarrow{\sim} \overline{G}_x(B') \times \overline{G}_x(B[I])$$

inducing the identity on the first factors; this describes the structure of principal homogeneous space if $\overline{G}_x(B') \neq \emptyset$. □

The functor $\overline{H}$, being representable, also commutes with fibre products of infinitesimal extensions; thus, under the hypotheses of (10.3), the set $\overline{H}_x(B[I])$ has the structure of a group and $\overline{H}_x(B')$ has the structure of a principal homogeneous space over $\overline{H}_x(B[I])$ whenever $\overline{H}_x(B')$ is nonempty. It follows from the definitions of these structures that the maps $\overline{\xi}_{B[I]}\colon \overline{G}_x(B[I]) \to \overline{H}_x(B[I])$ and $\overline{\xi}_{B'}\colon \overline{G}_x(B') \to \overline{H}_x(B')$ are compatibles.

Proposition 10.4. *If $\overline{H}_x(B') \neq \emptyset$, then $\overline{G}_x(B') \neq \emptyset$.*

Proof. Let (X, ρ) represent $x \in \overline{G}(B)$ and let M be the graded Cartier $\mathcal{O}[\Pi]$-module over B associated to X. One reduces easily by localisation to the case where M/VM is a free B-module. In this case let (γ_0, γ_1) be a homogeneous V-basis for M and let

$$\Pi\gamma_i = [a_{0,i}]\gamma_{\overline{i+1}} + \sum_{m>0} V^m[a_{m,i}]\gamma_{\overline{m+i+1}} \quad (i = 0, 1)$$

be the equations defining M. To lift X to X' over B', it suffices (2.3) to lift the $a_{m,i} \in B$ to some $a'_{m,i} \in B'$ satisfying $a'_{0,0} \cdot a'_{0,1} = \pi$.

The functor $\overline{\xi}$ associates to (X, ρ) the isomorphism class of $T = M/VM$ equipped with Π. The image $(\overline{\gamma_0}, \overline{\gamma_1})$ of (γ_0, γ_1) is a homogeneous basis for T such that $\Pi\overline{\gamma}_0 = a_{0,0}\overline{\gamma}_1$ and $\Pi\overline{\gamma}_1 = a_{0,1}\overline{\gamma}_0$. If $\overline{H}_x(B')$ is nonempty, then there exists (T', Π) over B' lifting (T, Π), since there then exist $a'_{0,0}$ and $a'_{0,1} \in B'$ lifting $a_{0,0}$ and $a_{0,1}$ such that $a'_{0,0} \cdot a'_{0,1} = \pi$. □

We are now able to establish the essential result of this subsection:

Proposition 10.5. *To show that $\overline{\xi}\colon \overline{G} \to \overline{H}$ is an isomorphism, it suffices to show that its restriction to the category of $\overline{k}$-algebras is an isomorphism.*

Proof. Let $\overline{\mathbf{Nilp}}_n$ denote the full subcategory of $\overline{\mathbf{Nilp}}$ of $\mathcal{O}^{nr}$-algebras such that the image of π^n is zero. In particular, $\overline{\mathbf{Nilp}}_1$ is the category of $\overline{k}$-algebras. Suppose by induction that $\overline{\xi}|_{\overline{\mathbf{Nilp}}_n}$ is an isomorphism. Let $B' \in \operatorname{Ob}\overline{\mathbf{Nilp}}_{n+1}$ and $B = B'/\pi^n B'$, $I = \pi^n B'$. Then B and $B[I]$ are objects in $\overline{\mathbf{Nilp}}_n$, and so $\overline{\xi}_B$ and $\overline{\xi}_{B[I]}$ are bijective. It follows from the material above that $\overline{\xi}_{B'}$ is bijective. □

11. **Tangent spaces.** Let $\overline{k}[\varepsilon]$ with $\varepsilon^2 = 0$ denote the dual numbers. For $x \in \overline{G}(\overline{k})$, the tangent space to $\overline{G}$ at x is $t_{\overline{G}}(x) = \{x' \in \overline{G}(\overline{k}[\varepsilon])$ mapping to $x \in \overline{G}(\overline{k})\}$. It follows from proposition (10.2) that $t_{\overline{G}}(x)$ is a $\overline{k}$-vectorspace in a canonical way, and that the tangent map $t_{\overline{\xi}}(x)\colon t_{\overline{G}}(x) \to t_{\overline{H}}(\overline{\xi}(x))$ induced by $\overline{\xi}$ is $\overline{k}$-linear. Our goal in this section is to prove the following

Proposition 11.1. *For every $x \in \overline{G}(\overline{k})$, the tangent map $t_{\overline{\xi}}(x)\colon t_{\overline{G}}(x) \to t_{\overline{H}}(\overline{\xi}(x))$ is bijective.*

To prove the proposition, we will both need to calculate the tangent space $t_{\overline{G}}(x)$, as well as identify the tangent map $t_{\overline{\xi}}(x)$ to show that it is injective.

Let (X, ρ) be a representative of x and let M be the graded Cartier $\mathcal{O}[\Pi]$-module of X over $\overline{k}$. By (10.1), a deformation of (X, ρ) is simply a deformation of X, or in other words a deformation of M. Thus:

$$t_{\overline{G}}(x) = \{\text{deformations of } M \text{ to a graded Cartier } \mathcal{O}[\Pi]\text{-module over } \overline{k}[\varepsilon]\}.$$

11.2. We first recall how to calculate the deformations of Cartier modules of p-divisible groups (cf. [No], [Zi 3] 5.41). If M' is a deformation of M over $\overline{k}[\varepsilon]$, then there is an exact sequence:

$$0 \to M_\varepsilon \to M' \to M \to 0,$$

where $M_\varepsilon = \bigoplus_{i=0}^{\infty} V^i[\varepsilon](M/VM)$, since $[\varepsilon]V = V[\varepsilon^q] = 0$.

This exact sequence splits, with a section of M being given by

$$\widetilde{M} = \{m \in M' \mid \text{there exists } l \text{ with } V^l m \in FM' \oplus \bigoplus_{i=0}^{l-1} V^i[\varepsilon](M/VM)\}.$$

This lifting of M to $\widetilde{M}$ extends the obvious lifting of FM to FM' with $FM' \cap M_\varepsilon = \{0\}$; it is obtained by noting that the action of V on M/FM is nilpotent.

The splitting thus defined is $W_{\mathcal{O}}(\overline{k})[F]$-equivariant; on the other hand, writing V' for the action of V on M', we have $V'\widetilde{M} \subset \widetilde{M} \oplus [\varepsilon](M/VM)$. The structure of M' is determined by the $\overline{k}$-linear map $\beta\colon VM/\pi M \to M/Vm$ such that $V'm = Vm + [\varepsilon]\beta(Vm)$ for $m \in \widetilde{M}$. Conversely, such a map uniquely determines a deformation M' of M.

Lemma 11.3. *The tangent space $t_{\overline{G}}(x)$ is canonically identified with the space of $\overline{k}$-linear maps $\beta\colon VM/\pi M \to M/VM$ of degree zero and such that $\beta\Pi = \Pi\beta$. To such a map corresponds the module $M' = M \oplus M_\varepsilon$ where $V'(m, 0) = (Vm, [\varepsilon]\beta(Vm))$.*

Proof. In this case M and M' are equipped with an action of $\mathcal{O}_D$, or equivalently with a grading and an action of Π, and the exact sequence above is compatible with this action. Since the action of $\mathcal{O}_D$ commutes with $W_{\mathcal{O}}(\overline{k}[\varepsilon])[F, V]$, we have $\mathcal{O}_D \cdot \widetilde{M} \subset$

$\widetilde{M}$; in other words the splitting is compatible with the grading and the action of Π. Otherwise it is determined over M_ε by its value on M/VM. Thus the splitting $M' = \widetilde{M} \oplus M_\varepsilon$ determines a grading and action of Π on M' depending on those of M. Moreover M' is a graded Cartier $\mathcal{O}[\Pi]$-module over $\overline{k}[\varepsilon]$ if and only if V' is of degree 1 and commutes with Π, that is if β is of degree 0 and commutes with Π. $\square$

Lemma 11.4. *(i) If M has a single critical index then the tangent space $t_{\overline{G}}(x)$ is one dimensional. More precisely, if i is critical and $i+1$ non-critical, then we neccessarily have $\beta_{i+1} = 0$ and $t_{\overline{G}}(x)$ is identified with the space of $\overline{k}$-linear maps $\beta_i \colon VM_{i+1}/\pi M_i \to M_i/VM_{i+1}$.*

(ii) If M has two critical indices, then the tangent space $t_{\overline{G}}(x)$ is two dimensional. More precisely, $t_{\overline{G}}(x)$ is identified with the space of pairs of $\overline{k}$-linear maps $\beta_i \colon VM_{i+1}/\pi M_i \to M_i/VM_{i+1}$ ($i = 0, 1$).

Proof. (cf. [Zi 2], 3.10). The following conditions are equivalent:

a) the index i is critical for M,
b) the map $\Pi \colon M_i/VM_{i+1} \to M_{i+1}/VM_i$ is zero,
c) the map $\Pi \colon VM_i/\pi M_{i+1} \to VM_{i+1}/\pi M_i$ is zero.

Indeed, there are inclusions between submodules of the same index in M_i or M_{i+1} if and only if there is equality; the assertion thus follows from the equivalence between the conditions $\Pi M_i = VM_i$ and $\pi M_i = \Pi V M_i$.

Thus in the first case, the relations $\Pi\beta_i = \beta_{i+1}\Pi$ [13] are satisfied if and only if $\beta_{i+1} = 0$, while in the second case they are always satisfied. $\square$

Lemma 11.5. *Suppose that i is critical for M. Then i is critical for M' if and only if $\beta_{i+1} = 0$.*

Proof. We first suppose $\beta_{i+1} = 0$ and show that i is critical for M', or in other words, that $\Pi M'_i \subset V'M'_i$. We verify this separately for each factor of $M'_i = \widetilde{M}_i \oplus M_{\varepsilon,i}$.

We have $\Pi M_{\varepsilon,i} \subset V'M_{\varepsilon,i}$. Indeed $M_{\varepsilon,i} = [\varepsilon](M_i/VM_{i+1}) \oplus V'M_{\varepsilon,i+1}$ and Π is trivial on M_i/VM_{i+1}.

Now we must show $\Pi\widetilde{M}_i \subset V'\widetilde{M}_i$. For every $m \in M_i$, we have $\Pi(m,0) = (\Pi m, 0)$; there exists $m_1 \in M_i$ such that $\Pi m = Vm_1$ and, since $\beta_{i+1} = 0$, we have $(Vm_1, 0) = V'(m_1, 0)$.

Conversely suppose that i is critical for M'. Let m and m_1 in M_i be such that $\Pi m = Vm_1 \notin \pi M_{i+1}$. We have $\Pi(m,0) = (\Pi m, 0) = (Vm_1, 0)$; but $(Vm_1, 0)$ is not contained in $V'M'$ if $\beta_{i+1}(Vm_1) = 0$, hence $\beta_{i+1} = 0$. $\square$

Lemma 11.6. *Suppose that i is critical for M'. Then we have $(M'_i)^{V'^{-1}\Pi} = \widetilde{M}_i^{V^{-1}\Pi}$.*

Proof. For $m \in \widetilde{M}_i$ and $n \in M_{\varepsilon,i}$, we have $V'(m,n) = (Vm, Vn)$, since $\beta_{i+1} = 0$. Also $\Pi(m,n) = (\Pi m, \Pi n)$. Since $(m,n) \in M_i'^{V'^{-1}\Pi}$ if and only if $m \in \widetilde{M}_i^{V^{-1}\Pi}$ and $n \in M_{\varepsilon,i}^{V^{-1}\Pi}$.

But $M_{\varepsilon,i}^{V^{-1}\Pi} = 0$. In fact $M_{\varepsilon,i} = \bigoplus_j V^j[\varepsilon](M_{i+j}/VM_{i+j+1})$ is $\mathbf{N}$-graded, and the same is true for $M_{\varepsilon,i+1}$; the map $V \colon M_{\varepsilon,i} \to M_{\varepsilon,i+1}$ is of degree 1 for the gradings, while Π is of degree 0. Thus $n = \sum n_j$ satisfies $Vn = \Pi n$ if and only if $\Pi n_0 = 0$ and $\Pi n_j = Vn_{j-1}$ for $j \geq 1$. Moreover $\Pi n_j = 0$ for even j, since i is critical for M. We deduce that $Vn_{j-1} = 0$, thus $n_{j-1} = 0$ and $Vn_{j-2} = \Pi n_{j-1} = 0$, thus $n_{j-2} = 0$; hence $n = 0$. $\square$

[13] the index was j in the original text, but that must have been a typo

Let (η, T, u, r) be a rigidified admissible triple (5.8 and 5.16) representing $\overline{\xi}(x)$. Let $\overline{\eta} = \eta \otimes_{\mathcal{O}} \overline{k}$ and let $\overline{u} \colon \overline{\eta} \to T$ be the $\overline{k}$-linear map induced by u.

Lemma 11.7. *(i) If (η, T, u) has a single critical index i, the tangent space $t_{\overline{H}}(\overline{\xi}(x))$ is one dimensional and is identified with the space of $\overline{k}$-linear maps $\delta_i \colon \ker \overline{u}_i \to T_i$.*

(ii) If (η, T, u) has two critical indices, the tangent space $t_{\overline{H}}(\overline{\xi}(x))$ is two dimensional and is identified with the set of pairs of $\overline{k}$-linear maps $\delta_i \colon \ker \overline{u}_i \to T_i$ ($i = 0, 1$).

Proof. Let (η', T', u') be a deformation of (η, T, u) over $\overline{k}[\varepsilon]$, let $\overline{\eta}' = \eta' \otimes_{\mathcal{O}} \overline{k}$ and $\overline{u}' \colon \overline{\eta}' \to T'$ be the $\overline{k}$-linear map induced by u'. The map $\eta' \to \eta$ is an isomorphism and the diagram:

$$\begin{tikzcd} \overline{\eta}' \arrow[r, "\sim"] \arrow[d, "\overline{u}'"'] & \overline{\eta} \arrow[d, "\overline{u}"] \\ T' \arrow[r] & T \end{tikzcd}$$

determines a $\overline{k}$-linear map $\delta \colon \ker \overline{u} \to \varepsilon T = \ker(T' \to T)$ of degree zero, such that $\Pi\delta = \delta\Pi$. Each such map determines an isomorphism between a deformation of (η, T, u) and the rigidification r of η defines a rigidification of η'.[14]

If i is a critical index, the maps $\Pi \colon T_i \to T_{i+1}$ and $\Pi \colon \overline{\eta}_i \to \overline{\eta}_{i+1}$ are zero; otherwise if i is not critical then they are isomorphisms. Thus if there is a single critical index, Π identifies δ_i and δ_{i+1}; if there are two critical indices, then δ_i and δ_{i+1} are independent. □

Remark 11.8. a) The dimension of the tangent space is obvious from the geometric interpretation of $\overline{H}|_{\overline{k}}$ as represented by a tree of projective lines. The points of intersection of two such lines correspond precisely with the triples having two critical indices.

b) Note that $\delta_i = 0$ if and only if the isomorphism $\overline{\eta}'_i \xrightarrow{\sim} \overline{\eta}_i$ induces an isomorphism $\ker \overline{u}'_i \xrightarrow{\sim} \ker \overline{u}_i$.

