Nearly rigid analytic modular forms and their values at CM points

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Abstract

In this thesis we expose a p-adic analogue of a classical result of Shimura on the algebraicity of CM values of modular forms and certain of their nonholomorphic derivatives. More specifically, we define an analogue of the Shimura-Maass differential operator for rigid analytic modular forms on the Cerednik-Drinfeld p-adic upper half plane. This definition leads us to define the space of nearly rigid analytic modular forms, which is a p-adic analogue of the space of complex valued nearly holomorphic modular forms. Our main theorem is a statement about the algebraicity of values of nearly rigid analytic modular forms at CM points in the p-adic upper half plane.

Abrégé

Nous démontrons un variante p-adique d'un résultat classique de Shimura sur l'algébricité des valeurs des formes modulaires et de certaines de leurs dérives non-holomorphes aux points CM. Plus précisément, nous définissons un analogue rigide analytique de l'opérateur différentiel de Shimura-Maass pour les formes modulaires rigide analytiques sur le demi-plan p-adique de Cerednik-Drinfeld. Cette définition nous conduit à définir l'espace des formes modulaires presque rigide analytiques, qui correspond dans notre analogie à l'espace des formes modulaires presque holomorphes. Notre résultat principal est un énoncé d'algébricité des valeurs des formes modulaires presque rigide analytiques en des points CM dans le demi-plan p-adique de Cerednik-Drinfeld.

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Contents

Chapter 1

Introduction

> As every mathematician knows, the parallelism between algebraic number theory and transcendental number theory exists only in their appellations and not in their contents. Indeed, the aim of the latter theory is to prove the transcendence of a given number, while algebraic numbers are there from the beginning in the former. Therefore, if one proves the algebraicity of an analytically defined number, it cannot be viewed as a theorem of either theory. It belongs to a new area of investigation for which this lecture is intended and to which I can give no good designation.
>
> – Goro Shimura, [40]

This chapter discusses background material which inspired this thesis. We also describe the main theorem of this work.

1.1 Values of modular forms at CM points

Let $\mathcal{H}$ denote the set of complex numbers with strictly positive imaginary part. Let $\tau = x + iy$ denote the usual complex coordinate on $\mathcal{H}$. Let $k \geq 0$ be an integer. Recall that the group of matrices in $\mathrm{GL}_2(\mathbf{R})$ of positive determinant, denoted $\mathrm{GL}_2(\mathbf{R})^+$, acts on $\mathcal{H}$ via fractional linear transformation, and hence on functions $f\colon \mathcal{H} \to \mathbf{C}$ on the right via the weight k slash operator:

$$(f|_k\gamma)(\tau) = \det(\gamma)^{k/2}(c\tau + d)^{-k}f(\gamma\tau) \qquad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{R})^+.$$

Here $\det(\gamma)^{k/2}$ indicates the positive branch of the square root. As is customary, for any ring R and

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R),$$

we write $j(\gamma, \tau) = c\tau + d$. We will only be interested in groups Γ consisting of matrices of determinant one. In this case the weight k slash operator is given by the formula

$$(f|_k\gamma)(\tau) = j(\gamma, \tau)^{-k} f(\gamma\tau).$$

Recall that holomorphic modular forms are holomorphic functions on $\mathcal{H}$ which are invariant under some arithmetic subgroup $\Gamma \subset \mathrm{GL}_2(\mathbf{R})$ with respect to some fixed integer weight k, and which satisfy a growth hypothesis at the cusps of $\Gamma\backslash\mathcal{H}$. Denote the space of holomorphic modular forms of weight k for Γ by $M_k(\Gamma)$. Put $M(\Gamma) = \bigoplus_{k\geq 0} M_k(\Gamma)$.

Definition 1.1.1. The weight k *Shimura-Maass differential operator* is

$$\delta_k = \frac{1}{2\pi i}\left(\frac{d}{d\tau} + \frac{k}{\tau - \overline{\tau}}\right),$$

where the operator $k/(\tau - \overline{\tau})$ above denotes multiplication: $f(\tau) \mapsto kf(\tau)/(\tau - \overline{\tau})$.

Set $\delta_k^0 f = f$ and for $r \geq 1$ put:

$$\delta_k^r = \delta_{k+2(r-1)} \circ \delta_{k+2(r-2)} \circ \cdots \circ \delta_k.$$

We refer to the δ_k^r's also as Shimura-Maass differential operators.

The operators δ_k^r preserve modularity, in the sense that if a (real) differentiable function f on $\mathcal{H}$ is invariant under the classical weight k slash operator, then $\delta_k^r f$ is weight $(k + 2r)$ invariant. If f is moreover holomorphic, for instance if f is a modular form, then the presence of the term $kf(\tau)/(\tau - \overline{\tau})$ in $\delta_k f$ prevents the Shimura-Maass derivative from being holomorphic. It is thus natural to study the smallest ring of functions which contains all analytic functions on $\mathcal{H}$ and which is closed under the Shimura-Maass differential operators.

Definition 1.1.2. A function $f\colon \mathcal{H} \to \mathbf{C}$ is said to be *nearly holomorphic* if it is of the form:

$$f(\tau) = \sum_{i=0}^{r} \frac{f_i(\tau)}{(\tau - \overline{\tau})^i}$$

for holomorphic functions $f_i(\tau)\colon \mathcal{H} \to \mathbf{C}$. A function $f\colon \mathcal{H} \to \mathbf{C}$ is said to be a *nearly holomorphic modular form* of weight k for Γ if the following three conditions are satisfied:

1. f is weight k invariant for Γ,
2. f is nearly holomorphic, and
3. f is slowly increasing, that is, there exist real constants A, $B > 0$ such that for every $\gamma \in \mathrm{SL}_2(\mathbf{Z})$, one has $|(f|_k\gamma)(x+iy)| < A(1+y^{-B})$ for y large enough; the implied constant here depends on f, A, B and γ.

Remark 1.1.3. Nearly holomorphic modular forms have been extensively studied by Shimura; for example, see his text [42]. They are closely related to *quasi-modular forms*, which are holomorphic functions that are *almost* (in a precise sense which we won't define) invariant under the slash operators. A standard example of a quasi-modular form is the Eisenstein series E_2 of weight 2. Quasi-modular forms were first given a formal definition by Kaneko and Zagier in [25]. For more information on the relationship between nearly-holomorphic modular forms and quasi-modular forms, consult chapters 5 and 6 of Zagier's lectures in [5].

Let $\overline{\mathbf{Q}} \subset \mathbf{C}$ denote an algebraic closure of $\mathbf{Q}$. Shimura proved the following remarkable result concerning the values of Shimura-Maass derivatives of modular forms at CM points:

Theorem 1.1.4 (Shimura). *Let $K \subset \overline{\mathbf{Q}}$ denote a quadratic imaginary extension of $\mathbf{Q}$. There exists $\Omega_K \in \mathbf{C}^\times$ such that for every $\tau \in K \cap \mathcal{H}$, every congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$, every k, $r \geq 0$ and every $f \in M_k(\Gamma)$ with algebraic Fourier coefficients,*

$$\frac{(\delta_k^r f)(\tau)}{\Omega_K^{k+2r}} \in \overline{\mathbf{Q}}.$$

Before turning to the results proved in this thesis, we would like to provide some further explanation concerning the arithmetic significance of Shimura's result. For this, let $K/\mathbf{Q}$ denote a quadratic imaginary extension of discriminant d_K. Embed K into $\mathbf{C}$ and consider the series

$$A(k,r) = \sum_{\substack{\alpha \in \mathcal{O}_K \\ \alpha \neq 0}} \frac{\overline{\alpha}^r}{\alpha^{k+r}}$$

for integers k and r. The series converges absolutely for $k \geq 3$ and $r \geq 0$. Note that $A(k,r)$ is related to the *Hurwitz zeta function*

$$\zeta_K(n_1, n_2) = \sum_{\substack{\alpha \in \mathcal{O}_K \\ \alpha \neq 0}} \alpha^{-n_1} (\overline{\alpha})^{-n_2}$$

by the formula $A(k,r) = \zeta_K(k+r, -r)$. We adopt the notation $A(k,r)$ since we are following Katz's wonderful ICM paper [29].