Let M' be a deformation of M, corresponding to (β_0, β_1), and (η', T', u'), corresponding to (δ_0, δ_1), the deformation of (η, T, u) which is the image of M' under $t_{\overline{\xi}(x)}$.

Lemma 11.9. *Suppose that i is critical for M'. Then $\beta_i = 0$ if and only if $\delta_i = 0$.*

Proof. Since i is critical for M and M', the diagram:

$$\begin{tikzcd} \eta'_i \arrow[r] \arrow[d, "u'_i"'] & \eta_i \arrow[d, "u_i"] \\ T'_i \arrow[r] & T_i \end{tikzcd}$$

is identified after (4.5) with the diagram:

$$\begin{tikzcd} (M'_i)^{V'^{-1}\Pi} \arrow[r] \arrow[d] & M_i^{V^{-1}\Pi} \arrow[d] \\ M'_i/V'M'_{i-1} \arrow[r] & M_i/VM_{i-1}. \end{tikzcd}$$

[14] Not so sure about this last sentence

Moreover $(M_i')^{V'^{-1}\Pi} = \widetilde{M}_i^{V^{-1}\Pi}$ by (11.6). Thus the diagram:

$$\begin{tikzcd} \overline{\eta}_i' \arrow[r] \arrow[d, "\overline{u}_i'"'] & \overline{\eta}_i \arrow[d, "\overline{u}_i"] \\ T_i' \arrow[r] & T_i \end{tikzcd}$$

can be identified with the diagram

$$\begin{tikzcd} \widetilde{M}_i/\pi\widetilde{M}_i \arrow[r] \arrow[d] & M_i/\pi M_i \arrow[d] \\ M_i'/V'M_{i-1}' \arrow[r] & M_i/VM_{i-1}. \end{tikzcd}$$

We have $\delta_i = 0$ if and only if the isomorphism $\overline{\eta}_i' \xrightarrow{\sim} \overline{\eta}_i$ induces a bijection $\ker \overline{u}_i' \xrightarrow{\sim} \ker \overline{u}_i$ (11.8). By the previous identifications, this is the case if and only if we have $V'M_{i-1}' \cap \widetilde{M}_i = (VM_{i-1})^{\sim}$. By the description in (11.3) of V' as a function of β_i, this is equivalent with $\beta_i = 0$. $\square$

11.10. Proposition (11.1) now follows from the preceding lemmas:

If i is the sole critical index for x, then the tangent spaces $t_{\overline{G}}(x)$ and $t_{\overline{H}}(\overline{\xi}(x))$ are one dimensional. We have $\beta_{i+1} = 0$ (11.4) and i is critical for M' (11.5). Finally $t_{\overline{\xi}}(x)$[15] is injective (11.9), thus bijective.

If x has two critical indices, then the tangent spaces are two dimensional. The two subspaces of $t_{\overline{G}}(x)$ of dimension one given by the equations $\beta_{i+1} = 0$ for $i = 0, 1$ are characterised by the equivalent condition: i is critical for M' (11.5). Moreover $t_{\overline{\xi}}(x)$ is injective when restriced to these subspaces (11.9) and their images are distinct in $t_{\overline{H}}(\overline{\xi}(x))$; thus $t_{\overline{\xi}}(x)$ is bijective.

12. **End of the proof.** In this section we concldue the proof of Drinfeld's theorem. By (10.5), it suffices to show that $\overline{\xi}\colon \overline{G} \to \overline{H}$ is an isomorphism when restricted to the category of $\overline{k}$-algebras. We now confine ourselves to working in this category.

We have seen previously that the map $\overline{\xi}(\overline{k})\colon \overline{G}(\overline{k}) \to \overline{H}(\overline{k})$ on geometric points is bijective (5.17), and the same is true on the tangent maps $t_{\overline{\xi}}$ for each of these points (11.1). To conclude the proof it suffices now to show that $\overline{\xi}$ is representable by a morphism of finite type.

Definition 12.1. For n and m integers ≥ 0, we define the subfunctor $G_{n,m}$ of $\overline{G}$ which, for a $\overline{k}$-algebra B, associates the set of isomorphism classes of pairs (X, ρ), as in (8.1), such that:

1) $\pi^n\rho\colon \Phi_B \to X$ is an isogeny,
2) $\ker(\pi^n\rho) \subset \Phi_B(\pi^{n+m})$.

We note here that $\Phi_B(\pi^{n+m})$ is the kernel of π^{n+m} in Φ_B. Condition 2) is thus equivalent with:

2') there exists an isogeny $\beta\colon X \to \Phi_B$ such that $\beta\pi^n\rho = \pi^{n+m}$.

Proposition 12.2. *For every choice of n and m, the functor $G_{n,m}$ is representable by a projective $\overline{k}$-scheme.*

[15] the x was subscripted in BC, but I think this was a typo

Proof. Let $\mathcal{A}$ be the algebra of $\Phi(\pi^{n+m})$ over $\overline{k}$. Giving a couple (X, ρ) over a $\overline{k}$-algebra B is equivalent with giving the kernel Z of $\pi^n\rho$. The algebra $\mathcal{O}_Z$ is a locally free B-algebra of rank q^{4n} which is a quotient of $\mathcal{A}_B$. Since $G_{n,m}$ is a subfunctor of the Hilbert scheme $\mathrm{Hilb}(\mathcal{A}, q^{4n})$.

Moreover the inclusion of $G_{n,m}$ into $\mathrm{Hilb}(\mathcal{A}, q^{4n})$ is representable by a closed immersion. In fact the condition that Z is a subscheme of $\mathcal{O}_D$-modules of $\Phi_B(\pi^{n+m})$ is closed.

To say for example that Z is stable under multiplication means that the map

$$\mu_Z\colon Z \times Z \to \Phi_B(\pi^{n+m}) \times \Phi_B(\pi^{n+m}) \to \Phi_B(\pi^{n+m})$$

factors through the immersion $Z \hookrightarrow \Phi_B(\pi^{n+m})$. Stated in terms of algebras of functions, the homomorphism

$$\mu_Z^*\colon \mathcal{A}_B \to \mathcal{A}_B \otimes_B \mathcal{A}_B \to \mathcal{O}_Z \otimes_B \mathcal{O}_Z$$

kills the kernel J_Z of the surjection $\mathcal{A}_B \to \mathcal{O}_Z$. The B-modules J_Z and $\mathcal{O}_Z \otimes_B \mathcal{O}_Z$ are locally free of finite rank and their formation commutes with arbitrary basechange $B \to B'$. The condition $\mu_Z^*(J_Z) = 0$ defines a closed subscheme of $\mathrm{Spec}(B)$.

One verifies in the same way that Z containing the identity and inverse are closed conditions, as well as the stability of Z by the action of $\mathcal{O}_D$. □

It is clear that, for $n \geq n'$ and $m \geq m'$, the functor $G_{n,m}$ is a subfunctor of $G_{n',m'}$. Moreover:

Lemma 12.3. *For every $\overline{k}$-algebra B, we have*

$$\overline{G}(B) = \bigcup_{n,m} G_{n,m}(B).$$

Proof. Let (X, ρ) represent an element of $\overline{G}(B)$. By definition $\rho\colon \Phi_B \to X$ is a quasi-isogeny, so there exists an integer n such that $\pi^n\rho$ is an isogeny. For this isogeny, there exists an integer m and an isogeny $\beta\colon X \to \Phi_B$ such that $\beta\pi^n\rho = \pi^{n+m}$ ([Zi 3], Satz 5.25). □

Lemma 12.4. *For $x \in G_{n,m}(\overline{k})$, the tangent map $t_{G_{n',m'}}(x) \to t_{\overline{G}}(x)$ is bijective whenever $n' > n$ and $m' > m$.*

Proof. The tangent map of x is injective since $G_{n',m'}$ is a subfunctor of $\overline{G}$. We now show that it is surjective.

Let (X, ρ) represent x and let (X', ρ') be a deformation of (X, ρ) over $\overline{k}[\varepsilon]$. By hypothesis $\pi^n\rho$ is an isogeny of $\Phi_{\overline{k}}$ onto X. As $\pi^{n+1}\rho$ lifts to an isogeny of $\Phi_{\overline{k}[\varepsilon]}$ over X' ([Zi 3], 4.47) and by the rigidity of π-divisible groups, this isogeny is necessarily $\pi^{n+1}\rho'$.

Further assume $\ker(\pi^n\rho) \subset \Phi(\pi^{n+m})$, or in other words that there exists an isogeny β of X over $\Phi_{\overline{k}}$ such that $\beta\pi^n\rho = \pi^{n+m}$. As $\pi\beta$ lifts to an isogeny β' of X' over $\Phi_{\overline{k}[\varepsilon]}$ such that $\beta'\pi^{n+1}\rho' = \pi^{n+m+2}$, or equivalently $\ker(\pi^{n+1}\rho') \subset \Phi(\pi^{n+m+2})$.

Thus (X', ρ') represents an element of $G_{n',m'}(\overline{k}[\varepsilon])$ whenever $n' \geq n+1$ and $m' \geq m+1$. □

12.5. Write $\xi_{n,m}$ for the morphism of $G_{n,m}$ in $\overline{H}$ obtained by composing the inclusion of $G_{n,m}$ into $\overline{G}$ with $\overline{\xi}$. It is a morphism of finite type since $G_{n,m}$ is a scheme of finite type over $\overline{k}$.

For $y \in \overline{H}(\overline{k})$ there exists, by (5.17) and (12.3), a unique $x \in \overline{G}(\overline{k})$ such that $y = \overline{\xi}(x)$ and indices n_y and m_y such that $x \in G_{n_y,m_y}(\overline{k})$. For $n = n_y+1$ and $m = m_y+1$, the tangent map at x of $\xi_{n,m}$ is bijective, by (11.1) and (12.4). Thus $\xi_{n,m}$ is etale in a neighbourhood of x; more precisely, since the map induced by $\xi_{n,m}$ on geometric points is injective, it is an open immersion in a neighbourhood of x. In other words, there exists an open neighbourhood $\mathcal{V}_y$ of y in $\overline{H}$ above which $\xi_{n,m}$ is an isomorphism.

For $n' > n$ and $m' > m$, the morphism $\xi_{n',m'}$ restricted to $\mathcal{V}_y$ induces bijections on geometric points and the tangent spaces at these points, and it is thus therefore also an isomorphism. Thus we have $G_{n',m'}|_{\mathcal{V}_y} = G_{n,m}|_{\mathcal{V}_y}$ and, by (12.3), $\overline{G}|_{\mathcal{V}_y} = G_{n,m}|_{\mathcal{V}_y}$; the morphism $\overline{\xi}$ coincides with $\xi_{n,m}$ above $\mathcal{V}_y$, and is thus an isomorphism.

Since this is true in a neighbourhood of every point of $\overline{H}$, it follows that $\overline{\xi}$ is an isomorphism.

Remark 12.6. One can show easily by analogous reasoning that, for every n, there exists m such that $G_{n,m'} = G_{n,m}$ for all $m' > m$. The subfunctor G_n of $\overline{G}$ defined by the single condition that $\pi^n\rho$ is an isogeny is representable by a projective $\overline{k}$-scheme. Geometrically G_n is represented by a finite subtree of the infininte tree of projective lines representing $\overline{G}$ over $\overline{k}$.

13. **Construction of a system of coverings of $\Omega \otimes_K \widehat{K}^{nr}$.**

13.1. Since it represents the functor G of (8.3), $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{\mathrm{nr}}$ is equipped with a "universal" formal $\mathcal{O}_D$-module, denoted X. For all integers $n \geq 1$, we write π^n for the endomorphism of X over $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{\mathrm{nr}}$ induced by $\pi^n \in \mathcal{O}_D$. For every geometric point s of $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{\mathrm{nr}}$ (that is to say, its special fiber), the restriction of π^n to the fiber X_s is an isogeny, with constant height equal to $4n$. Using the results of Th. Zink ([Zi 1]) or (10.1), we deduce that π^n is an *isogeny*: hence its kernel, which we write as X_n, is *representable by a formal group scheme, finite and locally free over* $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{\mathrm{nr}}$*, of rank* q^{4n}. It is clear that $(\mathcal{O}_D/\pi^n\mathcal{O}_D)$ acts on X_n. Note also that *the module of differentials* $\Omega^1_{X_n/\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{\mathrm{nr}}}$ *is killed by* π^n: it suffices in fact to verify this on the zero section, and this case follows from the definition of a formal $\mathcal{O}_D$-module; in fact, π^n operates on $\mathrm{Lie}(X)$ via the structural morphism and it follows that, locally on $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{\mathrm{nr}}$, the affine algebra X_n is generated by two "coordinates" x_1 and x_2 satisfying equations $F_i(x_1, x_2) = 0$ ($i = 1$ or 2), with:

$$F_i = \pi^n x_i + (\text{terms of degree } \geq 2).$$

We thus have, on the zero section: $\pi^n dx_i = 0$.

This can be phrased essentially in the same way by saying that X_n is "formally etale over $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{\mathrm{nr}}$ outside the special fiber" ([El]).

13.2. Let $\mathcal{X}_n$ denote the *rigid space* associated to X_n, that is to say its generic fiber in the sense of Raynaud ([Ra 1]). By what we have just seen, $\mathcal{X}_n$ is a finite etale covering of $\Omega \otimes_K \widehat{K}^{nr}$ (the rigid space over $\widehat{K}^{nr}$ obtained from Ω by extension of scalars), fibered in $(\mathcal{O}_D/\pi^n\mathcal{O}_D)$-modules[16]. We have inclusions $\mathcal{X}_{n-1} \hookrightarrow \mathcal{X}_n$, where $\mathcal{X}_{n-1}$ is identified with the subspace of points of $\mathcal{X}_n$ killed by π^{n-1}. We write $\mathcal{X}_{n-1/2}$ for the intermediate space of points of $\mathcal{X}_n$ killed by Π^{2n-1}.

We know furthermore that the cardinality of the fibers of $\mathcal{X}_n$ are all equal to q^{4n}, which is the cardinality of $(\mathcal{O}_D/\pi^n\mathcal{O}_D)$. One deduces immediately that these fibers

[16]Should this say instead that $\mathcal{X}_n$ is an $(\mathcal{O}_D/\pi^n\mathcal{O}_D)$-module?

are free $(\mathcal{O}_D/\pi^n\mathcal{O}_D)$-modules of rank 1 (each element of $\mathcal{X}_n - \mathcal{X}_{n-1/2}$ makes up a basis).

Now consider the complement: $\Sigma_n := \mathcal{X}_n - \mathcal{X}_{n-1/2}$ consisting of the points of $\mathcal{X}_n$ "killed by π^n exactly". It follows from the preceding work that Σ_n is an *etale galois covering of the Galois group* $(\mathcal{O}_D/\pi^n\mathcal{O}_D)^*$. As n varies, the Σ_n make up a *projective system*, via the isogeny π which induces the morphisms: $X_{n+1} \to X_n$, $\mathcal{X}_{n+1} \to \mathcal{X}_n$, $\Sigma_{n+1} \to \Sigma_n$; the Galois group of this system is the profinite completion $\widehat{\mathcal{O}}_D^*$ of $\mathcal{O}_D^*$.

Finally, it is important to note that the coverings just constructed are *equivariant* relative to the action of $\mathrm{GL}_2(K)$ considered in (9.3): it is clear in fact that this action lifts to an action on the universal formal $\mathcal{O}_D$-module X, and hence to an action on X_n, $\mathcal{X}_n$ and Σ_n.

13.3. *Some remarks.* (a) Using for example the results of Elkik ([El]), one can show that the category of finite etale coverings of $\Omega \otimes_K \widehat{K}^{nr}$ is equivalent with the category of finite etale coverings of $\Omega \otimes_K K^{nr}$. The construction just considered then defines a system of etale coverings, also denoted Σ_n, of $\Omega \otimes_K K^{nr}$.

(b) The construction of Σ_n is purely rigid analytic: abstractly Σ_n should come from a certain formal scheme $\widehat{\Sigma}_n$, but this is not how it was constructed (this would require the definition of a "Drinfeld basis" in this case).

(c) The article [Ca 2] provides one method which allows one to calculate globally – by using the theorem of Cerednik-Drinfeld – the cohomology of the coverings Σ_n: they furnish a geometric realisation of the Jacquet-Langlands correspondence (between representations of $\mathrm{GL}_2(K)$ and D^*) and the Langlands correspondence (between representations of $\mathrm{GL}_2(K)$ and the Weil group W_K).

III. The Cerednik-Drinfeld theorem

0. **Introduction and notation.**

0.1. In what follows we fix an indefinite division quaternion algebra Δ over $\mathbf{Q}$. It defines a reductive group over $\mathbf{Q}$, denoted Δ^* ("the multiplicative group of Δ") such that one has, for all $\mathbf{Q}$-algebras R:

$$\Delta^*(R) = (\Delta \otimes R)^* \quad \text{(and so } \Delta^*(\mathbf{Q}) = \Delta^*\text{)}.$$

The group $\Delta^*(\mathbf{R})$, in particular, is isomorphic with $\mathrm{GL}_2(\mathbf{R})$. By fixing such an isomorphism, one obtains an action of the group $\Delta^*(\mathbf{R})$ on the "double" Poincare upper-half plane:

$$\mathfrak{H}^{\pm} = \mathbf{P}^1(\mathbf{C}) - \mathbf{P}^1(\mathbf{R}).$$

Let also $U \subseteq \Delta^*(\mathbf{A}_F)$ be a compact open subgroup of the group of points of Δ^* with values in the finite adeles. One associates to this a *Shimura curve*, denoted S_U, defined over $\mathbf{Q}$, such that its collection of complex points is defined by the following formula:

$$S_U(\mathbf{C}) = \Delta^*(\mathbf{Q})\backslash[\mathfrak{H}^{\pm} \times \Delta^*(\mathbf{A}_f)/U].$$

The quotient that we have just written is none other than the union of a finite number of quotients $\Gamma_i \backslash \mathfrak{H}$ of the Poincare upper-half plane $\mathfrak{H}$ by arithmetic subgroups Γ_i (that is, commensurable with $\Delta^*(\mathbf{Z})$ for an arbitrary integral structure). It is in particular a compact Riemann surface, but in general disconnected. The $\mathbf{Q}$-structure has been defined by Shimura, which is part of a much more general and extremely beautiful theory; see [De]. Here we confine ourselves (in section 1) to a moduli problem, defined over $\mathbf{Q}$, and which is represented by S_U. More general cases, where

Δ is a quaternion algebra with center equal to a totally real field, have been described in [Mi] and [Br-La] (in the totally definite case), as well as in [Ca 1] (in the case of curves).

0.2. Write δ for the product of the ramified primes of Δ. The methods developed in [Mi], [Br-La] and [Ca 1] describe the reduction of S_U at a prime which is relatively prime to δ. We will study the reduction of S_U at a prime p (fixed in all that follows) which *divides* δ.

The group $\Delta^*(\mathbf{Q}_p) = \Delta_p^*$ is the mutiplicative group of the field $\Delta_p = \Delta \otimes \mathbf{Q}_p$. It contains a unique maximal compact subgroup denoted U_p^0 (which consists of the units of the unique maximal order of Δ_p). This maximal subgroup is filtered by a descending family $(U_p^n)_{n\geq 0}$ of distinguished subgroups, where U_p^n consists of the units which are congruent to 1 modulo the nth power of the maximal ideal. We will always assume that our subgroup $U \subseteq \Delta^*(\mathbf{A}_f)$ decomposes as a product $U = U_p^n \cdot U^p$ where $U^p \subseteq \Delta^*(\mathbf{A}_f^p)$ is an arbitrary compact open subgroup.

0.3. In its original form, the theorem of Cerednik gives, in the case $n = 0$ (which is to say the case where U is "maximal at p"), a p-adic uniformisation of $S_U \otimes \mathbf{Q}_p$ as a quotient of the Mumford ([Mu 2]) non-archimedean half-plane $\Omega = \Omega_{\mathbf{Q}_p}$. The modular proof of Drinfeld, which we explain here, allows one to recover this result, by comparing the universal family of abelian varieties parameterised by S_U and the universal family of formal groups parameterised by $\widehat{\Omega}$. From this one easily deduces, for $n > 0$, a uniformisation of $S_U \otimes \mathbf{Q}_p$ in terms of mysterious coverings Σ_n of Ω which the fundamental local theorem allows one to define.

0.4. We place ourselves in the case $n = 0$. We begin by stating (§1) the moduli problems over $\mathbf{Q}$ and over $\mathbf{Q}_p$. Then in §3 we define a moduli problem over $\mathbf{Z}_p$ which is the natural "extension". This extension is possible because the "level structure" is concentrated away from p (because $n = 0$), and it therefore makes sense in characteristic p. The only thing that one adds to pass from $\mathbf{Q}_p$ to $\mathbf{Z}_p$ is a single condition, in characteristic p, imposed on the formal group of the abelian variety, a condition which is analogous to one we encounter for the functor represented by $\widehat{\Omega}$.

Then we show that the extended functor is representable by a proper curve over $\mathbf{Z}_p$ (but non-smooth) $\mathbf{S}_U$ with generic fiber $S_U \otimes \mathbf{Q}_p$; to prove this result, one must show that the abelian varieties under consideration are equipped with a canonical principle polarisation compatible with a well-chosen positive involution (which we will define) of Δ: this is the subject of §4. Before this, we prove in §2 that all the $\mathbf{F}_p$-points of $\mathbf{S}_U$ are in a single isogeny class: this is a fundamental difference from the case when p does not divide δ.

After this we are able (in §5) to state the theorem of Cerednik-Drinfeld and its variants. We prove this in §6 using the fundamental local theorem of chapter II.

0.5. Fix in all that follows a *maximal order* $\mathcal{O}_\Delta$ in Δ (recall that all such orders are conjugate, which follows by strong approximation). We impose that it is *stable under the canonical involution* $x \mapsto \overline{x}$ of Δ (this is not imposed in [Br - La], nor in [Mi]); it is possible to find such an order since one can find such orders locally at all places (giving a global maximal order $\mathcal{O}_\Delta$ is equivalent to giving, for all finite places v, a maximal order $\mathcal{O}_{\Delta_v} = \mathcal{O}_\Delta \otimes \mathbf{Z}_v$ in $\Delta_v = \Delta \otimes \mathbf{Q}_v$, such that the local orders agree almost everywhere with an arbitrary fixed global order).

A positive involution $x \mapsto x^*$ of Δ is obtained, as is explained in (loc. cit.), by conjugating the canonical involution by an element $t \in \Delta^*$ such that the square t^2 is a negative element of $\mathbf{Q}$:

$$x^* = t^{-1}\overline{x}t.$$

It is useful to make our choice for t explicit now. Our choice is made as in the following easy lemma, whose proof we leave to the reader:

Lemma 0.6. *One can choose t such that: $t \in \mathcal{O}_\Delta$; $t^2 = -\delta$. [It suffices to remark that $\mathbf{Q}(\sqrt{-\delta}))$ is a splitting field for Δ].*

We suppose henceforth that t is fixed as above, and so we have defined the involution $x \mapsto x^*$; note that it stabilises the order $\mathcal{O}_\Delta$.

To end this list of notations, we write W for the order $\mathcal{O}_\Delta$, but endowed with the structure of a left $\mathcal{O}_\Delta$-module, and $V = W \otimes \mathbf{Q}$ (a left Δ-module). We are particularly interested in the different completions $W_l = W \otimes \mathbf{Z}_l$ (viewed as left $\mathcal{O}_{\Delta_l}$-modules) and in $V_l = W \otimes \mathbf{Q}_l$ (Δ_l-modules). Note that the group $\mathrm{Aut}_{\Delta_l}(V_l)$ is identified with $\Delta^*(\mathbf{Q}_l) = \Delta_l^*$, where an element g acts by multiplication on the right by g^{-1}.

Finally, we use, for A an abelian variety, the standard notations: A_n designates the n-torsion, $T_l(A)$ denotes the Tate module and $V_l(A) = T_l(A) \otimes \mathbf{Q}$. Finally we write $T_f(A)$ for the product of all the Tate modules for all primes numbers l, and $V_f(A)$ for the restricted product of all the $V_l(A)$.

1. **The moduli problem over C; polarisations.**

1.1. A moduli problem represented by S_U is described in [Mi] and [Gi] in the most general case where Δ is an indefinite quaternion algebra over a totally real field. In our particular case, one obtains:

Theorem 1.2. *The curve $S_U/\mathbf{C}$ represents, if U is small enough (see below), the functor $\mathcal{M}_U\colon \mathbf{Sch}/\mathbf{C} \to \mathbf{Set}$ defined as follows: for $S \in \mathbf{Sch}/\mathbf{C}$, $\mathcal{M}_U(S)$ is the collection of isomorphism classes of triples $(A, \iota, \overline{\nu})$ such that:*

(i) *A is an abelian scheme over S of relative dimension 2.*
(ii) *$\iota\colon \mathcal{O}_\Delta \to \mathrm{End}_S A$ is an action of $\mathcal{O}_\Delta$ on A.*
(iii) *$\overline{\nu}$ is a level U structure on A (see below).*

Remark 1.3. A pair (A, ι) as above is sometimes called a "false elliptic curve". In the introduction to [De-Ra], they explain why the reduction of such an object, at a prime not dividing δ, is composed of usual elliptic curves.

1.4. We begin by recalling, following [Bo], how to define a "level U structure":

(a) In the case where $U = U(N)$ is the subgroup of units of $\mathcal{O}_\Delta \otimes \widehat{\mathbf{Z}}$ which are congruent to 1 modulo an integer N, a level U (or N) structure consists of an $\mathcal{O}_\Delta$-linear isomorphism:

$$\nu\colon A_N \cong W \otimes (\mathbf{Z}/N\mathbf{Z}).$$

(b) In the general case one chooses an integer N such that $U(N) \subseteq U$. A structure of level U is then the giving, locally for the etale topology, of a class $\overline{\nu}$ modulo U of isomorphisms ν as above. [One verifies without difficulty that this definition is independent of the choice of N.]

(c) For S the spectrum of an algebraically closed field, one can also describe the level structure as the giving of a class $\overline{\nu}$ modulo U of isomorphisms:

$$\nu\colon T_f(A) \cong W \otimes \widehat{\mathbf{Z}}.$$

It is also possible, as explained in [De], to work in the category of abelian varieties up to isogeny, and the level structures are given by classes modulo U of isomorphisms $V_f(A) \cong W \otimes \mathbf{A}_f$. In this way one can easily write down the action of the group $\Delta^*(\mathbf{A}_f)$ on the projective system of S_U – or if one prefers, the action of the Hecke operators.

1.5. *Remarks on the condition "U small enough":* This is the condition which assures the level U structure is rigid enough to eliminate nontrivial automorphisms. It suffices for example (see [Bo]) that U is contained in $U(M)$ for M an integer ≥ 3. When U is of the form $U_p^0 \cdot U^p$, it evidently suffices to suppose U^p is small enough in order for U to be so: since the theorem of Cerednik that we will prove is "invariant" upon replacing U^p by a subgroup, we may henceforth suppose that this condition is satisfied.

1.6. The moduli problem that we are describing is in fact defined and representable over $\mathbf{Q}$, and one can use this to define a $\mathbf{Q}$-structure on S_U. In [Mi] and [Bo], they show that it is the generic fiber of a proper smooth curve over $\mathbf{Z}[1/\delta N]$, where N is an integer such that $U(N) \subseteq U$. However, we would like to study the curve S_U over p (recall that p divides δ). In what follows we define a moduli problem over $\mathbf{Z}_p$, which is representable by a proper and flat curve (but which is not smooth!).

1.7. *Polarisations.* Recall (cf. loc. cit.) that, for all triples $(A, \iota, \overline{\nu})$ as above, a $*$-polarisation is a polarisation λ of A such that, for all geometric points s of S, the Rosatti involution on $\mathrm{End}^0(A_s)$ induces, via ι, the involution $*$ on $\mathcal{O}_\Delta$. One can equivalently require that, for all $d \in \mathcal{O}_\Delta$, the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "\lambda"] \arrow[d, "\iota(d^*)"'] & A^* \arrow[d, "\iota(d)^*"] \\ A \arrow[r, "\lambda"] & A^* \end{tikzcd}$$

Proposition 1.8. *Let S be a scheme in characteristic 0 and $(A, \iota, \overline{\nu}) \in \mathcal{M}_U(S)$. Then there exists a principal $*$-polarisation of A. Such a polarisation is unique.*

Proof. One reduces immmediately to the case where $S = \mathrm{Spec}(\mathbf{C})$. In loc. cit. (cf. for example [Mi], Lemma 1.1 or [Bo] §8), they show the existence of a $*$-polarisation, and its unicity up to a rational number. We must now consider the possibility of choosing such a polarisation which is also principal (and, evidently, that there is a unique such choice). This result is clear by the following lemma, which gives the existence of an $\mathcal{O}_\Delta$-linear isomorphism $T_l A \cong W_l$. □

Lemma 1.9. *For all prime numbers l, consider the collection of bilinear antisymmetric maps:*

$$\psi \colon W_l \times W_l \to \mathbf{Z}_l,$$

which satisfy:

$$\forall d \in \mathcal{O}_{\Delta_l}, \quad \psi(dx, y) = \psi(x, d^* y).$$

This collection is naturally a free $\mathbf{Z}_l$-module of rank 1, and every generator ψ_0 of this module defines a perfect self-duality on W_l.