Let $G_k(\tau)$ denote the non-normalized Eisenstein series for $\mathrm{SL}_2(\mathbf{Z})$ of weight $k \geq 4$:

$$G_k(\tau) = \sum_{\substack{(m,n) \in \mathbf{Z}^2 \\ (m,n) \neq (0,0)}} \frac{1}{(m+n\tau)^k}.$$

Note that G_k is not algebraic, but $\pi^{-k} G_k$ has rational Fourier coefficients, and so is in fact rational. Write $\mathcal{O}_K = \mathbf{Z} \oplus \mathbf{Z}\omega$ where we take $\omega \in \mathcal{H}$. Then a straightforward computation allows one to prove the following:

Theorem 1.1.5. *With notation as above, for all integers $r \geq 1$ one has*

$$(\delta_k^r G_k)(\omega) = \frac{(k+r-1)!}{(\omega - \overline{\omega})^r (k-1)!} A(k,r).$$

Thus, if one combines this with Shimura's result, one deduces the algebraicity of the ratios

$$\frac{A(k,r)}{\pi^k \Omega_K^{k+2r}}.$$

This was first proved by Damerell [10], and in fact, one can prove more: if one writes

$$B(k,r) = \frac{(-1)^k (k+r-1)! (2\pi i)^r}{2(\omega - \overline{\omega})^r \Omega_K^{k+2r}} \cdot A(k,r),$$

then for any integer $b \geq 1$, the values

$$b^k (b^k - 1) (\sqrt{d_K})^r B(k,r)$$

are algebraic *integers*.

Katz studied the p-adic properties of these integers in [27] and later work. Let $E(\mathbf{C}) = \mathbf{C}/\mathcal{O}_K$, and note that by the theory of complex multiplication, $E(\mathbf{C})$ is the set of complex points of an elliptic curve E defined over the ring $\mathcal{O}_{\overline{\mathbf{Q}}}$ of algebraic integers, and which has everywhere good reduction there. In fact, one can choose an invariant differential ω on E such that the reduction of ω modulo every prime in $\mathcal{O}_{\overline{\mathbf{Q}}}$ is a nonzero regular differential on the reduced curve. The *period lattice* for (E, ω), that is, the collection of integrals of ω against the integral homology of $E(\mathbf{C})$, has the form $\Omega_K \cdot \mathcal{O}_K$ for a value $\Omega_K \in \mathbf{C}^\times$. This period Ω_K is well-defined up to multiplication by a unit in $\mathcal{O}_{\overline{\mathbf{Q}}}$ and is a valid choice for the period appearing in Shimura's algebraicity theorem 1.1.4 above.

Let K'/K denote a finite extension such that (E, ω) is defined over K' and has good reduction at all the primes of K'. Let $\mathfrak{p}$ denote a prime of K', let $K'_\mathfrak{p}$ denote the completion of K' at $\mathfrak{p}$, and let W denote the ring of integers in the completion of the maximal unramified extension of $K'_\mathfrak{p}$. Let p denote the rational prime beneath $\mathfrak{p}$.

Theorem 1.1.6 (Katz). *If p splits in K, then there exists a unit $c \in W^\times$ and, for all rational integers b prime to p, a W-valued p-adic measure $\mu(c, b)$ on $\mathbf{Z}_p \times \mathbf{Z}_p$, whose moments are given by the formula*

$$\int_{\mathbf{Z}_p \times \mathbf{Z}_p} x^{k-3} y^r \, d\mu(c, b) = 2c^{k+2r}(b^k - 1)B(k, r),$$

which is valid for integers $k \geq 3$ and $r \geq 0$.

Proof. See the Theorem on page 367 of [29]. □

Katz used this measure to construct a two-variable p-adic L-function attached to K, and showed that it interpolates certain L-values of the Dedezind zeta function of $K/\mathbf{Q}$ twisted by higher weight characters.

1.2 A rigid analytic analogue

The main theorem of this work is a rigid analytic analogue of Shimura's Theorem 1.1.4. Let $\mathcal{H}_p = \mathbf{P}_1(\mathbf{C}_p) - \mathbf{P}_1(\mathbf{Q}_p)$ denote the p-adic upper half plane; see Chapter 3 for more on the definition of $\mathcal{H}_p$ and for a brief summary of its properties. The group $\mathrm{GL}_2(\mathbf{Q}_p)$ acts on $\mathcal{H}_p$ by fractional linear transformations, and one has a weight k slash operator defined on the space of functions $\mathcal{H}_p \to \mathbf{C}_p$ using the same formula as in the previous section. Rigid analytic modular forms are rigid analytic functions on $\mathcal{H}_p$ which are invariant under this action with respect to some arithmetic subgroup $\Gamma \subset \mathrm{SL}_2(\mathbf{Q}_p)$; for more on this subject the reader can consult Chapter 4. Let $S_k(\Gamma)$ denote the collection of rigid analytic modular forms for some arithmetic group Γ. Note that we use the letter S rather than M because the groups Γ which we consider arise from definite quaternion algebras and they contain no parabolic elements.

Let $\widehat{\mathbf{Q}_p^{\mathrm{un}}} \subset \mathbf{C}_p$ denote the completion of the maximal unramified extension of $\mathbf{Q}_p$ inside $\mathbf{C}_p$, and let $\mathcal{H}_p(\widehat{\mathbf{Q}_p^{\mathrm{un}}}) = \mathbf{P}_1(\widehat{\mathbf{Q}_p^{\mathrm{un}}}) - \mathbf{P}_1(\mathbf{Q}_p)$ denote the $\widehat{\mathbf{Q}_p^{\mathrm{un}}}$-rational points of $\mathcal{H}_p$. Let $\tau \mapsto \overline{\tau}$ denote the arithmetic Frobenius automorphism of $\widehat{\mathbf{Q}_p^{\mathrm{un}}}$. Let $\mathcal{O}$ denote the ring of rigid analytic functions on $\mathcal{H}_p$. Then $\mathcal{O}$ injects into $C(\mathcal{H}_p(\widehat{\mathbf{Q}_p^{\mathrm{un}}}), \mathbf{C}_p)$, the space of continuous functions $\mathcal{H}_p(\widehat{\mathbf{Q}_p^{\mathrm{un}}}) \to \mathbf{C}_p$. Note that the arithmetic group Γ acts on $C(\mathcal{H}_p(\widehat{\mathbf{Q}_p^{\mathrm{un}}}), \mathbf{C}_p)$ via the slash operator. In Section 4.3 we define the ring $N_k(\Gamma)$ of *nearly rigid analytic modular forms* to be the subring of $C(\mathcal{H}_p(\widehat{\mathbf{Q}_p^{\mathrm{un}}}), \mathbf{C}_p)$ consisting of all functions of the form

$$f(\tau) = \sum_{i=0}^{r} \frac{f_i(\tau)}{(\tau - \overline{\tau})^i},$$

where $f_i(\tau) \in \mathcal{O}$ for all i and such that $f|_k\gamma = f$ for all $\gamma \in \Gamma$. We show that $f \in N_k(\Gamma)$ determines the rigid analytic coefficients $f_i(\tau)$ uniquely, and that the formula

$$(\delta_k f)(\tau) = \frac{df}{d\tau}(\tau) + \frac{kf(\tau)}{\tau - \overline{\tau}},$$

defines a differential operator

$$\delta_k \colon N_k(\Gamma) \to N_{k+2}(\Gamma).$$

We define δ_k^r by iterating δ's as in the previous section.

For a certain class of arithmetic subgroups $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Q}_p)$ (see Section 2.1.4), we use the Cerednik-Drinfeld theorem (see Section 3.7) to give a definition for the algebraicity of a rigid analtyic modular form. We then show that if $K/\mathbf{Q}$ is a quadratic imaginary extension for which there are CM-points for K inside $\mathcal{H}_p$ (see Section 3.6), then there exists a period $\Omega_K \in \mathbf{C}_p^\times$ such that the following algebraicity property holds: for all $f \in S_k(\Gamma)$ which are algebraic, for all CM-points $\tau \in \mathcal{H}_p$ by an order in $K/\mathbf{Q}$, and for all integers $r \geq 0$, one has

$$\frac{\delta_k^r f(\tau)}{\Omega_K^{k+2r}} \in \overline{\mathbf{Q}}.$$

This is the main result of this paper; for a precise statement, see Chapter 5.