Proof. (cf. the proof of lemma 1.1 in [Mi]). Let ψ be as in the lemma. Remembering that W_l can be identified with $\mathcal{O}_{\Delta_l}$, we can write:

$$\psi(x, y) = \phi(x^* y)$$

where ϕ is the linear form $\psi(1,u)$. One can express ϕ in terms of the reduced trace trd, by the expression:

$$\phi(u) = \mathrm{trd}(d_0tu^*) = \mathrm{trd}(d_0\overline{u}t),$$

with d_0 an element of Δ_l. The antisymmetry of ψ follows from the identity $\phi(u^*) = -\phi(u)$, and a brief calculation shows that this is equivalent with $\overline{d_0} = d_0$, where $d_0 \in \mathbf{Q}_l$. One concludes the proof of this lemma by the following one. $\square$

Lemma 1.10. *The map $\psi_0\colon W_l \times W_l \to \mathbf{Q}_l$ defined by:*

$$\begin{cases} \textit{if } (l,\delta) = 1, & \psi_0(x,y) = \mathrm{trd}(ty^*x) \\ \textit{if } (l,\delta) \neq 1, & \psi_0(x,y) = l^{-1}\,\mathrm{trd}(ty^*x) \end{cases}$$

takes values in $\mathbf{Z}_l$, and induces a perfect self-duality on W_l.

Proof. This sub-lemma can be verified immediately: in the first case ($l \not| \delta$), one can suppose that $\mathcal{O}_{\Delta_l} = M_2(\mathbf{Z}_l)$, and it is well-known that the trace defines a perfect self-duality. The involution $*$ defines an automorphism of $M_2(\mathbf{Z}_l)$, and our hypotheses on t ensure that $t \in \mathrm{GL}_2(\mathbf{Z}_l)$, which proves the lemma in this case. In the other case ($l|\delta$), the maximal order $\mathcal{O}_{\Delta_l}$ is a division quaternion algebra and t is a "uniformiser" for this order; it is well-known that the reduced trace gives a duality between $\mathcal{O}_{\Delta_l}$ and $t^{-1}\mathcal{O}_{\Delta_l}$. The lemma follows, and hence also the preceding lemma and proposition. $\square$

Remark 1.11. Note that the truth of Lemma 1, and thus of Proposiiton 1, depends entirely on the particular choice of the element t, and on the involution $*$.

2. **Application of the Tate-Honda theorem.** The next paragraph will be devoted to extending the preceding moduli problem $\mathcal{M}_U$ (with $U = U_p^0U^p$) into a moduli problem $\mathbf{M}_U$ defined over $\mathbf{Z}_p$. The following proposition, which we place here for expositional purposes, states that all points with values in $\overline{\mathbf{F}}_p$ are in the same isogeny class.

Proposition 2.1. *There exists a single isogeny class of pairs (A,ι) where A is an abelian variety of dimension 2 over $\overline{\mathbf{F}}_p$ with an action ι of the order $\mathcal{O}_\Delta$. In particular, the variety A is isogenous to a product of supersingular elliptic curves. The algebra $\mathrm{End}^0_{\mathcal{O}_\Delta}(A) = \mathrm{End}_{\mathcal{O}_\Delta}(A)\otimes\mathbf{Q}$ of endomorphisms of a pair (in the category "up to isogeny") is isomorphic with the quaternion algebra $\overline{\Delta}$ obtained from Δ by interchanging the invariants p and ∞ (that is to say $\overline{\Delta}$ is definite, unramified at p, and $\overline{\Delta}_l$ is isomorphic with Δ_l for all $l \neq p,\infty$).*

The proof of this theorem is a standard application of the theorem of Honda-Tate.

(a) One begins by showing that A is isogeneous to a product of two supersingular elliptic curves:

– The p-divisible group A_{p^∞} of A has trivial etale part (and so, by duality, no multiplicative part); in fact, if the etale component is nontrivial, it is of dimension 1 or 2. But $\mathcal{O}_\Delta$ cannot operate on such a group, because there do not exist algebra homomorphisms $\Delta_p \to \mathbf{Q}_p$, nor $\Delta_p \to M_2(\mathbf{Q}_p)$. This contradiction proves that A_{p^∞} is isogeneous to two copies of the p-divisible group of a supersingular elliptic curve.

– One applies the theorem of Honda-Tate (cf. [Br] or [Ta 2]): if A is not isogeneous to a product of two elliptic curves (necessarily supersingular by the preceding argument), it is simple. Suppose it is defined over a finite extension $\mathbf{F}_q$ of $\mathbf{F}_p$, and write π for the Frobenius endomorphism of F_q. From the structure of the p-divisible group, one deduces that the element π^2/q is a unit in the field $\mathbf{Q}(\pi)$: since this unit is of absolute value one at every place, it is a root of unity: we may thus suppose that

$\pi = \sqrt{q} \in \mathbf{Q}$. But then A is a supersingular elliptic curve! This contradiction proves that A is isogeneous to a product of two supersingular elliptic curves, as was claimed.

(b) It follows that $\mathrm{End}^0(A)$ is isomorphic with the algebra $M_2(H)$, where H denotes the quaternion algebra over $\mathbf{Q}$ ramified exactly at p and ∞. The unicity up to isogeny of a pair (A, ι) is equivalent to all the embeddings $\Delta \hookrightarrow M_2(H)$ being conjugate, and this follows from the theorem of Skolem-Noether.

(c) Place by place, one checks that $\Delta \otimes \overline{\Delta}$ is isomorphic with $M_2(H)$. This proves both the existence of an embedding $\Delta \hookrightarrow M_2(H)$, and the fact that the algebra $\mathrm{End}^0_{\mathcal{O}_\Delta}(A)$ (which is identified with the commutant of Δ in $M_2(H)$) is isomorphic with $\overline{\Delta}$.

Remark 2.2. For A as above, if $l \neq p$, one sees that $V_l(A)$ is isomorphic with V_l as Δ_l-modules (because the dimesion is 4 over $\mathbf{Q}_l$), with an action (Δ_l-linear) of the algebra $\mathrm{End}^0_{\mathcal{O}_\Delta}(A)$. It is clear that $\mathrm{End}_{\Delta_l}(V_l)$ can be naturally identified with the algebra Δ_l^{opp} which is *opposite* to Δ_l (operating by multiplication on the right). The choice of isomorphisms $V_l(A) \cong V_l$ and $\overline{\Delta} \cong \mathrm{Aut}^0_{\mathcal{O}_\Delta}(A)$ thus determine an isomorphism, for all $l \neq p$: $\overline{\Delta}_l \cong \Delta_l^{\mathrm{opp}}$, and an isomorphism:

$$\overline{\Delta} \otimes \mathbf{A}_f^p \cong \Delta^{\mathrm{opp}} \otimes \mathbf{A}_f^p.$$

3. The moduli problem over $\mathbf{Z}_p$.

3.1. We consider the case where the compact subgroup U is of the form $U_p^0 U^p$, that is, it is "maximal at p", and in this case we will extend the preceding moduli problem to $\mathbf{Z}_p$ (or, what is the same, over the localisation $\mathbf{Z}_{(p)}$ of $\mathbf{Z}$ at p). We begin by remarking that, in this setting, there exists an integer N which is *prime to* p such that one has: $U(N) \subseteq U$ (cf. §1.2). It follows that the notion of a level U structure, given in §1, makes sense in characteristic p. This allows us to define a moduli problem $\mathbf{M}_U$ over $\mathbf{Z}_p$, in the same way that it was defined in characteristic 0; the only difference is the introduction of a supplementary condition on the points of characteristic p.

Definition 3.2. If S is a $\mathbf{Z}_p$-scheme, then $\mathbf{M}_U(S)$ is the set of isomorphism classes of triples $(A, \iota, \overline{\nu})$ such that:

(i) A is an abelian scheme over S of relative dimension 2.

(ii) $\iota\colon \mathcal{O}_\Delta \to \mathrm{End}_S(A)$ is an action of $\mathcal{O}_\Delta$ on A.

We impose the following condition, for all geometric points $s = \mathrm{Spec}\, k(s)$ of characteristic p of S: write $\mathbf{Z}_p^{(2)}$ for the ring of integers of the quadratic unramified extension of $\mathbf{Q}_p$. The ring embeds into $\mathcal{O}_{\Delta_p}$ (the inclusion is well-defined up to conjugation). We require that the action of $\mathbf{Z}_p^{(2)}$ on $\mathrm{Lie}(A_s)$ decomposes into a sum of two injections $\mathbf{Z}_p^{(2)} \otimes \mathbf{F}_p \cong \mathbf{F}_{p^2} \hookrightarrow k(s)$. (One says that the pair (A, ι) is "special").

(iii) $\overline{\nu}$ is a level U structure on A.

3.3. *Some remarks on the "special" condition.* (a) One sees that the condition is really a condition on the formal completion of A at the origin – or if one prefers, on the p-divisible group A_{p^∞}; it is none other than the condition that we encountered in the preceding chapter: to say that A is a special $\mathcal{O}_\Delta$-variety is the same as saying that the associated formal group is a special formal $\mathcal{O}_{\Delta_p}$-module (chap. II §2.1).

(b) Suppose that S is a $\mathbf{Z}_p^{(2)}$-scheme (a condition which we may always, perhaps after an etale base-change, suppose holds). The $\mathcal{O}_S$-module $\mathrm{Lie}(A)$ is equipped with an action of the ring $\mathbf{Z}_p^{(2)} \otimes \mathbf{Z}_p^{(2)}$, which is isomorphic with $\mathbf{Z}_p^{(2)} \oplus \mathbf{Z}_p^{(2)}$. This action decomposes $\mathrm{Lie}(A)$ into a direct sum of two projective $\mathcal{O}_S$-modules $\mathrm{Lie}^1(A) \oplus \mathrm{Lie}^2(A)$, such that $\mathbf{Z}_p^{(2)} \subseteq \mathcal{O}_{\Delta_p}$ operates on the first term via the structural morphism $\mathbf{Z}_p^{(2)} \to \mathcal{O}_S$, and on the second via the composition of this morphism with conjugation on $\mathbf{Z}_p^{(2)}$. The "special" condition, as was reformulated above, means that the rank of each of these $\mathcal{O}_S$-modules is equal to 1 at each geometric point of characteristic p of S. This condition also makes sense at points of characteristic 0, but then the condition is automatically satisfied [because, for the unique representation $\Delta_p \hookrightarrow M_2(\overline{\mathbf{Q}_p})$, it is true that the action $\mathbf{Z}_p^{(2)} \hookrightarrow \Delta_p$ involves each of the two embeddings $\mathbf{Z}_p^{(2)} \hookrightarrow \overline{\mathbf{Q}}_p$].

Since the rank of an $\mathcal{O}_S$-module is locally constant, one sees that, for S connected, *the condition is satisfied once it is satisfied at a single geometric point*. In particular, *the condition is automatic for flat $\mathbf{Z}_p$-schemes S* (in fact, it is satisfied at the points of characteristic 0). In other words, it is satisfied for all pairs (A, ι) which "come from characteristic 0". This condition is thus necessary if one wants M_U to be representable by a flat scheme over $\mathbf{Z}_p$.

3.4. *Representability*. To prove the representability of the functor M_U, we need the following proposition, which generalizes Proposition 1. The notion of $*$-polarisation is defined as in $\S 1$ (cf. also [Bo] $\S 8$).

Proposition 3.5. *Let $(A, \iota, \overline{\nu}) \in \mathrm{M}_U(S)$, for S a $\mathbf{Z}_p$-scheme. Then there exists a principal $*$-polarisation, and it is uniquely determined.*

We admit this result for the time being – the proof is the subject of the following subsection. For now we use it to prove the following:

Theorem 3.6. *The functor M_U is representable – if U^P is small enough – by a projective $\mathbf{Z}_p$-scheme S_U with generic fiber $S_U \otimes \mathbf{Q}_p$.*

Remark 3.7. One can show, for example by using the theorem of Cerednik that we would like to prove, that S_U is *flat* over $\mathbf{Z}_p$.

Proof of the theorem:

(a) The preceding proposition defines a morphism of functors between M_U and the functor "principally polarized abelian varieties". To prove that M_U is representable by a quasi-projective scheme, it suffices to prove that it is relatively representable above the Siegel moduli stack. But this is an easy consequence of the theory of Hilbert schemes.

(b) The projectivity follows, using the valuative criterion of properness, from the following lemma ("potential good reduction"):

Lemma 3.8. *Let V be a $\mathbf{Z}_p$-algebra which is a discrete valuation ring with fraction field denoted L, and let $x = (A, \iota, \overline{\nu})$ be a point of $\mathrm{M}_U(L)$. Then there exists a finite extension L' of L and a point $\widetilde{x} = (\widetilde{A}, \widetilde{\iota}, \widetilde{\overline{\nu}})$ of M_U with values in the integral closure V' of V in L', such that the image of $\widetilde{x}$ in $\mathrm{M}_U(L')$ coincides with x.*

The proof of this lemma is standard: by the semi-stable reduction theorem, there exists an extension L' of L such that the Neron model of $A_{L'}$ has special fiber equal to an extension of an abelian variety by a torus T.

One proceeds by showing that T *is trivial*: indeed $\mathcal{O}_\Delta$ operates on the Neron model by functoriality. If T is not trivial, then $X_*(T)$ is a $\mathbf{Z}$-module of rank 1 or 2 with an action of $\mathcal{O}_\Delta$, which is impossible.

Thus the Neron model is an abelian scheme $\widetilde{A}$ over V'. By functoriality, it is equipped with an action $\widetilde{i}$ of $\mathcal{O}_\Delta$, which satisfies the "special" condition by virtue of remark (b) of (3.3). As for the level structure, it extends in a unique way, and this concludes the proof of the lemma.

4. **Polarisations [Proof of proposition (3.5)].**

4.1. We begin by proving the existence of a $*$-polarisation (not necessarily principal) when the base is $\overline{\mathbf{F}}_p$.

Lemma 4.2. *Let A be an abelian variety over $\overline{\mathbf{F}}_p$, with an action of $\mathcal{O}_\Delta$. Then there exists a $*$-polarisation of A. Such a polarisation is unique up to multiplication by a positive rational constant.*

Proof. It suffices to prove the existence, and the uniqueness up to a scalar ($\in \mathbf{R}^{*+}$), of an element in $\mathrm{NS}(A) \otimes \mathbf{R}$, contained in the positive cone of polarisations, such that the involution associated to $\mathrm{End}(A) \otimes \mathbf{R}$ induces the involution $*$ on Δ.

Following §2, A is isogeneous to a product of two supersingular elliptic curves. It follows that we may indentify $\mathrm{End}(A) \otimes \mathbf{R}$ with the algebra $M_2(\mathbf{H})$, and $\mathrm{NS}(A) \otimes \mathbf{R}$ with the subspace of elements of $M_2(\mathbf{H})$ that are fixed by the involution $z \mapsto \overline{z}^T$ [where $z \mapsto \overline{z}$ is the canonical involution of $\mathbf{H}$]. With this identification, the Rosatti involution of $M_2(\mathbf{H})$ associated to a symmetric element ($\beta = \overline{\beta}^T$) is given by:

$$x \mapsto \beta \overline{z}^T \beta^{-1}.$$

Fix also an isomorphism between $\Delta \otimes \mathbf{R}$ and $M_2(\mathbf{R})$, such that the positive involution $*$ corresponds to the transposition $m \mapsto m^T$. The action of $\mathcal{O}_\Delta$ on A defines an injection $\iota\colon M_2(\mathbf{R}) \to M_2(\mathbf{H})$, which is conjugate by a certain $\alpha \in \mathrm{GL}_2(\mathbf{H})$ to the obvious inclusion $M_2(\mathbf{R}) \hookrightarrow M_2(\mathbf{H})$ (in other words: $\iota(m) = \alpha m \alpha^{-1}$).

The condition for the involution associated to β, symmetric, to induce the involution $*$ on Δ is:

$$\forall m \in M_2(\mathbf{R}) : \alpha m^T \alpha^{-1} = \beta \overline{\alpha m \alpha^{-1}}^T \beta^{-1}$$

which is to say:

$$(\alpha^{-1} \beta \overline{\alpha^T}^{-1}) m (\overline{\alpha}^T \beta^{-1} \alpha) = m.$$

This is satisfied if and onlyif β is of the form

$$\beta = \lambda \alpha \overline{\alpha}^T, \ (\lambda \in \mathbf{R}^*).$$

Moreover, β is the in the positive cone of polarisations if and only if $\lambda \in \mathbf{R}^{*+}$ (for this see [Mu 1], §21), and the lemma follows. □

4.3. *The following lemma is crucial.*

Lemma 4.4. *Let X be a formal group of dimension 2 and height 4 over a local artinian ring B with residue field $\overline{\mathbf{F}}_p$. Suppose that X is equipped with an action ι of the ring $\mathcal{O}_\Delta \otimes \mathbf{Z}_p = \mathcal{O}_{\Delta_p}$, such that the "special" condition is satisfied. We consider the collection of symmetric morphisms $\lambda\colon X \to X^*$ of X into its Cartier dual, which are compatible with the involution $*$, in other words such that the diagram of (1.5) commutes (if one prefers: formal polarisations). Then this collection is a free $\mathbf{Z}_p$-module of rank 1, and the generators are the isomorphisms $X \cong X^*$.*

We momentarily admit this lemma and explain how Proposition 3.5 follows.

(i) In the case where the base $S_0 = \mathrm{Spec}\,\overline{\mathbf{F}}_p$, the lemma 4.2 yields the existence of a polarisation, which we must adjust to obtain a principal one. As in the case of an earlier proposition (1.5), the possibility to make such an adjustment is controlled place by place: if $l \neq p$, one proceeds exactly as in the case of the field $\mathbf{C}$, and uses lemma (1.6); otherwise, for $l = p$, one replaces that lemma with the one above.

(ii) Next we treat the case where the base S is the spectrum of an artinian ring with residue field $\overline{\mathbf{F}}_p$. The possibility to deform the principal $*$-polarisation above the closed points of S follows, by the theorem of Serre-Tate, from the preceding lemma. By passing to the limit, this applies also to the case when the base is the spectrum of a complete local ring with residue field $\overline{\mathbf{F}}_p$.

(iii) For the remaining cases, suppose S is of finite type over $\mathbf{Z}_p$. It follows from preceding remarks that the problem admits a unique solution in formal neighbourhoos of every geometric point of S. Using the unicity, one sees that the local formal solutions are algebraic and may be glued.

4.5. We now prove Lemma (4.4). Drinfeld proved it using a very original method, by verifying the lemma in the case of a particular formal group $\Phi/\overline{\mathbf{F}}_p$, and then using the representability theorem (chapter II, §8) to prove the lemma in general.

We begin by defining Φ: write $\mathcal{E}$ for "the" formal group over $\overline{\mathbf{F}}_p$ of dimension 1 and height 2, and choose an isomorphism $\mathrm{End}(\mathcal{E}) \cong \mathcal{O}_{\Delta_p}$. One may choose a "formal polarisation" $\mu_0\colon \mathcal{E} \xrightarrow{\sim} \mathcal{E}^*$ (as in the statement of the lemma), and the associated Rosatti involution is identified with the canonical involution of Δ_p (to fix ideas, one may take $\mathcal{E}$ to be the formal group of a supersingular elliptic curve, and take the formal polarisation associated to the principal polarisation of the curve). We consider the product $\Phi = \mathcal{E} \times \mathcal{E}$ of two copies of $\mathcal{E}$, on which $\mathcal{O}_{\Delta_p}$ acts in the following way: an element $u \in \mathcal{O}_{\Delta_p}$ acts (via the isomorphism $\mathcal{O}_{\Delta_p} \cong \mathrm{End}(\mathcal{E})$) by u on the first factor, and by tut^{-1} on the second (recall that the element $t \in \Delta^*$ is a "uniformiser" of Δ_p). In this way, the condition to be "special" is satisfied: Φ is a special formal $\mathcal{O}_{\Delta_p}$-module, in the sense of the previous chapter (note that both indices 0 and 1 are critical).

Proof that lemma (4.4) holds for Φ: using the above identification of $\mathcal{E}$ with $\mathcal{E}^*$, and of $\mathcal{O}_{\Delta_p}$ with $\mathrm{End}(\mathcal{E})$, one sees that the collection of formal polarisations compatible with the involution $*$ is identified with the matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathcal{O}_{\Delta_p}), \quad \text{which are Hermitian symmetric:}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ \overline{\gamma} & \overline{\delta} \end{pmatrix}, \quad \text{and which satisfy, for all } u \in \mathcal{O}_{\Delta_p}, \text{ the relation:}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & tu^*t^{-1} \end{pmatrix} = \begin{pmatrix} \overline{u} & 0 \\ 0 & \overline{tut^{-1}} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Using the formulae: $u^* = t^{-1}\overline{u}t = t\overline{u}t^{-1} = \overline{tut^{-1}}$[17], one sees that the preceding conditions are equivalent to: $\alpha = \delta = 0$, $\beta = \gamma \in \mathbf{Z}_p$. The set under consideration is thus indeed a free $\mathbf{Z}_p$-module of rank 1, generated for example by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We fix such a generator $\lambda_0\colon \Phi \xrightarrow{\sim} \Phi^*$, that is, a "principal formal $*$-polarisation".

Remark 4.6. One knows, by chapter II, §5, that $\mathrm{End}^0_{\Delta_p}(\Phi)$ is isomorphic with $M_2(\mathbf{Q}_p)$. In this case the isomorphism is realised by the inclusion of $M_2(\mathbf{Q}_p)$ in $M_2(\Delta_p) =$

[17] Check if this and the above formula are correct; it's different from in BC, but theirs looks funny

$\mathrm{End}^0(\Phi)$ defined by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & bt \\ ct^{-1} & d \end{pmatrix}.$$

A calculation immediately shows that the Rosatti involution associated to λ_0 induces the canonical involution on $M_2(\mathbf{Q}_p)$.

4.7. We now treat the general case: in light of what we have just seen, it suffices to establish – using the notations of lemma (4.4) – the existence of a formal principal $*$-polarisation $\lambda\colon X \xrightarrow{\sim} X^*$. Broadening the hyptheses of the lemma a bit, suppose we are given a $\mathbf{Z}_p^{nr}$-algebra B such that the image of p is nilpotent, and a formal group X, of dimension 2 and height 4 over B, with a special action of the ring $\mathcal{O}_{\Delta_p}$. Suppose moreover that we are given a quasi-isogeny ρ (compatible with the action of $\mathcal{O}_{\Delta_p}$), of height 0, of $\Phi_{B/pB}$ into $X_{B/pB}$. That that such a ρ exists due to the hypotheses of lemma (4.4): for $B = \overline{\mathbf{F}}_p$ this follows from the fact that all special formal $\mathcal{O}_{\Delta_p}$-modules over $\overline{\mathbf{F}}_p$ are isogeneous with Φ (chapter II, §5), and so one may choose (by composing with a suitable endomorphism of Φ) a quasi-isogeny of height 0. For B artinian of residue field $\overline{\mathbf{F}}_p$, it follows from the previous case by deforming quasi-isogenies (see for example [Zi 3] 5.31).

The lemma thus clearly follows from the following precise assertion: *there exists an isomorphism $\lambda\colon X \xrightarrow{\sim} X^*$ such that the restriction $\lambda_{B/pB}$ to special fibers makes the following diagram commute:*

$$\begin{array}{ccc} X_{B/pB} & \xrightarrow{\lambda_{B/pB}} & X^*_{B/pB} \\ {\scriptstyle\rho}\uparrow & & \downarrow{\scriptstyle\rho^*} \\ \Phi_{B/pB} & \xrightarrow{\lambda_0} & \Phi^*_{B/pB} \end{array}$$

(if such a λ exists, it is unique, and symmetric (since λ_0 is); moreover, it is necessarily compatible with the involution $*$).

Recall that, by the definitions of chapter II, §8, giving the isomorphism class of a pair (X,ρ) is equivalent to giving a B-valued point of the functor $\overline{G}$ (which is represented by the formal $\widehat{\mathbf{Z}}_p^{nr}$-scheme $\widehat{\Omega}\widehat{\otimes}\widehat{\mathbf{Z}}_p^{nr}$). The preceding assertion says that the pairs (X,ρ) and $(X^*, (\rho^*)^{-1}\circ\lambda_0)$ are isomorphic. The formula:

$$j(X,\rho) = (X^*, (\rho^*)^{-1}\circ\lambda_0)$$

defines an automorphism of the functor $\overline{G}$. To show that X^* is special when X is, we may work over an algebraically closed field k of characteristic p, and use the fact that the reduction modulo p of the Dieudonne module of X is an extension of $\mathrm{Lie}(X^*)$ by the dual of $\mathrm{Lie}(X)$; one easily sees that, for the action of $\mathbf{Z}_p^{(2)}$ on the Dieudonne module, each of the two embeddings $\mathbf{Z}_p^{(2)} \hookrightarrow W(k)$ appears twice (one may also verify the last assertion on Φ, because all special formal $\mathcal{O}_{\Delta_p}$-modules over k are isogeneous).

In this way one obtains an automorphism, denoted j, of the $\widehat{\mathbf{Z}}_p^{nr}$-scheme $\widehat{\Omega}\widehat{\otimes}\widehat{\mathbf{Z}}_p^{nr}$. To prove the preceding assertion, and thus finally lemma (4.4), *it suffices for us to show that this automorphsm is the identity*.

4.8. We note that this automorphism – which is involutive – *commutes with the natural action of* $\mathrm{SL}_2(\mathbf{Q}_p)$ on $\widehat{\Omega}\widehat{\otimes}\widehat{\mathbf{Z}}_p^{nr}$: in fact, it follows from a result of chapter

II §9.3 that the action of an element $g \in \mathrm{SL}_2(\mathbf{Q}_p)$ on the functor $\overline{G}$ is given by: $g(X,\rho) = (X, \rho \circ g^{-1})$. Hence:

$$j \circ g(X, rho) = (X^*, ((\rho \circ g^{-1})^*)^{-1} \circ \lambda_0) = (X^*, (\rho^*)^{-1} \circ g^* \circ \lambda_0);$$
$$g \circ j(X,\rho) = (X^*, (\rho^*)^{-1} \circ \lambda_0 \circ g^{-1}).$$

The commutativity follows from the relation: $\lambda_0^{-1} g^* \lambda_0 = \overline{g} = g^{-1}$ for $g \in \mathrm{SL}_2(\mathbf{Q}_p)$ (by the remark in (4.3)).

The following lemma will allow us to conclude the proof:

Lemma 4.9. *An automorphism j of $\widehat{\Omega}\widehat{\otimes}\widehat{\mathbf{Z}}_p^{nr}$, which commutes with the action of $\mathrm{SL}_2(\mathbf{Q}_p)$, is necessarily the identity.*

In fact, j operates on the special fiber $\widehat{\Omega} \otimes \overline{\mathbf{F}}_p$, and also on its "dual graph", which is isomorphic with the tree of $\mathrm{PGL}_2(\mathbf{Q}_p)$. It is a simple exercise to see that any automorphism of the tree that commutes with the action of $\mathrm{SL}_2(\mathbf{Q}_p)$ is necessarily the identity: therefore, j stabilises each irreducible component of the special fiber, and fixes all points of intersection of the components. Each of the components is a projective line, and has $p+1 \geq 3$ points of intersection with its neighbouring components: it follows that j acts trivially on the special fiber.

The "deviation" of j from the identity on the first infinitesimal deformation $\widehat{\Omega} \otimes (\mathbf{Z}_p^{nr}/(p^2))$ of the special fiber is measured by a derivation of the structure sheaf of the special fiber; this is equivalent to giving a stack of tangent spaces, trivial at each singuar point. Such a stack is necessarily trivial, and j is thus the identity on the first deformation. Continuing on successive deformations by induction proves that j is the identity.

Remark 4.10. A result analogous to Proposition (3.5) is proved in a more general case in [Zi 2]. The method used by Zink is more direct than the one we just used.

5. The Cerednik-Drinfeld theorem: statement, variants and remarks.

5.1. We suppose throughout that our compact open subgroup U is of the form $U_p^0 U^p$, with U^p a compact open subgroup of $\Delta^*(\mathbf{A}_f^p)$. When U^p is small enough, the curve corresponding curve $\mathbf{S}_U$ is a projective system (with an action of the group $\Delta^*(\mathbf{A}_f^p)$).