In the final Chapter 6 we discuss several questions which evolved from this work, as well as possible applications of our result. The appendix includes computations which illustrate our main theorem.

Chapter 2

Shimura curves

2.1 Quaternion algebras

Our exposition in this section is brief; for more details consult [18], [44] or the book in preparation [45].

Let K be a field, and assume that it is of characteristic zero for simplicity.

Definition 2.1.1. A *quaternion algebra* over K is a central simple algebra B with $\dim_K B = 4$.

In this work we are only concerned with the case when K is either $\mathbf{Q}$ or a local field of characteristic zero. Let L/K be a field extension and let B/K be an algebra. We write $B_L = B \otimes_K L$; note that B is a quaternion algebra if and only if B_L is so. In the special case that $K = \mathbf{Q}$ and $L = \mathbf{Q}_p$ for some prime p, we will often write $B_p = B \otimes_{\mathbf{Q}} \mathbf{Q}_p$.

Definition 2.1.2. Let A be a commutative integral domain with field of fractions K. An A-*order* in a quaternion algebra B/K is a subring R of B which satisfies the following two conditions: (1) R is finitely generated as an A-module, and (2) one has $KR = B$. A *maximal* A-*order* is an A-order not contained in any other A-order.

Remark 2.1.3. We say that elements in B which are roots of a monic polynomial over A are *integral*. If one assumes that A is noetherian in the definition above, then for every A-order $R \subseteq B$, and for every $x \in R$, the subring $A[x] \subseteq R$ is finitely generated. Thus, x is a root of the characteristic polynomial of multiplication by x on $A[x]$, that is, x is integral. Note, though, that the sum or product of an arbitrary pair of integral elements of B need not be integral, unlike what happens in the setting of commutative rings.

The following definition will be useful later, when we wish to give a criterion for the existence of CM-points.

Definition 2.1.4. Let B/K be a quaternion algebra and let F/K be a field extension. Then F is said to *split* B if $B \otimes_K F \cong M_2(F)$.

Proposition 2.1.5. *A quadratic extension F/K splits a quaternion algebra B/K if and only if F embeds into B as a K-algebra.*

Proof. See Proposition 1.2.3 of [18]. □

We end this section on generalities by discussing the reduced norm and reduced trace of a quaternion algebra. Recall that Hamilton viewed his quaternion algebra as a 4-dimensional generalization of the complex numbers. There one can easily define a conjugation $z \mapsto \overline{z}$ by negating the "purely imaginary" part of a Hamilton quaternion. Then one can define a norm and trace:

$$n(z) = z\overline{z}, \quad t(z) = z + \overline{z}$$

for Hamilton quaternions. A similar fact is true for every quaternion algebra: that is, every quaternion algebra possesses a canonical anti-involution which is linear over K. If B/K is a quaternion algebra over a field K of characteristic 0, and $z \mapsto \overline{z}$ denotes the canonical anti-involution, then the *reduced norm* of B is defined by $n(z) = z\overline{z}$, and the *reduced trace* of B is defined by $t(z) = z + \overline{z}$. Both n and t have image in the center of B, and so we view them as maps to K. The reduced norm is important for us becase a quaternion is invertible if and only if its reduced norm is nonzero.

Example 2.1.6. In the case $B = M_2(K)$ then the canonical involution is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

In this case the reduced norm is the determinant, while the reduced trace is the trace.

Remark 2.1.7. Do not confuse the reduced trace and norm with the algebra trace and norm of B/K. Recall that if B/K is an algebra, if $z \in B$ and $m_z\colon B \to B$ denotes the K-linear map given by left multiplication by z, then one defines the norm $N(z) = \det m_z$. If B/K is a quaternion algebra, then the algebra norm is the square of the reduced norm, which explains the terminology. Similarly for the reduced trace.

2.1.1 Quaternion algebras over local fields

Let K be a local field of characteristic zero. If $K = \mathbf{C}$ then since $\mathbf{C}$ is algebraically closed, one can show that, up to isomorphism, $M_2(\mathbf{C})$ is the unique quaternion algebra over $\mathbf{C}$. Over $K = \mathbf{R}$ there are precisely two isomorphism classes of quaternion algebras: the classes of $M_2(\mathbf{R})$ and of $\mathbf{H}$, where the latter denotes the so-called Hamilton quaternions.

Over $K = \mathbf{Q}_p$ the situation is similar to the case $K = \mathbf{R}$: besides $M_2(\mathbf{Q}_p)$ there is a second isomorphism class of quaternion algebras over K, and this class consists of division algebras. Such a division algebra can be described as follows: let $\mathbf{Q}_q$ denote the quadratic unramified extension of $\mathbf{Q}_p$. Let $B = \mathbf{Q}_q \oplus \mathbf{Q}_q\pi$ and define a multiplication on B by setting $\pi^2 = p$ and for $x \in \mathbf{Q}_q$, set $\pi x = \overline{x}\pi$, where the overline denotes the nontrivial Galois automorphism in $\mathrm{Gal}(\mathbf{Q}_q/\mathbf{Q}_p)$. One can show that the valuation of $\mathbf{Q}_q$ extends to B by setting $v(\pi) = 1/2$. Then the valuation ring of B is the unique maximal $\mathbf{Z}_p$-order of B. We summarize this below:

Proposition 2.1.8. *Over $\mathbf{Q}_p$ there exists a unique isomorphism class of division quaternion algebras. Such an algebra contains a unique maximal $\mathbf{Z}_p$-order.*

2.1.2 Quaternion algebras over the rationals

In this section $Z \subseteq \mathbf{Q}$ will denote a finitely generated subring. For example, we could take $Z = \mathbf{Z}$ or $Z = \mathbf{Z}[1/p]$.

Definition 2.1.9. Let $B/\mathbf{Q}$ be a quaternion algebra and let p be a prime of $\mathbf{Q}$ ($p = \infty$ is allowed). Then B is said to be *unramified* at p if $B_p \cong M_2(\mathbf{Q}_p)$. Otherwise B is said to be *ramified* at p. In the case $p = \infty$ one also says that B is *indefinite* if it is unramified at ∞, and *definite* otherwise.

The *discriminant* $\Delta = \Delta(B)$ of B is defined to be the product of all the ramified primes in B.

Quaternion algebras over $\mathbf{Q}$ are determined up to isomorphism by their discriminant:

Theorem 2.1.10. *Every quaternion algebra over $\mathbf{Q}$ ramifies at an even number of primes. If $\Delta > 1$ is a squarefree positive integer, then there exists a single isomorphism class of quaternion algebras over $\mathbf{Q}$ of discriminant Δ, and all such algebras are division algebras. If $\Delta = 1$ then $M_2(\mathbf{Q})$ is a representative for the unique class of unramified quaternion algebras over $\mathbf{Q}$.*

Proof. See Theorem 3.1 of Chapter III in [44]. □

Over $\mathbf{Q}$ a quaternion algebra B contains many maximal orders. For example, if $B = M_2(\mathbf{Q})$ then $R = M_2(\mathbf{Z})$ is a maximal $\mathbf{Z}$-order. For any $M \in \mathrm{GL}_2(\mathbf{Q})$, one obtains another maximal $\mathbf{Z}$-order $M^{-1}RM$ which need not be equal to R.

Definition 2.1.11. Let $B/\mathbf{Q}$ be a quaternion algebra. An *Eichler Z-order* in B is an intersection of two maximal Z-orders. If $R = R_1 \cap R_2$ is an Eichler Z-order with the R_i's two maximal Z-orders, then the *level* of R is defined to be the index of the lattice R in R_1.

Since $R_1 \cap R_2 = R_2 \cap R_1$, this definition of level does not appear to be well-defined. We thus state the following lemma.

Lemma 2.1.12. *Let $B/\mathbf{Q}$ be a quaternion algebra, let R_1 and R_2 be two maximal Z-orders, and let $R = R_1 \cap R_2$. Then $(R_1\colon R) = (R_2\colon R)$.*

Proof. See Section 1 in Chapter III of [37]. □

Reconsider the example $B = M_2(\mathbf{Q})$. In this case all maximal $\mathbf{Z}$-orders are conjugate by elements of $B^\times = \mathrm{GL}_2(\mathbf{Q})$: just choose a basis for a maximal order to give a conjugating element which takes the order to $M_2(\mathbf{Z})$. This need not hold in general for quaternion algebras over $\mathbf{Q}$, but it does hold if the following condition is satisfied:

Definition 2.1.13. A quaternion algebra $B/\mathbf{Q}$ is said to satisfy the *Eichler condition* for Z if B is indefinite, or if there is a rational prime that splits B and which is invertible in Z.