We write $\overline{\Delta}^*$ to denote the reductive group over $\mathbf{Q}$ defined – in the same way as Δ^* is obtained from Δ – as the multiplicative group of the algebra $\overline{\Delta}$ considered in §2. We fix an isomorphism between the groups:

$$\Delta^*(\mathbf{A}_f^p) = (\Delta \otimes \mathbf{A}_f^p)^* \quad \text{and} \quad \overline{\Delta}^*(\mathbf{A}_f^p) = (\overline{\Delta} \otimes \mathbf{A}_f^p)^*,$$

obtained via an *anti-isomorphism* between the algebras $\Delta \otimes \mathbf{A}_f^p$ and $\overline{\Delta} \otimes \mathbf{A}_f^p$ (composed with the inversion $g \mapsto g^{-1}$). Via this isomorphism, the group U^p that we are considering corresponds to a subgroup of $\overline{\Delta}^*(\mathbf{A}_f^p)$.

We also fix an isomorphism $\overline{\Delta}^*(\mathbf{Q}_p) \cong \mathrm{GL}_2(\mathbf{Q}_p)$ obtained from an isomorphism between $\overline{\Delta} \otimes \mathbf{Q}_p$ and $M_2(\mathbf{Q}_p)$.

Consider the following collection of double cosets, denoted Z_U or Z_{U^p}:

$$Z_U = U^p \backslash \overline{\Delta}^*(\mathbf{A}_f)/\overline{\Delta}^*(\mathbf{Q}).$$

The group $\overline{\Delta}^*(\mathbf{Q}_p)$ acts on the left of this group, and the quotient by action is finite. All orbits contain the double coset of an element x whose pth component x_p is equal to 1. The stabiliser Γ_x of x is given by:

$$\Gamma_x = \overline{\Delta}^*(\mathbf{Q}) \cap x^{-1} U^p x,$$

where the intersection is taken in $\overline{\Delta}^*(\mathbf{A}_f^p)$ then, seen as a subgroup of $\overline{\Delta}^*(\mathbf{Q})$, is injected into $\overline{\Delta}^*(\mathbf{Q}_p) \cong \mathrm{GL}_2(\mathbf{Q}_p)$. One verifies without difficulty that the stabilisers are discrete and cocompact subgroups of $\overline{\Delta}^*(\mathbf{Q}_p)$, and that they contain a power of the matrix $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$.

For U^p small enough, the collection Z_U is a projective system on which the group $\overline{\Delta}^*(\mathbf{A}_f^p)$ acts.

5.2. *The Cerednik-Drinfeld theorem.* Retain the preceding notation and conventions, in particular the isomorphisms:

$$\Delta^*(\mathbf{A}_f^p) \cong \overline{\Delta}^*(\mathbf{A}_f^p), \quad \overline{\Delta}^*(\mathbf{Q}_p) \cong \mathrm{GL}_2(\mathbf{Q}_p).$$

Theorem 5.3. *For every compact open subgroup $U^p \subseteq \Delta^*(\mathbf{A}_f^p)$ which is small enough, one has (putting $U = U_p^0 U^p$) an isomorphism of formal $\mathbf{Z}_p$-schemes:*

$$\widehat{\mathbf{S}}_U \cong \mathrm{GL}_2(\mathbf{Q}_p) \backslash [\widehat{\Omega} \widehat{\otimes} \widehat{\mathbf{Z}_p}^{nr} \times Z_U]$$

where $\widehat{\mathbf{S}}_U$ denotes the formal completion of $\mathbf{S}_U$ along its special fiber. This isomorphism is compatible, as U^p varies, with the projection maps. The isomorphism of the two projective systems thus obtained is compatible with the action on the two members of the group $\Delta^*(\mathbf{A}_f^p) \cong \overline{\Delta}^*(\mathbf{A}_f^p)$. Finally, this isomorphism lifts to an isomorphism between the special formal $\mathcal{O}_{\Delta_p}$-modules naturally associated to the two formal schemes.

The preceding theorem calls for a number of remarks: we'll start with some background and some explanatory details that may be necessary for its formal understanding; then, we explain why the quotient which figures, with a somewhat monstrous appearance, in the theorem above, is none other than the finite union of twisted forms of some Mumford curves. We then compute the graph of the irreducible components of the special fiber. Finally we generalize the theorem to the case of a subgroup of the form $U = U_p^n U^p$.

We begin by giving some clarifying remarks regarding the statement of the theorem:

a) Recall that the action of $\mathrm{GL}_2(\mathbf{Q}_p)$ on $\widehat{\Omega} \,\widehat{\otimes}\, \widehat{\mathbf{Z}_p}^{nr}$ that we consider is obtained from the natural action on $\widehat{\Omega}$ and from the action $g \mapsto \widetilde{Fr}^{-v(\det g)}$ on $\widehat{\mathbf{Z}_p}^{nr}$ (cf. chapter II, §9). This action is defined solely over $\mathbf{Z}_p$ and not over $\mathbf{Z}_p^{nr}$.

b) The natural action of the group $\Delta^*(\mathbf{A}_f^p)$ on the projective system of $\mathbf{S}_U$ is a right action, whil the action of the group $\overline{\Delta}^*(\mathbf{A}_f^p)$ on the system of Z_U is a left action. In order to compare them, one must change the side of one of the actions: we use the anti-isomorphism between the two groups associated to the anti-isomorphism of the corresponding algebras.

c) The formal $\mathcal{O}_{\Delta_p}$-module defined by $\widehat{\mathbf{S}}_U$ is the formal completion of the universal abelian variety given by the moduli problem $\mathbf{M}_U$. The one associated to the formal scheme on the right of the theorem comes from the moduli description of $\widehat{\Omega} \,\widehat{\otimes}\, \widehat{\mathbf{Z}_p}^{nr}$ (chapter II, §8).

5.4. *Remarks.* 3.5.3.1 – Because the action of $\overline{\Delta}^*(\mathbf{Q}_p) \cong \mathrm{GL}_2(\mathbf{Q}_p)$ on Z_U decomposes the last collection into a finite number of orbits, we see that the quotient which appears in the statement of the theorem is the union of a finite number of quotients of the form:

$$\Gamma_i \backslash \widehat{\Omega} \,\widehat{\otimes}\, \widehat{\mathbf{Z}_p^{nr}},$$

where the $\Gamma_i = \Gamma_{x_i}$ are the different stabilisers, which were described in (5.1). Since each of these stabilisers contains a power $p^{n_i} \cdot 1$ of $p \cdot 1$, we may begin by passing to a quotient by the action of $p^{n_i} \cdot 1$ (which acts trivially on $\widehat{\Omega}$), thus obtaining the tensor product of $\widehat{\Omega}$ by the unramified extension of $\mathbf{Z}_p$ of degree $2n_i$, denoted $\mathbf{Z}_p^{(2n_i)}$. The quotient can thus be written:

$$\Gamma_i \backslash \widehat{\Omega} \,\widehat{\otimes}\, \mathbf{Z}_p^{(2n_i)}.$$

After extending scalars to $\mathbf{Z}_p^{(2n_i)}$, is becomes isomorphic to a finite union of "Mumford quotients", of the form $\Gamma_i' \backslash \widehat{\Omega}$ (cf. [Mu 2] or [Ra 2]), where Γ_i' denotes the image in the group $\mathrm{PGL}_2(\mathbf{Q}_p)$ of the subgroup Γ_i consisting of elements of determinant equal to a unit. Under the hypothesis that U^p is small enough (it suffices, for example that the same condition as in (1.3) holds), one sees that the groups Γ_i' are Schottky subgroups of $\mathrm{PGL}_2(\mathbf{Q}_p)$ – in particular, they act freely on the tree I and on $\widehat{\Omega}$. If we suppose also that U^p is sufficiently small so that the Γ_i' operate "very" freely on I (all vertices are shifted by a distance ≥ 2 by all nontrivial elements), then the Mumford quotients are obtained simply from standard affines, indexed by the vertices of the tree, as described in chapter I.

In what follows, we suppose that the quotients that appear in theorem are the union of Galois twisted forms (by unramified extensions) of Mumford quotients.

3.5.3.2 – *The theorem (5.2) is true in a partial sense without the hypothesis that "U^p is small enough"*: there exists in fact in every case a distinguished subgroup of finite index $U_1^p \subseteq U^p$ which is small enough. By applying the theorem to the group U_1^p, then passing to the quotient by the finite group U^p/U_1^p, one obtains an analogous isomorphism:

$$\widehat{\mathbf{S}}_U \cong \mathrm{GL}_2(\mathbf{Q}_p) \backslash [\widehat{\Omega} \,\widehat{\otimes}\, \widehat{\mathbf{Z}_p^{nr}} \times Z_U]$$

where $\mathbf{S}_U$ is the quotient $\mathbf{S}_{U_1}/(U/U_1)$, and the right side is a union of twisted forms of quotients of Mumford curves by finite groups. In other words, the set on the right is always isomorphic to the formal completion of an integral model of S_U. Note that, without the hypothesis that U^p is small enough, neither of the two sides of the isomorphism above carry a natural formal $\mathcal{O}_{\Delta_p}$-module.

Note finally that, in all cases, the isomorphism of the theorem yields a p-adic uniformisation:

$$S_U^{an} \cong \mathrm{GL}_2(\mathbf{Q}_p) \backslash [\Omega \,\widehat{\otimes}\, \widehat{\mathbf{Q}_p^{nr}} \times Z_U]$$

(where S_U^{an} denotes the rigid analytic space over $\mathbf{Q}_p$ associated to S_U). Moreover we see that it is not necessary to complete (that is, we may write only $\otimes \mathbf{Q}_p^{nr}$ in the previous formula).

3.5.3.3 – There is a particular case where the above quotient takes a very simple form; since this case – student in [Jo-Li] – is relevant for the work of Ribet ([Ri 2]), we will say a few words about it. We impose in this paragraph the following hypotheses on the group U^p (we do not suppose that U^p is "small enough"):

a) The image of U^p by the reduced norm is maximal, that is, equal to $\prod_{l \neq p} \mathbf{Z}_l^*$.

b) The p-adic valuation maps the intersection of U^p and the center $\mathbf{Q}^*$ of $\Delta^*(\mathbf{Q})$ surjectively onto $\mathbf{Z}$.

Under hypothesis (a), it is easy to see, by using the strong approximation theorem, that $\overline{\Delta}^*(\mathbf{Q}_p)$ acts transitively on Z_U. The quotient that we are interested in thus takes the form:

$\Gamma\backslash\widehat{\Omega}\,\widehat{\otimes}\,\widehat{\mathbf{Z}_p^{nr}}$, with $\Gamma = \overline{\Delta}^*(\mathbf{Q}) \cap U^p$ (seen as a subgroup of $\overline{\Delta}^*(\mathbf{Q}_p) \cong \mathrm{GL}_2(\mathbf{Q}_p)$). Moreover, thanks to the second hypothesis, the quotient of $\mathrm{Spf}\,\mathbf{Z}_p^{nr}$ by the intersection $\Gamma \cap \mathbf{Q}^*$ is identified with $\mathrm{Spf}\,\mathbf{Z}_p^{(2)}$.

Write Γ_+ for the subgroup of Γ consisting of elements whose reduced norm have even p-adic valuation: one checks that Γ_+ is of index 2 in Γ. Write $W = \Gamma/\Gamma_+$ (a group of two elements), and note that the above quotient can be written as:

$$W\backslash[(\Gamma_+\backslash\widehat{\Omega}) \otimes \mathbf{Z}_p^{(2)}].$$

In other words, one obtains a twisted form of the quotient $\Gamma_+\backslash\widehat{\Omega}$ (which is the quotient of a Mumford curve by a finite group). The cocycle which describes the twist in $H^1(\mathrm{Gal}(\mathbf{Q}_p^{(2)}/\mathbf{Q}_p, \mathrm{Aut}(\Gamma_+\backslash\widehat{\Omega}))$ maps the nontrivial element in the Galois group to the automorphism of $\Gamma_+\backslash\widehat{\Omega}$ defined by $\omega \in \Gamma - \Gamma_+$. This can be described easily in the adelic language: our quotient takes the form

$$W\backslash[\overline{\Delta}^*(\mathbf{Q}_p)_+\backslash\widehat{\Omega} \times Z_U] \otimes \mathbf{Z}_p^{(2)},$$

where $\overline{\Delta}^*(\mathbf{Q}_p)_+$ is the subgroup of $\overline{\Delta}^*(\mathbf{Q}_p)$ consisting of elements such that the valuation of their determinant is even, and where W (isomorphic with $\mathbf{Z}/2\mathbf{Z}$) denotes the quotient $\overline{\Delta}^*(\mathbf{Q}_p)/\overline{\Delta}^*(\mathbf{Q}_p)_+$. We thus obtain the twisted form of the quotient

$$\overline{\Delta}^*(\mathbf{Q}_p)_+\backslash\widehat{\Omega} \times Z_U$$

where the automorphism which describes the twist is defined by any element $w \in \overline{\Delta}^*(\mathbf{Q}_p) - \overline{\Delta}^*(\mathbf{Q}_p)_+$.

5.5. *Graphs.* 3.5.4.1 – We have seen – in chapter I – that the "dual graph" of special fiber of $\widehat{\Omega}$ is isomorphic with the tree I of $\mathrm{PGL}_2(\mathbf{Q}_p)$. If one considers a Mumford quotient $\Gamma\backslash\widehat{\Omega}$, so that the image $\overline{\Gamma}$ of Γ in the projective group operates freely on I, then it is well-known, and easy to see, that the graph of the special fiber of $\Gamma\backslash\widehat{\Omega}$ is identified with the quotient graph $\Gamma\backslash I$. Kurihara ([Ku], see also [Jo-Li]) has also obtained an analogous result in the more general situation (as in (5.3.2)) of a subgroup $\overline{\Gamma}$, discrete and cocompact in $\mathrm{PGL}_2(\mathbf{Q}_p)$, which is not necessarily Schottky (although there always exists a subgroup of finite index in $\overline{\Gamma}$ which is Schottky). Below we discuss the result of Kurihara.

3.5.4.2 – Let R be a discrete valuation ring, with residue field k, and let ϖ denote a uniformiser. We adopt the terminology of Jordan and Livne and say that a curve $\mathcal{C}/\operatorname{Spec}(R)$ is *admissible* if it satisfies the following conditions:

a) $\mathcal{C}$ is proper and flat over $\operatorname{Spec}(R)$, with smooth generic fiber.

b) The special fiber $\mathcal{C}_k$ is reduced; its singularities are ordinary double points which are rational over k, as are the two branches which cross there. The normalizations of the irreducible components of $\mathbf{C}_k$ are rational curves.

c) For all singular points $x \in \mathcal{C}_k$, there exists an integer m such that the completion $\widehat{\mathcal{O}}_{\mathcal{C},k}$ of the local ring at x is R-isomorphic with the completion of the local ring $R[\![X,Y]\!]/(XY - \varpi^m)$; it is easy to see that m is well-defined by x.

To such an admissible curve, one associates a *graph* in the following way: the vertices of the graph are the irreducible components of $\mathcal{C}_k$; the oriented edges of the graph are the branches of the singularties of $\mathcal{C}_k$; the inverse of an edge a corresponds to the other branch $\overline{a}$ meeting at the same singular double point. That is to say, the origin of a is the component which contains a, and its endpoint is the origin of $\overline{a}$. One thus obtains a structure of a graph in the sense of Serre ([Se]). This graph admits an additional structure, which is a map m on from the collection of edges to the integers ≥ 1: it is simply the map which assigns an edge a to the integer $m = m(a)$ associated, by condition (c) above, to the corresponding singular point. We say that the *length* of an edge a is the integer $m(a)$, and it satisfies: $m(\overline{a}) = m(a)$. We thus obtain an additional structure on the graph, which Kurihara calls *a graph with lengths*.

3.5.4.3 – We next define a quotient structure associated to a discrete and cocompact subgroup $\overline{\Gamma} \subseteq \mathrm{PGL}_2(\mathbf{Q}_p)$ operating on the tree I: the natural idea is to consider the combinatorial object $\overline{\Gamma}\backslash I$ whose vertices (resp. edges) are the quotient by $\overline{\Gamma}$ of the collection of vertices (resp. edges) of I, with the obvious incidence and inversion relations. However, one does not always obtain in this way a tree in the sense of Serre, for example, if there exists an element of $\overline{\Gamma}$ which transforms an edge into its inverse, then the quotient contains an edge which is equal to its inverse. *We write* $(\overline{\Gamma}\backslash I)^*$ *for the graph obtained from* $(\overline{\Gamma}\backslash I)$ *by removing the edges which are equal to their own inverses* (note that this does not prevent $(\overline{\Gamma}\backslash I)^*$ from eventually containing its "lacets" [18]).

We also define the length $m(a)$ of an edge a of $(\overline{\Gamma}\backslash I)^*$: it's the order of the stabiliser $\overline{\Gamma}_{\widetilde{a}}$ of a lift $\widetilde{a}$ of a to I.

We are now ready to give Kurihara's result: let $\overline{\Gamma} \subseteq \mathrm{PGL}_2(\mathbf{Q}_p)$ denote a discrete and cocompact subgroup as above, and let $(\Gamma\backslash\widehat{\Omega})$ denote the associated curve (the quotient by a finite group of a Mumford curve).

Theorem 5.6 (Kurihara)**.** *The curve* $(\overline{\Gamma}\backslash\widehat{\Omega})$ *is an admissible curve over* $\mathbf{Z}_p$*. The associated dual graph "with lengths" coincides with the quotient* $(\overline{\Gamma}\backslash I)^*$ *defined above.*

For the proof, we refer to the article of Kurihara, where they also explain how to obtain the dual graph of a regular model of the curve (see also [Jo-Li]). We are content to illustrate the proof, in an intuitive fashion, of why it is necessary to remove from $(\overline{\Gamma}\backslash)$ the edges which are their own inverses, and why the "length" is given by the order of the stabiliser:

a) If there exists an inversion $\gamma \in \overline{\Gamma}$ which exchanges two edges of the tree, then the passage to the quotient identifies the two corresponding components of the special fiber of $\widehat{\Omega}$. One sees that the singularity disappears in the quotient.

b) An example of a stabiliser of a group which operates on the singular equation $XY = p$ is a cycle group of order m, with a generator of this group operating by:

$$\begin{cases} X & \rightarrow & \zeta X \\ Y & \rightarrow & \zeta^{-1} Y, \end{cases}$$

where ζ is a primitive mth root of unity.

Thus one sees that the quotient is the singular equation $X'Y' = p^m$, where the projection is given by: $X' = X^m$, $Y' = Y^m$.

3.5.4.4 – We apply the preceding work to a curve S_U associated to a subgroup U satisfying the simplifying hypotheses (5.3.3), and we adopt the notation of that section. One obtains an integral model (defined over $\mathbf{Z}_p$) of the curve, and this model is admissible after extension of scalars to $\mathbf{Z}_p^{(2)}$. The graph "with lengths" associated to the special fiber is equal to $(\overline{\Gamma}_+\backslash\widehat{\Omega})^*$. Since one is dealing with a twisted form of

[18] what is this?

the quotient $(\overline{\Gamma}_+\backslash\widehat{\Omega})$, this leads to a nontrivial action of the Frobenius automorphism Frob_p: this action on $(\overline{\Gamma}_+\backslash I)^*$ is defined by any element $w \in \Gamma - \Gamma_+$. If one prefers an adelic expression, the graph above can also be described:

$$[\overline{\Delta}^*(\mathbf{Q}_p)_+\backslash(I \times Z_U)]^*,$$

and the action of Frobenius can then be defined by any element $w \in \overline{\Delta}^*(\mathbf{Q}_p) - \overline{\Delta}^*(\mathbf{Q}_p)_+$.

5.7. *Generalisation to the case when U_p is not maximal.* 3.5.5.1 – One of the advantages of the approach of Drinfeld is that it yields also a p-adic uniformisation of the curves S_U, for U of the form $U_p^n U^p$ (cf. (0.2)), in termms of the coverings Σ_n of $\Omega\widehat{\otimes}\widehat{K}^{nr}$ defined in chapter II, § 13.

Theorem 5.8. *For U of the form $U_p^n U^p$, there exists an isomorphism of rigid analytic spaces:*

$$S_U^{an} \cong \mathrm{GL}_2(\mathbf{Q}_p)\backslash[\Sigma_n \times Z_{U^p}].$$

This isomorphism is compatible, as U and n vary, with the projection operations, and the isomorphism thus obtained between the two projective systems is equivariant for the action of the group $\Delta^*(\mathbf{A}_f)$.

Remark 5.9. As above, one sees that the set on the right of the formula above is the finite union of quotients $\Gamma_i\backslash\Sigma_n$, for congruence subgroups $\Gamma_i \subseteq \mathrm{GL}_2(\mathbf{Q}_p)$. One easily sees, moreover, that it changes nothing to consider Σ_n as a covering of $\Omega\widehat{\otimes}\widehat{K}^{nr}$ or of $\Omega \otimes K^{nr}$ (cf. II, 13.3 Remark (a)).

Remark 5.10. In the article [Ca 2], they show how to use the theorem above to compute the rigid analytic cohomology of the spaces Σ_n.

3.5.5.2 – We now explain how the theorem above is deduced from theorem (5.2). One may suppose that U^p is small enough. Writeing $U_0 = U_P^0 \times U^p$, the moduli problem $\mathbf{M}_{U_0}$ defines over the $\mathbf{Z}_p$-scheme $\mathbf{S}_{U_0}$ a "universal" abelian variety $\mathbf{A}$. Thus S_U (which exists only in characteristic zero), seen as an S_{U_0}-scheme, classifies $\mathcal{O}_\Delta$-linear isomorphisms $\overline{\nu}$ between the p^n-torsion A_{p^n} of the generic fiber A of $\mathbf{A}$, and $\mathcal{O}_\Delta \otimes (\mathbf{Z}/p^n\mathbf{Z})$. The giving of an isomorphism is simply equivalent to the giving of a point of exact order p^n in A_{p^n}.

Using the notations and definitions of chapter II, § 13, we have, after the last assertion of theorem (5.2), an isomorphism of formal $\mathcal{O}_{\Delta_p}$-modules over $\widehat{\mathbf{S}}_{U_0}$:

$$\widehat{\mathbf{A}} \cong \mathrm{GL}_2(\mathbf{Q}_p)\backslash[X \times Z_{U^p}],$$

and thus

$$\widehat{\mathbf{A}}_{p^n} \cong \mathrm{GL}_2(\mathbf{Q}_p)\backslash[X_n \times Z_{U^p}].$$

We thus obtain an isomorphism over $S_{U_0}^{an}$:

$$\widehat{\mathbf{A}}_{p^n}^{an} \cong \mathrm{GL}_2(\mathbf{Q}_p)\backslash[\mathcal{X}_n \times Z_{U^p}].$$

Finally, one has:

$$S_U^{an} \cong \mathrm{GL}_2(\mathbf{Q}_p)\backslash[\Sigma_n \times Z_{U^p}].$$

5.11. To conclude the commentary on theorem (5.2) we remark that the article [Ri 1] gives a canonical description for the collection of the components and the points over $\overline{\mathbf{F}}_p$ of the curve $\mathbf{S}_U$: one sees in particular that the irreducible components are parameterized by the collection of $\mathcal{O}_\Delta$-abelian varieties of dimension 2 over $\overline{\mathbf{F}}_p$, with an U^p-level structure, and which do not satisfy the "special" condition.

6. **Proof of the Cerednik-Drinfeld theorem.**

6.1. We begin by fixing an abelian variety A_0 over $\overline{\mathbf{F}}_p$, of dimension 2, with a "special" action of the ring cO_Δ (to see that A_0 exists, one could take the abelian variety associated to a point over $\overline{\mathbf{F}}_p$ of $\mathbf{S}_U$; or for an explicit construction, take the product of two supersingular elliptic curves and define the action analogously to (4.3) to exhibit a special formal $\mathcal{O}_{\Delta_p}$-module). Write Φ for the associated formal group, which is a special formal $\mathcal{O}_{\Delta_p}$-module. We also choose an identification (cf. § 2): $\overline{\Delta} = \mathrm{End}^0_\Delta(A_0)$ (and therefore: $\overline{\Delta}^*(\mathbf{Q}) = \overline{\Delta}^* = \mathrm{Aut}^0_\Delta(A_0)$).

This induces an identification:

$$\overline{\Delta}_p = \overline{\Delta} \otimes \mathbf{Q}_p = M_2(\mathbf{Q}_p) = \mathrm{End}^0_{\Delta_p}(\Phi)$$
$$(\text{where: } \overline{\Delta}^*(\mathbf{Q}_p) = \overline{\Delta}^*_p = \mathrm{GL}_2(\mathbf{Q}_p) = \mathrm{Aut}^0_{\Delta_p}(\Phi)).$$

We fix finally isomorphisms, for $l \neq p$:

$$\nu_{0,l} \colon V_l(A_0) \xrightarrow{\sim} V_l,$$

compatible (in the sense of the final remark of § 2) with the fixed isomorphism between $\Delta \otimes \mathbf{A}^p_f$ and $(\overline{\Delta} \otimes \mathbf{A}^p_f)^{opp}$: this means that, via $\nu_{0,l}$, the action of $\overline{\Delta} = \mathrm{End}^0_\Delta(A_0)$ on V_l is given by the composition:

$$\overline{\Delta} \hookrightarrow \overline{\Delta}_l \cong \Delta_l^{opp} \to \mathrm{End}_{\Delta_l}(V_l)$$

[acting by multiplication on the right].

6.2. *Algebrisation.* 3.6.2.1 – Let S be a $\mathbf{Z}_p$-scheme such that the image of p is nilpotent, and let X be a special formal $\mathcal{O}_{\Delta_p}$-module on S.

Definition 6.3. An algebrisation of X is the giving of a pair (A, ε) consisting of an abelian scheme A over S, with an action of $\mathcal{O}_\Delta$, and a $\mathcal{O}_\Delta$-equivariant isomorphism $\varepsilon \colon \widehat{A} \xrightarrow{\sim} X$ between X and the formal group associated to A. When A is moreover equipped with a level structure U (or U^p, which amounts to the same), one says it's an algebrisation with U-level structure.

3.6.2.2 – In particular, we write $\mathcal{A}\updownarrow\}_{\mathrm{U}}(\Phi)$ for the collection of isomorphism classes of algebrisations, with U-level structure, of Φ. It is fundamental for what follows to determine this set: it is the collection of isomorphism classes of triples $(A, \varepsilon, \overline{\nu})$ where A is an $\mathcal{O}_\Delta$-abelian variety over $\overline{\mathbf{F}}_p$, where ε is an equivariant isomorphism between $\widehat{A}$ and Φ, and $\overline{\nu}$ is an $\mathcal{O}_\Delta$-linear isomorphism class modulo U^p:

$$\nu \colon \prod_{l \neq p} T_l(A) \xrightarrow{\sim} \prod_{l \neq p} W_l.$$

The usual yoga (see for example [Mi]) allows one to realize this collection as the collection of isogeny classes of triples $(A, \varepsilon, \overline{\nu})$ where A is an abelian variety, with an action of Δ by isogenies, where ε is an (equivariant) *quasi-isogeny* between $\widehat{A}$ and Φ, and where finally $\overline{\nu}$ is a class modulo U^p of Δ-linear isomorphisms:

$$\nu\colon \prod_{l\neq p} V_l(A) \xrightarrow{\sim} \prod_{l\neq p} V_l.$$

The projective limit $\mathcal{A}\updownarrow\}_\infty(\Phi)$ of $\mathcal{A}\updownarrow\}_{\mathrm{U}}(\Phi)$ is the collection of triples (A, ε, ν); this description yields a left action of the group $\overline{\Delta}^*(\mathbf{A}_f) = \overline{\Delta}^*(\mathbf{Q}_p) \times \overline{\Delta}^*(\mathbf{A}_f^p)$ on $\mathcal{A}\updownarrow\}_\infty(\Phi)$, such that the first component $\overline{\Delta}^*(\mathbf{Q}_p)$ acts by composition with ε, and the right term $\overline{\Delta}^*(\mathbf{A}_f^p)$ acts by composition with ν. This action is *transitive*, as follows from the uniqueness of the isogeny class of A (§ 2).