The following proposition explains the importance of this condition:

Proposition 2.1.14. *Let $B/\mathbf{Q}$ denote a quaternion algebra, let $Z \subseteq \mathbf{Q}$ denote a finitely generated subring, and suppose that B satisfies the Eichler condition for Z. Then there is a unique $B^\times$-conjugacy class of maximal Z-orders in B. Similarly, any two Eichler Z-orders of the same level are conjugate.*

Proof. See Corollaire 5.17 bis in Chapter III of [44]. Note that the result stated there is for a quaternion algebra over a general number field K, and the number h appearing there is the narrow class number of K. In the case $K = \mathbf{Q}$, of course $h = 1$ and so there is a unique conjugacy class of Eichler orders for any given level in this case. □

In applications we will be given a quaternion algebra $B/\mathbf{Q}$ split at a finite prime p, and we will consider Z-orders in B for $Z = \mathbf{Z}[1/p]$. Such pairs (B, Z) satisfy the Eichler condition, and so in our applications there will always be a *unique* Eichler Z-order of any given level, up to conjugacy.

Proposition 2.1.15 (Hasse-Brauer-Noether-Albert)**.** *Let $B/\mathbf{Q}$ denote a quaternion algebra and let $K/\mathbf{Q}$ denote a quadratic extension. Then B contains an isomorphic copy of K if and only if each prime which divides the discriminant of B is either inert or ramified in K.*

Proof. This theorem is proved in many places and in many different formulations. The expository paper [38] of Shemanske gives a nice discussion of this theorem and its reformulations. The statement above is equivalent with Theorem 2.4 in Shemanske's paper. □

2.1.3 Adelizations

As in the previous section, let $Z \subseteq \mathbf{Q}$ denote a finitely generated subring. Let $\widehat{\mathbf{Q}}$ denote the finite adele ring of $\mathbf{Q}$. If we let $\widehat{\mathbf{Z}}$ denote the profinite completion of $\mathbf{Z}$, then $\widehat{\mathbf{Q}} \cong \widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q}$. Let $B/\mathbf{Q}$ be a quaternion algebra and let R be an Eichler Z-order of level N in B. Put

$$\widehat{R} = R \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}, \qquad \widehat{B} = B \otimes_{\mathbf{Q}} \widehat{\mathbf{Q}}.$$

We refer to these as the *adelisations* of R and B.

Proposition 2.1.16. *The set of Eichler Z-orders of level N in B is in natural bijection with the coset space $\widehat{B}^\times/\widehat{\mathbf{Q}}^\times\widehat{R}^\times$ via the map which sends a finite idele (b_l) to the order*

$$(b_l)\widehat{R}(b_l^{-1}) \cap B.$$

Proof. See the *global-adelic dictionary* on page 87 of [44]. □

Corollary 2.1.17. *The set of conjugacy classes of Eichler Z-orders of level N in B is in natural bijection with the double coset space $B^\times\backslash\widehat{B}^\times/\widehat{R}^\times$.*

Below we will use a finite splitting prime of B to generate a discrete subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$. It will thus be useful, in particular for our discussion of CM-points on the p-adic upper half plane, to note the following p-adic description of the set of conjugacy classes of an Eichler order.

Theorem 2.1.18 (Strong approximation)**.** *Let p be a finite splitting prime for B. Let R be an Eichler Z-order of level N in B. Then there is a natural bijection*

$$R[1/p]^\times\backslash B_p^\times/R_p^\times \to B^\times\backslash\widehat{B}^\times/\widehat{R}^\times$$

sending a class $(R[1/p]^\times)b_p(R_p^\times)$ to the class of the idele with all entries equal to 1, save for in the pth spot where it equals b_p.

Proof. See Théorème 4.3 of [44]. □

2.1.4 Congruence subgroups

Let p be a rational prime and N^- a positive squarefree integer with an odd number of prime divisors which is coprime to p. Up to isomorphism there is a unique quaternion algebra $B/\mathbf{Q}$ of discriminant N^-; fix such an algebra $B/\mathbf{Q}$. Note that B is definite and unramified at p. Hence, it satisfies the Eichler $\mathbf{Z}[1/p]$-order condition.

Let $R_0 \subseteq B$ denote the maximal $\mathbf{Z}$-order whose local component for each finite prime l which does not divide N^- is simply $M_2(\mathbf{Z}_l)$. Let $R = R_0[1/p]$ be the corresponding $\mathbf{Z}[1/p]$-order, and note that we have not changed any of the local components other than at p. If N^+ is a positive integer coprime to pN^-, then let $R(N^+) \subseteq R$ denote the $\mathbf{Z}[1/p]$-order whose local components equal those of R, save for at the places l dividing N^+. At such places we impose the condition that

$$R(N^+)_l = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}_l) \,\middle|\, c \equiv 0 \pmod{l^t} \right\},$$

where l^t divides N^+ exactly. Let $R(N^+)_1$ denote the subgroup of elements in $R(N^+)$ of reduced norm equal to 1.

Definition 2.1.19. Let B_1 denote the elements of reduced norm 1 in the quaternion algebra B. A subgroup $\Gamma \subseteq B_1$ is said to be a *congruence subgroup* for B if it contains a finite index subgroup of the form $R(N^+)_1$ for some $N^+ \geq 1$ coprime to pN^-.

Lemma 2.1.20. *Let Γ and Γ' be congruence subgroups for B. Then $\Gamma \cap \Gamma'$ is also a congruence subgroup. Similarly, if $b \in B^\times$ then $b^{-1}\Gamma b$ is a congruence subgroup.*

Proof. In the first case, let Γ and Γ' be congruence subgroups. Then if M^+ and N^+ are integers coprime to pN^- such that $R(M^+)_1 \subseteq \Gamma$ and $R(N^+)_1 \subseteq \Gamma'$, and if $L^+ = \gcd(N^+, M^+)$, then $R(L^+)_1 \subseteq \Gamma \cap \Gamma'$.

For the second part, note that $b \in B^\times$ is a local unit at all but finitely many primes. For such primes conjugating by b leaves $R(N^+)_1$ invariant. At the primes where b is not a local unit one adopts an argument analogous to the case of modular curves and congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$. $\square$

2.2 Shimura curves

2.2.1 Definition

Let p be a rational prime, let N^- denote a squarefree positive integer which is relatively prime to p and which has an odd number of prime factors, and let N^+ denote an integer which is corpime to pN^-. Let $\mathcal{B}$ denote a quaternion algebra over $\mathbf{Q}$ of discriminant pN^-. Since pN^- has an even number of distinct prime divisors, $\mathcal{B}$ is indefinite. This means that $\mathcal{B}_\infty \cong M_2(\mathbf{R})$. Note also that since the discriminant pN^- of $\mathcal{B}$ is larger than 1, the algebra $\mathcal{B}$ is a division algebra over $\mathbf{Q}$. This is essentially the only role played by the prime p in this section – it ensures that $\mathcal{B}$ is a division algebra. This prime will play a more important role later in our discussion of the Cerednik-Drinfeld theorem.

Let $\mathcal{R}$ denote a maximal **Z**-order in $\mathcal{B}$ and let $\mathcal{R}(N^+)$ denote an Eichler **Z**-order in $\mathcal{R}$ of level N^+. Let Γ_{N^+,pN^-} denote the group of units of $\mathcal{R}(N^+)$ of reduced norm equal to 1. We fix an identification

$$\iota\colon \mathcal{B}_\infty \xrightarrow{\sim} M_2(\mathbf{R}),$$

and identify Γ_{N^+,pN^-} as a subgroup of $\mathrm{SL}_2(\mathbf{R})$. Let $\mathcal{H}$ denote the classical complex upper half plane. Then Γ_{N^+,pN^-} acts on $\mathcal{H}$ via fractional linear transformation.

Proposition 2.2.1. *The group Γ_{N^+,pN^-} acts discretely and with compact quotient on $\mathcal{H}$.*

Proof. For the statement about the discreteness see Theorem 5.2.7 of [26]. For the statement about the compact quotient see Section 5.4 of [26]. □

It follows, say via the theory developed in Chapter 1 of [41], from the above proposition that the quotient $\Gamma_{N^+,pN^-}\backslash\mathcal{H}$ has a natural structure of a compact Riemann surface. There thus exists a projective algebraic curve over X_{N^+,pN^-} whose complex analytification is isomorhic with $\Gamma_{pN^+,N^-}\backslash\mathcal{H}$. In fact, Shimura showed that X_{N^+,pN^-} is defined over **Q**, and that it has an integral model which has good reduction away from the primes dividing pN^-N^+.

2.2.2 Moduli interpretation

Let S be a **Q**-algebra and let A be an abelian scheme over S of relative dimension 2. Then A is said to have *quaternionic multiplication* by the maximal order $\mathcal{R}$ in $\mathcal{B}$ if $\mathcal{R}$ acts on A; that is, there is a ring homomorphism

$$\iota\colon \mathcal{R} \to \mathrm{End}_S A.$$

A *level N^+-structure* on (A,ι) is a subgroup scheme C of A which is locally isomorphic to the constant scheme $\mathbf{Z}/N^+\mathbf{Z}$, and which is stable and locally cyclic for the action of $\mathcal{R}(N^+)$ defined via ι.

Theorem 2.2.2. *The curve $X_{N^+,pN^-}/\mathbf{Q}$ is the coarse moduli scheme for the following moduli problem: to each **Q**-algebra S, the point set $X_{N^+,pN^-}(S)$ is in functorial bijection with the set of triples (A,ι,C) where A is an abelian scheme over S of relative* *dimension 2, the map ι endows A with quaternionic multiplication by $\mathcal{R}$, and C is a subgroup scheme of A defining a level N^+-structure.*

For abelian surfaces with quaternionic multiplication, one has a notion of complex multiplication analogous to the case of elliptic curves:

Definition 2.2.3. Let $F/\mathbf{Q}$ be a field extension. Let A/F denote an abelian surface with quaternionic multiplication $\iota\colon \mathcal{R} \to \mathrm{End}_F(A)$. Then $\mathrm{End}_{\mathcal{R}}(A)$ is either isomorphic with $\mathbf{Z}$ or an order $\mathcal{O}$ in a quadratic imaginary field $K/\mathbf{Q}$. In the latter case, A is said to have *complex multiplication* by $\mathcal{O}$, or simply is said to have CM by $\mathcal{O}$.

Definition 2.2.4. Let $F/\mathbf{Q}$ denote a field extension. Let (A, ι, C) denote a point of $X_{N^+,pN^-}(F)$ such that (A, ι) has CM by an order $\mathcal{O}$ in a quadratic imaginary field. Then (A, ι, C) is said to be a *CM point* of X_{N^+,pN^-}.

Remark 2.2.5. Suppose A has quaternionic multiplication by $\mathcal{R}$ and complex multiplication by $\mathcal{O} \subseteq K$, and let H denote the Hilbert class field of K. Then it can be shown that $A_H \cong E \times E$, where E/H is an elliptic curve with CM by $\mathcal{O}$, and $A_H = A \times \mathrm{Spec}(H)$.

2.2.3 The Jacquet-Langlands correspondence

Recall that $\Gamma_{N^+,pN^-} \backslash \mathcal{H}$ is a compact Riemann surface. For each integer $k \geq 0$ we write $S_k(\Gamma_{N^+,pN^-})$ for the space of modular forms of weight k for Γ_{N^+,pN^-}. Although the study of modular forms on groups arising from division quaternion algebras over $\mathbf{Q}$ differs qualitatively from the classical theory of modular forms on congruence subgroups of $\mathrm{SL}_2(\mathbf{R})$, many similarities and connections persist. Perhaps the most interesting connection is expressed by the Jacquet-Langlands correspondence [24]. Before we can explain what this correspondence says in a manner which adapted to our work, we must introduce a few concepts.

Write $N = pN^-N^+$ and let $S_k(\Gamma_0(N))$ denote the space of cusp forms on the congruence subgroup $\Gamma_0(N)$ of matrices in $\mathrm{SL}_2(\mathbf{Z})$ which are upper triangular modulo N. Recall that for proper divisors d of N, there are natural inclusions and degeneracy maps of $S_k(\Gamma_0(d))$ into $S_k(\Gamma_0(N))$. The image of all these maps spans the space of *oldforms* for $\Gamma_0(N)$, and its orthogonal complement for the Petersson inner product is the *new* subspace of $S_k(\Gamma_0(N))$.

Consider instead the inclusions and degeneracy maps for the proper divisors d of pN^- only. Then the span of the images of these maps is a subspace of the old space which we call the pN^-*-old space*. Its orthogonal complement with respect to the Petersson inner product is called the pN^--new space, and we denote it $S_k(\Gamma_0(N))^{pN^-\text{-new}}$.

Note that both $S_k(\Gamma_0(N))$ and $S_k(\Gamma_{pN^+,N^-})$ carry an action of the Hecke algebra $\mathbf{T}$ defined via double coset operators. The new space constructed above is a Hecke submodule. One has the following relationship between these Hecke modules:

Theorem 2.2.6 (Jacquet-Langlands)**.** *With notation as above, there is a Hecke-equivariant isomorphism*

$$S_k(\Gamma_0(N))^{pN^-\text{-}new} \cong S_k(\Gamma_{N^+,pN^-}).$$

Proof. Consult the original [24], or the final chapter of the more recent book [6]. □

Chapter 3

The p-adic upper half plane

In this chapter p denotes a fixed rational prime. Fix a p-adic completion $\mathbf{C}_p$ of an algebraic closure of $\mathbf{Q}_p$. Let K_p denote the unique unramified quadratic extension of $\mathbf{Q}_p$ contained in $\mathbf{C}_p$. We write $x \mapsto \overline{x}$ for the nontrivial element of $\mathrm{Gal}(K_p/\mathbf{Q}_p)$, in analogy with the complex setting.

In this section the phrase *rigid analytic variety* always refers to a rigid analytic variety in the sense of Tate; see [2] or [16] for generalities on rigid analytic spaces, or the notes [13] for a more focused discussion on the p-adic upper half plane. If X is a rigid analytic variety then we write $\mathcal{O}_X$ for its sheaf of rigid analytic functions.

3.1 Definition

Note that the classical upper half plane is one of the two connected components of $\mathbf{P}_1(\mathbf{C}) - \mathbf{P}_1(\mathbf{R})$. This suggests the following definition of the p-adic upper half plane *as a set*.

Definition 3.1.1. The *p-adic upper half plane* $\mathcal{H}_p$ is the set $\mathbf{P}_1(\mathbf{C}_p) - \mathbf{P}_1(\mathbf{Q}_p)$.

We write points of $\mathbf{P}_1(\mathbf{C}_p)$ in terms of homegeneous coordinates $(xy)^T$. Let $\mathrm{GL}_2(\mathbf{Q}_p)$ act on $\mathbf{P}_1(\mathbf{C}_p)$ on the left by matrix multiplication.

If K is a complete field satisfying $\mathbf{Q}_p \subseteq K \subseteq \mathbf{C}_p$ then we set

$$\mathcal{H}_p(K) = \mathbf{P}_1(K) - \mathbf{P}_1(\mathbf{Q}_p).$$

For such a field K we identify $\mathbf{P}_1(K)$ with $\mathbf{P}(\mathrm{Hom}_{\mathbf{Q}_p}(\mathbf{Q}_p^2, K))$. The latter space can be described as the set of $K^\times$-proportionality classes of nonzero $\mathbf{Q}_p$-linear maps of $\mathbf{Q}_p^2$ into K. Points of $\mathbf{P}_1(\mathbf{Q}_p) \subseteq \mathbf{P}_1(K)$ are precisely those (classes of) maps with one-dimensional image. Thus $\mathcal{H}_p(K)$ may be identified with the set of $\mathbf{Q}_p^\times$-proportionality classes of injective $\mathbf{Q}_p$-linear maps $\mathbf{Q}_p^2 \to K$. Explicitely, if $(x : y)$ represents a point of $\mathcal{H}_p(K)$, then the corresponding map $\phi\colon \mathbf{Q}_p^2 \to K$ is described by $\phi(e_1) = x$ and $\phi(e_2) = y$, where

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The corresponding action of $\mathrm{GL}_2(\mathbf{Q}_p)$ on classes of maps $\phi\colon \mathbf{Q}_p^2 \to \mathbf{C}_p$ is given by $\gamma \cdot \phi = \phi \circ \gamma^{-1}$, where γ^{-1} is regarded as a linear transformation on the space of $\mathbf{Q}_p^2$ of column vectors via matrix multiplication.