Using the identifications of (6.1), we consider the element of $\mathcal{A}\updownarrow\}_\infty(\Phi)$ given by triple:

$$A_0, \quad \varepsilon_0\colon \widehat{A}_0 = \Phi, \quad \prod \nu_{0,l}.$$

One sees that the *stabilizer* of this element is the subgroup $\overline{\Delta}^*(\mathbf{Q}_p) \subseteq \overline{\Delta}^*(\mathbf{A}_f)$. This follows from the commutativity of the diagrams below, where γ denotes an element of $\overline{\Delta}^*(\mathbf{Q})$ and γ_l its image in $\overline{\Delta}^*(\mathbf{Q}_l)$:

$$\begin{array}{ccc}
A_0 \\ \big\downarrow{\scriptstyle\gamma} \\ A_0
\end{array}
\qquad
\begin{array}{ccc}
\widehat{A}_0 & \xrightarrow{=} & \Phi \\ \big\downarrow{\scriptstyle\widehat{\gamma}} & & \big\downarrow{\scriptstyle\gamma_p} \\ \widehat{A}_0 & \xrightarrow{=} & \Phi
\end{array}
\qquad
\begin{array}{ccc}
V_l(A_0) & \xrightarrow[\nu_{0,l}]{\sim} & V_l \\ \big\downarrow{\scriptstyle V_l(\gamma)} & & \big\downarrow{\scriptstyle\gamma_l} \\ V_l(A_0) & \xrightarrow[\nu_{0,l}]{\sim} & V_l
\end{array}$$

3.6.3.4 – One obtains a bijection between $\mathcal{A}\updownarrow\}_\infty(\Phi)$ and the homgeneous space $\overline{\Delta}^*(\mathbf{A}_f)/\overline{\Delta}^*(\mathbf{Q})$. This yields a bijection:

$$\mathcal{A}\updownarrow\}_{\mathrm{U}}(\Phi) \cong U^p\backslash\overline{\Delta}^*(\mathbf{A}_f)/\overline{\Delta}^*(\mathbf{Q}) = Z_U.$$

6.4. The next step consists in defining a morphism Θ from the special fibre $[\widehat{\Omega}\,\widehat{\otimes}\,\widehat{\mathbf{Z}_p}^{nr} \otimes \mathbf{F}_p] \times Z_U = (\widehat{\Omega} \otimes \overline{\mathbf{F}}_p) \times Z_U$ to $\mathbf{S}_U \otimes \mathbf{F}_p$.

3.6.3.1 – Let S be a scheme of characteristic p. If A_1 and A_2 are two abelian schemes over S, we call "*p-quasi-isogeny*" a quasi-isogeny $g\colon A_1 \to A_2$ such that the product of g be a large enough power of p is an isogeny of order equal to a power of p. Such a quasi-isogeny induces, for all $l \neq p$, and isomorphism between $T_l(A_1)$ and $T_l(A_2)$.

We will use the following lemma:

Lemma 6.5. *Let X_1 and X_2 be two special formal $\mathcal{O}_{\Delta_p}$-modules over S, and $f\colon X_1 \to X_2$ a quasi-isogeny. Let (A_1, ε_1) be an algebrisation of X_1. Then there exists an algebrisation (A_2, ε_2) of X_2, and a p-quasi-isogeny $h\colon A_1 \to A_2$, such that the following diagram commutes:*

$$\begin{array}{ccc}
\widehat{A}_1 & \xrightarrow{\varepsilon_1} & X_1 \\ {\scriptstyle\widehat{h}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \widehat{A}_2 & \xrightarrow{\varepsilon_2} & X_2
\end{array}$$

The triple (A_2, ε_2, h) is uniquely determined, up to isomorphism, by this property. If moreover A_1 is given a U-level structure, then (via h) A_2 is also so equipped.

We leave the reader to convince themself of this essentially obvious lemma: if for example f is an isogeny, then one can take for A_2 the quotient $A_1/\varepsilon_1^{-1}(\ker f)$.

3.6.3.2 – Using the fundamental theorem of chapter II (§ 8.4), together with 6.2.3 above, one sees that giving a section of $(\widehat{\Omega}^2 \otimes \overline{\mathbf{F}}_p) \times Z_U$ over a connected scheme $S = \operatorname{Spec} B$ of characteristic p yields:

a) a homomorphism $\psi \colon \overline{\mathbf{F}}_p \to B$.

b) An isomorphism class of pairs (X, ρ) with:

* X a special formal $\mathcal{O}_{\Delta_p}$-module over S;

* $\rho \colon \psi_* \Phi \to X$ a quasi-isogeny of height 0.

c) An algebrisation $(A, \varepsilon, \overline{\nu})$ of Φ with U-level structure.

From the above data, one applies the lemma above with $X_1 = \psi_* \Phi$, $X_2 = X$, $f = \rho$, $A_1 = \psi_* A$, $\varepsilon_1 = \psi_* \varepsilon$; one obtains also an algebrisation of X with U-level structure, which is to say a point of $\mathbf{S}_U(B)$. This defines a morphism of functors, which gives the desired morphism of $\mathbf{F}_p$-schemes:

$$\Theta \colon (\widehat{\Omega} \otimes \overline{\mathbf{F}}_p) \times Z_U \to \mathbf{S}_U \otimes \mathbf{F}_p.$$

3.6.3.3 – We verify that Θ is *invariant under the left action of the group* $\mathrm{GL}_2(\mathbf{Q}_p)$; for this recall that, by (9.3) of chatper II, the action of an element g of this group on the functor G is given by (write $n = v(\det_g)$):

$$g(\psi, X, \rho) = (\psi \circ \mathrm{Fr}^{-n}, X, \rho \circ \psi_*(g^{-1} \operatorname{Frob}^n)).$$

Also, the action on Z_U as in (6.2.2) can also be described in terms of lemma (6.3.1): the image $(A_1, \varepsilon_1, \overline{\nu}_1) = g(A, \varepsilon, \overline{\nu})$ is characterised by the existence of a p-quasi-isogeny $h_g \colon A \to A_1$ making the following diagram commute:

$$\begin{tikzcd}
\widehat{A} \arrow[r, "\varepsilon", "\sim"'] \arrow[d, "\widehat{h}_g"'] & \Phi \arrow[d, "g"] \\
\widehat{A}_1 \arrow[r, "\varepsilon_1", "\sim"'] & \Phi.
\end{tikzcd}$$

We write (A_2, ε_2) for the algebrisation of X associated to the point defined by $(\psi, X, \rho, A, \varepsilon, \overline{\nu})$; one thus has a commutative diagram:

$$\begin{tikzcd}
\psi_* \widehat{A} \arrow[r, "\psi_* \varepsilon", "\sim"'] \arrow[d, "\widehat{h}"'] & \psi_* \Phi \arrow[d, "\rho"] \\
\widehat{A}_2 \arrow[r, "\varepsilon_2", "\sim"'] & X.
\end{tikzcd}$$

The point defined by:

$$(\psi_1 = \psi \circ \mathrm{Fr}^{-n}, X, \rho_1 = \rho \circ \psi_*(g^{-1} \operatorname{Frob}^n), A_1, \varepsilon_1, \overline{\nu}_1)$$

has image equal to the same algebrisation of X, which implies the commutativity of the diagram:

$$\begin{tikzcd}
\psi_{1*}\widehat{A}_1 \arrow[r, "\psi_{1*}\varepsilon_1", "\sim"'] \arrow[d, "\psi_* \operatorname{Frob}^n"'] & \psi_{1*}\Phi \arrow[d, "\psi_* \operatorname{Frob}^n"] \\
\psi_*\widehat{A}_1 \arrow[r, "\psi_*\varepsilon_1", "\sim"'] \arrow[d, "\psi_*\widehat{h}_g^{-1}"'] & \psi_*\Phi \arrow[d, "\psi_* g^{-1}"] \\
\psi_*\widehat{A} \arrow[r, "\psi_*\varepsilon", "\sim"'] \arrow[d, "\widehat{h}"'] & \psi_*\Phi \arrow[d, "\rho"] \\
\widehat{A}_2 \arrow[r, "\varepsilon_2", "\sim"'] & X
\end{tikzcd}$$

Finally, one sees that Θ *factors* via morphism of $\mathbf{F}_p$-schemes:

$$\overline{\Theta}\colon \operatorname{GL}_2(\mathbf{Q}_p)\backslash[(\widehat{\Omega}\otimes\overline{\mathbf{F}}_p)\times Z_U]\to \mathbf{S}_U\otimes\mathbf{F}_p.$$

6.6. *Showing that $\overline{\Theta}$ is an isomorphism.* 3.6.4.1 – The quotient above is none other than the special fiber of the quotient that appears in the statement of the Cerednik-Drinfeld theorem (recall that U is assumed small enough, and thus the group acts freely); comments anologus to those in (5.3) thus apply here. It is convenient to change scalars from $\mathbf{F}_p$ to $\overline{\mathbf{F}}_p$, and one sees easily that the scheme obtained from the quotient above can be identified with the quotient:

$$\operatorname{GL}_2'(\mathbf{Q}_p)\backslash(\widehat{\Omega}_{\overline{\mathbf{F}}_p}\times Z_U),$$

where one writes GL' for the subgroup of $\operatorname{GL}_2(\mathbf{Q}_p)$ consisting of the elements g such that $v(\det g)=0$. In terms of moduli, $\widehat{\Omega}_{\overline{\mathbf{F}}_p}$ represents the functor $\overline{G}$ which classifies pairs (X,ρ), and GL' operates by composition with ρ.

The morphism $\overline{\Theta}_{\overline{\mathbf{F}}_p}$ obtained from $\overline{\Theta}$ by extension of scalars yields a morphism of $\overline{\mathbf{F}}_p$-schemes:

$$\Theta_1\colon \widehat{\Omega}_{\overline{\mathbf{F}}_p}\times Z_U\to\mathbf{S}_U\otimes\overline{\mathbf{F}}_p.$$

This map associates to a point $(X,\rho,A,\varepsilon,\overline{\nu})$ defined over and $\overline{\mathbf{F}}_p$-algebra B the algebrisation of X obtained by applying lemma (6.3.1) with $X_1=\Phi_B$, $X_2=X$, $f=\rho$, $A_1=A_B$ and $\varepsilon=\varepsilon_B$.

3.6.4.2 – We next show that $\overline{\Theta}_{\overline{\mathbf{F}}_p}$ *induces a bijection between the collection of $\overline{\mathbf{F}}_p$-points of the two schemes*.

Injectivity: Suppose that two $\overline{\mathbf{F}}_p$-points $(X,\rho,A,\varepsilon,\overline{\nu})$ and $(X',\rho',A',\varepsilon',\overline{\nu}')$ have the same image A_2 (a special form $\mathcal{O}_\Delta$-abelian variety over $\overline{\mathbf{F}}_p$, with a level structure) under Θ_1. Then, since A_2 is both an algebrisation of X and X', one sees that X is naturally identified with X'. One sees also that ρ and ρ' differ by composition by an element $g\in\operatorname{GL}_2'(\mathbf{Q}_p)$: we may thus suppose $\rho=\rho'$. One notes finally that $(A,\varepsilon,\overline{\nu})$ [resp. $(A',\varepsilon',\overline{\nu}')$] is the algebrisation of Φ obtained by application of lemma (6.3.1) to the algebrisation A_2 of X and ρ^{-1} [resp. the algebrisation A_2 of X' and ρ'^{-1}].

The two points are thus indeed equivalent under the action of $\operatorname{GL}_2'(\mathbf{Q}_p)$.

Surjectivity: Given A_2, write X for its formal completion. Since every special formal $\mathcal{O}_{\Delta_p}$-module of height 4 over $\overline{\mathbf{F}}_p$ is isogeneous (chapter II, § 5), there exists a quasi-isogeny $\rho\colon\Phi\to X$. We may suppose, by composing with an appropriate endomorphism of Φ, that ρ is of height 0.

Apply lemma (6.3.1) to the algebrisation A_2 of X and ρ^{-1} to obtain an algebrisation $(A, \varepsilon, \overline{\nu})$ of Φ with level structure: it is clear that A_2 is the image of $(X, \rho, A, \varepsilon, \overline{\nu})$ by Θ_1.

3.6.4.3 – We now prove that Θ_1 is etale: let B be an $\overline{\mathbf{F}}_p$-alegebra, and $B' \to B$ a thickening of B, with a kernel of square zero; let $x = (X, \rho, A, \varepsilon, \overline{\nu})$ be a point with values in B of $\widehat{\Omega}_{\overline{\mathbf{F}}_p} \times Z_U$, and $y = A_2$ its image under Θ_1. Deforming x to a B'-point x' is the same as deforming the special formal $\mathcal{O}_{\Delta_p}$-module X: in fact, the quasi-isogeny ρ has a unique deformation (cf. [Zi 3] 5.31 for example), and the scheme Z_U is constant. One sees that Θ_1 givs a bijection between the deformations x' of x and the deformations y' of y: the inverse map associates to a deformation of A_2 the deformation of the corresponding formal group $\widehat{A}_2 \cong X$.

Therefore, $\overline{\Theta}_{\overline{\mathbf{F}}_p}$ is etale; since it is also bijective on $\overline{\mathbf{F}}_p$-points, it is an isomorphism. *Thus* $\overline{\Theta}$ *is an isomorphism.*

6.7. One thus obtains an isomorphism as predicted by theorem (5.2), *but so far only between the special fibres*; note that, by construction, it lifts to the formal $\mathcal{O}_{\Delta_p}$-modules naturally obtained from these two fibres. It is clear and formal to verify that this isomorphism is compatible with the projective system obtained by varying U^p, and that these two projective systems are $\Delta^*(\mathbf{A}_f^p)$-equivariantly isomorphic.

The possiblity to extend $\overline{\Theta}$ by an isomorphism between the two formal schemes follows from the theorem of Serre and Tate.

Let B be a $\mathbf{Z}_p$-algebra where p is nilpotent, and $B_0 = B/pB$. Giving a B_0-point x_0 of the scheme $\mathrm{GL}_2(\mathbf{Q}_p)\backslash[(\widehat{\Omega} \,\widehat{\otimes}\, \widehat{\mathbf{Z}_p}^{nr}) \times Z_U]$ endows (by inverse image) B_0 with a special formal $\mathcal{O}_{\Delta_p}$-module X_0. One sees that *deforming the point* x_0 *to* B *is the same as deforming* X_0: the question is in fact local, and so we are reduced to solving the problem for the functor G, which is trivial.

Finally, the giving of a B-point of the quotient scheme above is the same as giving its restriction x_0 to B_0, plus a deformation X of X_0 over B. Moreover it is clear that the giving of a B-point y of $\widehat{\mathbf{S}}_U$ is the same as giving its restriction y_0 and a deformation A of A_0 (the special abelian $\mathcal{O}_\Delta$-scheme over B_0 defined by y_0). As x_0 and y_0 correspond to one another under the isomorphism $\overline{\Theta}$, X_0 is identified with the completion $\widehat{A}_0$. The theorem of Serre and Tate ([Me], [Dr 2]) implies that the deformations of A_0 correspond bijectively with those of $\widehat{A}_0$, and therefore with X: this defines in a natural way an ismorphism between the formal scheme $\mathrm{GL}_2(\mathbf{Q}_p)\backslash[(\widehat{\Omega} \,\widehat{\otimes}\, \widehat{\mathbf{Z}_p}^{nr}) \times Z_U]$ and $\mathbf{S}_U$, which extends $\overline{\Theta}$, and which possesses all of the desired properties.

This concludes the proof of the Cerednik-Drinfeld theorem.

Remark 6.8. Consult the original paper of Boutot-Carayol for references.

Translated by CAMERON FRANC