We would like to endow $\mathcal{H}_p$ with the structure of a rigid analytic variety. To this end, define subsets $\mathcal{X}_n \subseteq \mathcal{H}_p$ for each integer $n \geq 0$ as follows:

$$\mathcal{X}_n = \left\{ z \in \mathbf{P}_1(\mathbf{C}_p) \;\middle|\; \begin{array}{ll} v_p(z-a) \geq p^{-n} & \text{for } a = 0, \ldots, p^{n+1}-1 \text{ and} \\ v_p\left(\frac{1}{z} - b\right) \geq p^{-n} & \text{for } b = 0, \ldots, p^n - 1 \end{array} \right\}.$$

Each $\mathcal{X}_n$ is a connected affinoid subset of $\mathbf{P}_1(\mathbf{C}_p)$, that is, $\mathcal{X}_n$ is the complement of a finite number of disjoint open disks centered on rational points of $\mathbf{P}_1$ with radii in $p^{\mathbf{Q}}$. Moreover $\mathcal{X}_n \subseteq \mathcal{X}_{n+1}$ for all n and $\cup_n \mathcal{X}_n = \mathcal{H}_p$. It is not hard to show that this realizes $\mathcal{H}_p$ as an admissible open subset of the rigid analytic variety $\mathbf{P}_1$:

Proposition 3.1.2. *The covering $\{\mathcal{X}_n\}$ of $\mathcal{H}_p$ is an admissible covering of the open subset $\mathcal{H}_p \subseteq \mathbf{P}_1(\mathbf{C}_p)$. Thus, $\mathcal{H}_p$ has the structure of a rigid analytic variety.*

Proof. See Proposition 1 of [35], where they offer three proofs of this result in the more general case of the higher Drinfeld half spaces. □

Recall the definition of a rigid analytic Stein space:

Definition 3.1.3. A rigid analytic variety X (over $\mathbf{C}_p$) is said to be *quasi-Stein* if there exists an admissible covering $\{U_n\}$ by affinoid subdomains with $U_n \subseteq U_{n+1}$ for all n and such that the image of the corresponding map $\mathcal{O}_X(U_{n+1}) \to \mathcal{O}_X(U_n)$ is closed. A Stein space X is said to be *Stein* if there exists an admissible covering $\{U_n\}$ of X by affinoid domains satisfying the following technical conditions: for each $U_n = \mathrm{Sp}(A_n)$, there exist topological generators $f_1^{(n)}, \ldots, f_{r_n}^{(n)}$ of A_n over $\mathbf{C}_p$, such that there exists $a_n \in \mathbf{C}_p$ with $0 < |a_n| < 1$ such that

$$U_{n-1} = \left\{ u \in U_n \;\middle|\; \left| \frac{f_i^{(n)}(u)}{a_i} \right| \leq 1 \text{ for } i = 1, \ldots, r_n \right\}.$$

Remark 3.1.4. In [30] it is shown that every Stein space is quasi-Stein.

Stein spaces behave like affine spaces in many ways. For example, one has the following important rigid analogue of Cartan's theorems A and B:

Theorem 3.1.5 (Theorems A and B). *Let $(X, \{U_n\})$ be a quasi-Stein space and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules on X. Then:*

1. *the image of $\mathcal{F}(X)$ in $\mathcal{F}(U_n)$ is closed for every n;*
2. *Theorem A: for each point x of X, the image of $\mathcal{F}(X)$ in the stalk $\mathcal{F}_x$ topologically generates the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$;*
3. *Theorem B: $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proof. See Satz 2.4 in [30]. □

Proposition 3.1.6. *The p-adic upper half plane is a Stein space.*

Proof. See Proposition 4 of [35]. □

The CM points that figure in this thesis are points of $\mathcal{H}_p$ defined over K_p, where K_p denotes the quadratic unramified extension of $\mathbf{Q}_p$ in $\mathbf{C}_p$. It will thus be worthwhile to have a simple algebraic description of the the K_p-valued points $\mathcal{H}_p(K_p)$.

Lemma 3.1.7. *The map $\psi \mapsto$ fixed point of ψ yields an identification*

$$\mathcal{H}_p(K_p) \cong \mathrm{Hom}_{\mathbf{Alg}_{\mathbf{Q}_p}}(K_p, M_2(\mathbf{Q}_p)).$$

Proof. If $\tau \in \mathcal{H}_p(K_p)$ then $K_p = \mathbf{Q}_p(\tau)$ and we can write $\tau^2 = b\tau + c$ for uniquely determined b and $c \in \mathbf{Q}_p$, such that $c \neq 0$. Then the map

$$x + y\tau \mapsto x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y \begin{pmatrix} b & c \\ 1 & 0 \end{pmatrix}$$

defines a $\mathbf{Q}_p$-algebra map $K_p \to M_2(\mathbf{Q}_p)$. Conversely, given such a map $\phi\colon K_p \to M_2(\mathbf{Q}_p)$, there exist exactly two points $\tau, \overline{\tau} \in \mathcal{H}_p(K_p)$ which are fixed by all element of $\phi(K_p^\times)$. Hence $\phi(K_P^\times) \subset \mathrm{GL}_2(\mathbf{Q}_p)$ acts on the tangent spaces at τ and $\overline{\tau}$ by differentiating the fractional linear transformation action on $\mathcal{H}_p$. If we elect to map ϕ to the choice of τ or $\overline{\tau}$ such that the corresponding tangent action is via the character $z \mapsto z/\overline{z}$, then this sets up a bijection between $\mathcal{H}_p(K_p)$ and the collection of $\mathbf{Q}_p$-algebra maps $K_p \to M_2(\mathbf{Q}_p)$. □

3.2 Bruhat-Tits tree for $\mathrm{GL}_2(\mathbf{Q}_p)$

3.2.1 Definition

A *lattice* in $\mathbf{Q}_p^2$ is a free $\mathbf{Z}_p$-submodule of rank 2. Two lattices are said to be *homothetic* if one is a $\mathbf{Q}_p^\times$-multiple of the other. Homothety is easily seen to be an equivalence relation. The set of homothety classes of lattices in $\mathbf{Q}_p^2$ is the vertex set for the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{GL}_2(\mathbf{Q}_p)$. Two vertices are joined by an edge if and only if they can be represented by lattices Λ, Λ' such that there are strict containments

$$p\Lambda \subset \Lambda' \subset \Lambda.$$

Proposition 3.2.1. *The Bruhat-Tits tree $\mathcal{T}$ is in fact a connected infinite tree which is regular of degree $p+1$.*

Proof. Let $\Lambda \subseteq \mathbf{Q}_p^2$ be a lattice, so that $\Lambda \cong \mathbf{Z}_p^2$ as $\mathbf{Z}_p$-modules. Thus

$$\Lambda/p\Lambda \cong \mathbf{Z}_p^2/p\mathbf{Z}_p^2 \cong (\mathbf{Z}/p\mathbf{Z})^2.$$

It follows that vertices adjacent to the homothety class of Λ in $\mathcal{T}$ correspond to proper subgroups of $(\mathbf{Z}/p\mathbf{Z})^2$. There are $p+1$ such subgroups, all of order p. To deduce that $\mathcal{T}$ is regular of degree $p+1$, it suffices now to argue that these subgroups correspond to distinct homothety classes of lattices. In order to have strict containments

$$p\Lambda \subset \Lambda_1, \Lambda_2 \subset \Lambda,$$

with $\Lambda_1 = \alpha\Lambda_2$ for some $\alpha \in \mathbf{Q}_p^\times$, it is necessary to have $v_p(\alpha) = 0$. But then multiplication by α gives an automorphism of Λ_2 and $\Lambda_1 = \Lambda_2$.

To see that $\mathcal{T}$ is connected, let v_0 denote the vertex corresponding to $\mathbf{Z}_p^2 \subseteq \mathbf{Q}_p^2$. Each homothety class of lattices contains a unique representative Λ such that $\Lambda \subseteq \mathbf{Z}_p^2$ but $\Lambda \not\subseteq p\mathbf{Z}_p^2$. Since $\mathbf{Z}_p$ is a PID, we may apply the elementary divisors theorem to obtain a basis e_1, e_2 for $\mathbf{Z}_p^2$ such that ae_1, be_2 is a basis for Λ, and where $a, b \in \mathbf{Z}_p$ are such that $b|a$. We may replace a and b by powers of p, say $a = p^r$, $b = p^t$ with $t \le r$, since multiplication by p-adic units won't change the $\mathbf{Z}_P$-bases. Since Λ is not contained in $p\mathbf{Z}_p^2$, it follows that $t = 0$. Now consider the chain of submodules

$$\Lambda = p^r\mathbf{Z}_pe_1 \oplus \mathbf{Z}_pe_2 \subset p^{r-1}\mathbf{Z}_pe_1 \oplus \mathbf{Z}_pe_2 \subset \cdots \subset p\mathbf{Z}_pe_1 \oplus \mathbf{Z}_pe_2 \subset \mathbf{Z}_p^2.$$

These correspond to the unique path without backtracking from Λ to v_0 in $\mathcal{T}$, so that $\mathcal{T}$ is a connected tree. $\square$

The group $\mathrm{GL}_2(\mathbf{Q}_p)$ acts on the set of lattices in $\mathbf{Q}_p^2$. The scalar matrices are precisely those which act by homothety, and so since the scalars are in the center of $\mathrm{GL}_2(\mathbf{Q}_p)$, this action descends to an action of $\mathrm{GL}_2(\mathbf{Q}_p)$ on the vertex set $V(\mathcal{T})$. This action is given simply by matrix multiplication and is thus seen to preserve the incidence relations $p\Lambda \subset \Lambda' \subset \Lambda$. In this way one obtains an action of $\mathrm{GL}_2(\mathbf{Q}_p)$ on $\mathcal{T}$. The action is transitive on $V(\mathcal{T})$, and the stabilizer of the vertex v_0 corresponding to the class of $\mathbf{Z}_p^2$ is the product $\mathbf{Q}_p^\times \cdot \mathrm{GL}_2(\mathbf{Z}_p)$ in $\mathrm{GL}_2(\mathbf{Q}_p)$. This gives an identification

$$V(\mathcal{T}) \cong \mathrm{GL}_2(\mathbf{Q}_p)/\mathbf{Q}_p^\times \cdot \mathrm{GL}_2(\mathbf{Z}_p).$$

3.2.2 The ends of $\mathcal{T}$

Before connecting the Bruhat-Tits tree to the p-adic upper half plane via the reduction map, we will define the "ends" of $\mathcal{T}$. This discussion will later aid us to identify rigid analytic modular forms as simple combinatorial objects, called harmonic cocycles, related to the Bruhat-Tits tree.

Definition 3.2.2. Let $\mathcal{E}(\mathcal{T})$ denote the collection of all infinite non-backtracking sequences of adjacent vertices of $\mathcal{T}$

$$(v_1, v_2, \ldots),$$

modulo the following equivalence relation: two sequences are identified if they differ by a finite initial sequence. Then $\mathcal{E}$ is the collection of *ends* of the Bruhat-Tits tree.

Note that $\mathrm{GL}_2(\mathbf{Q}_p)$ acts on the ends (it acts on everything in sight). Each end can be represented by a sequence of lattices

$$\Lambda_0 \supset \Lambda_1 \supset \Lambda_2 \supset \cdots$$

such that $\Lambda_n/\Lambda_{n+1} \cong \mathbf{Z}/p\mathbf{Z}$ for all n. The elementary divisors theorem shows that the intersection of such a sequence $\bigcap_n \Lambda_n$ is a $\mathbf{Z}_p$-submodule of $\mathbf{Q}_p^2$ spanning a $\mathbf{Q}_p$-line. In this way one obtains a natural map

$$N \;:\; \mathcal{E}(\mathcal{T}) \to \mathbf{P}_1(\mathbf{Q}_p).$$

If $e = [v \to u]$ is an *oriented* edge of $\mathcal{T}$, then associate to it the subset

$$V_e = \{\varepsilon \in \mathcal{E} \mid \varepsilon = (v, u, \cdots)\}.$$

These sets generate a topology on $\mathcal{E}$ making it into a compact Hausdorff space. In fact, one can prove the following without too much difficulty:

Proposition 3.2.3. *The map $N\colon \mathcal{E} \to \mathbf{P}_1(\mathbf{Q}_p)$ is a $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant homeomorphism.*

Proof. See Lemma 12 of [13]. □

3.2.3 Geometric realization

Identify each edge of $\mathcal{T}$ with an interval $[0,1]$ endowed with the usual Archimedean topology. Glue edges according to the incidence relations of $\mathcal{T}$ using the obvious quotient topology and let $\mathcal{T}_{\mathbf{R}}$ denote the corresponding *geometric realization* of $\mathcal{T}$. If v and v' are adjacent edges of $\mathcal{T}$, we will often denote points of $\mathcal{T}_{\mathbf{R}}$ on the edge joining v and v' in the form $u = (1-t)v + tv'$ for some $0 \le t \le 1$. Let $\mathcal{T}_{\mathbf{Q}}$ denote the subset of $\mathcal{T}_{\mathbf{R}}$ consisting of those points $u = (1-t)v + tv'$ with $t \in \mathbf{Q} \cap [0,1]$. In particular, $\mathcal{T}_{\mathbf{Q}}$ contains all of the vertices of $\mathcal{T}_{\mathbf{R}}$.

The topological space $\mathcal{T}_{\mathbf{R}}$ is connected to $\mathbf{R}_{>0}$-homothety classes of norms on $\mathbf{Q}_p^2$ in the following way:

If Λ is lattice in $\mathbf{Q}_p^2$, then there exists a unique norm on $\mathbf{Q}_p^2$ such that Λ is the corresponding unit ball. If one chooses a basis (e_1, e_2) for Λ, then this norm can be described explicitely:

$$|ae_1 + be_2|_\Lambda = \sup\{|a|, |b|\}.$$

Note that $|\cdot|_{\alpha\Lambda} = |\alpha|\,|\cdot|_\lambda$ for all $\alpha \in \mathbf{Q}_p^\times$, so that homothetic lattices produce homothetic norms.

Now suppose that Λ and Λ' are lattices with $p\Lambda \subsetneq \Lambda' \subsetneq \Lambda$. Choose a basis (e_1, e_2) for Λ such that (e_1, pe_2) is a basis for Λ'. Let $0 < t < 1$ represent a point of the geometric realisation $\mathcal{T}_{\mathbf{R}}$ between the adjacent vertices corresponding to Λ and Λ'. Then define a norm on $\mathbf{Q}_p^2$ via the formula:

$$|ae_1 + be_2|_{\Lambda,\Lambda',t} = \sup\{|a|\,, p^t\,|b|\}.$$

In this way one obtains a bijection:

Proposition 3.2.4. *The above definitions yield a bijection between the geometric realisation $\mathcal{T}_{\mathbf{R}}$ of the Bruhat-Tits tree for $\mathrm{GL}_2(\mathbf{Q}_p)$ and the space of $\mathbf{R}_{>0}$-homothety classes of norms on $\mathbf{Q}_p^2$.*

Proof. We describe the inverse map. If $|\cdot|$ is a norm on $\mathbf{Q}_p^2$ then for real $\alpha > 0$, the collection $\Lambda_\alpha = \{x \in \mathbf{Q}_p^2 \mid |x| \le \alpha\}$ is a lattice in $\mathbf{Q}_p^2$. One has $\Lambda_{\alpha'} \subset \Lambda_\alpha$ if $\alpha' \le \alpha$ and $\Lambda_{p^{-1}\alpha} = p\Lambda_\alpha$, thus $[\Lambda_\alpha]$ takes at most two values in $V(\mathcal{T})$ as α varies.

If $[\Lambda_\alpha] = v$ is constant, then $|\cdot|$ corresponds to v.

Otherwise $[\Lambda_\alpha]$ equals v or v' for two adjacent vertices of $\mathcal{T}$. After possibly replacing $|\cdot|$ by a proportional norm, one has $[\Lambda_\alpha] = v$ for $q^t \le \alpha < q$ and $[\Lambda_\alpha] = v'$ for $1 \le \alpha < q^t$, with $0 < t < 1$. Then $|\cdot|$ corresponds with the point $u = (1-t)v + tv'$ of the edge joining v and v'. □

Remark 3.2.5. The topological structure on the space of norms on $\mathbf{Q}_p^2$ dates back, at least, to the paper [19] of Goldman and Iwahori. See their work for more details, or see I.2 of [4].

There is an action of $\mathrm{GL}_2(\mathbf{Q}_p)$ on $\mathcal{T}_{\mathbf{R}}$ which is compatible with the action on $\mathcal{T}$. If $\alpha = |\cdot|$ is a norm on $\mathbf{Q}_p^2$ and if $\gamma \in \mathrm{GL}_2(\mathbf{Q}_p)$, then define $\gamma\alpha$ by setting $\gamma\alpha(x) = \alpha(\gamma^{-1}x)$. For example, if $v \in \mathcal{T}_{\mathbf{R}}$ is the vertex corresponding to the lattice $\mathbf{Z}_p^2 \subseteq \mathbf{Q}_p^2$, so that the corresponding norm α_v is the usual sup-norm

$$\alpha_v \begin{pmatrix} x \\ y \end{pmatrix} = \sup\{|x|\,, |y|\},$$

and if $\gamma \in \mathrm{GL}_2(\mathbf{Q}_p)$ is such that we may represent

$$\gamma^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$(\gamma\alpha_v)\begin{pmatrix} x \\ y \end{pmatrix} = \sup\{|ax+by|, |cx+dy|\}$$

by the definition above. On the other hand, note that the vertex γv corresponds with the homothety class of the lattice $\gamma\mathbf{Z}_p^2$. A basis for this lattice is given by the vectors

$$\frac{1}{ad-bc}\begin{pmatrix} d \\ -c \end{pmatrix} \quad \text{and} \quad \frac{1}{ad-bc}\begin{pmatrix} -b \\ a \end{pmatrix}.$$

Since

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad-bc}\left((ax+by)\begin{pmatrix} d \\ -c \end{pmatrix} + (cx+dy)\begin{pmatrix} -b \\ a \end{pmatrix}\right),$$

it follows by definition of the norm $\alpha_{\gamma v}$ associated to γv that

$$\alpha_{\gamma v}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{|ad-bc|}\sup\{|ax+by|, |cx+dy|\}.$$

Thus $\gamma\alpha_v$ and $\alpha_{\gamma v}$ are $\mathbf{Q}_p^\times$-multiples of one another, which shows that the two actions are compatible. An identical computation shows the same for an arbitrary vertex $v \in \mathcal{T}$.

3.3 The reduction map

We now describe a continuous map

$$\mathrm{red}\colon \mathcal{H}_p \to \mathcal{T}_{\mathbf{R}}$$

which is a very useful combinatorial device for understanding the geometry of the p-adic upper half plane.

Recall from Section 3.1 that points $\phi \in \mathcal{H}_p$ can be regarded as injective $\mathbf{Q}_p$-linear maps $\phi\colon \mathbf{Q}_p^2 \to \mathbf{C}_p$. Composing such a point ϕ with the p-adic norm on $\mathbf{C}_p$

yields a norm $|\cdot|_\phi$ on $\mathbf{Q}_p^2$. It was shown in Proposition 3.2.4 that homothety classes of norms are in bijection with the points of $\mathcal{T}_{\mathbf{R}}$. We thus define the *reduction map* $\mathrm{red}\colon \mathcal{H}_p \to \mathcal{T}_{\mathbf{R}}$ by

$$\mathrm{red}(\phi) = |\cdot|_\phi .$$

Proposition 3.3.1. *The reduction map is* $\mathrm{GL}_2(\mathbf{Q}_p)$*-equivariant, continuous when* $\mathcal{H}_p$ *is endowed with its* p*-adic topology, and* $\mathrm{im}(\mathrm{red}) = \mathcal{T}_{\mathbf{Q}}$*.*

Proof. That the image of the reduction map is $\mathcal{T}_{\mathbf{Q}}$ follows from our chosen normalization for the absolute value of $\mathbf{C}_p$, so that $\left|\mathbf{C}_p^\times\right| = p^{\mathbf{Q}}$. We omit the proof that reduction is continuous and content ourselves instead with showing that reduction is $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant.

Let $\alpha\colon \mathbf{C}_p \to \mathbf{R}$ denote the absolute value. If $\phi \in \mathcal{H}_p$, then $\mathrm{red}(\phi) = |\cdot|_\phi = \alpha \circ \phi$. Thus

$$\mathrm{red}(\gamma\phi) = \alpha \circ (\gamma\phi) = \alpha \circ \phi \circ \gamma^{-1} = \gamma(\alpha \circ \phi) = \gamma\,\mathrm{red}(\phi).$$

$\square$

Although the definition of the reduction map given above is quite elegant, it does not immediately help one to work with the rigid analytic topology on $\mathcal{H}_p$. If $v \in \mathcal{T}_{\mathbf{Q}}$ is a vertex, then $\mathrm{red}^{-1}(v)$ is a connected affinoid subset of $\mathbf{P}_1(\mathbf{C}_p)$ defined over $\mathbf{Q}_p$. For example, if we let v_0 correspond to the lattice $\mathbf{Z}_p^2 \subseteq \mathbf{Q}_p^2$, then

$$\mathrm{red}^{-1}(v_0) = \mathcal{A}_0 = \{z \in \mathbf{P}_1(\mathbf{C}_p) \mid\ |z| \le 1,\ \ |z-a| = 1 \text{ for } a = 0,\dots,p-1\}.$$

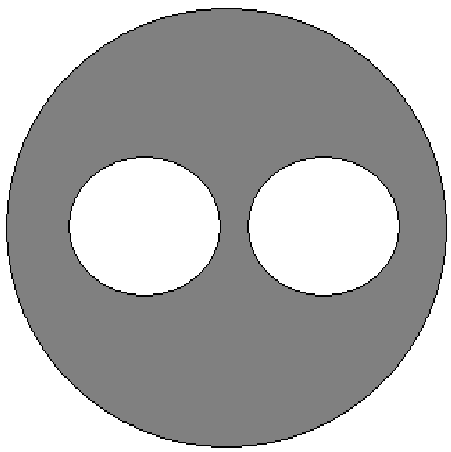
We will refer to this as the *standard affinoid*. If v and v' are adjacent edges of $\mathcal{T}$ and $x = (1-t)v' + tv$ is a point on the edge joining them, then $\mathrm{red}^{-1}(x)$ describes a circle in $\mathbf{P}_1(\mathbf{C}_p)$ about some $\mathbf{Q}_p$-rational point of radius p^t (and such circles are empty if $t \notin \mathbf{Q}$). More generally, if $U \subseteq \mathcal{T}_{\mathbf{R}}$ is an open subset, then $\mathrm{red}^{-1}(U)$ is a *wide open subset* of $\mathbf{P}_1(\mathbf{C}_p)$ (cf. Coleman [9]).

Definition 3.3.2. If v is a vertex of $\mathcal{T}$, then let $\mathcal{A}_v = \mathrm{red}^{-1}(v)$ denote the corresponding connected affinoid of $\mathcal{H}_p$ which reduces to v. If $e = [v \to v']$ is an edge of $\mathcal{T}$, then write $W_e = \mathrm{red}^{-1}(e)$ for the inverse image of the copy of the open interval $(0,1)$ in $\mathcal{T}_{\mathbf{R}}$ corresponding to e. Hence W_e is an open anulus connecting $\mathcal{A}_v$ and $\mathcal{A}_{v'}$. Let $\mathcal{A}_e = \mathcal{A}_v \cup W_e \cup \mathcal{A}_{v'}$ and note that $\mathcal{A}_e$ is a connected affinoid which is the complement of $2p$ disjoint open disks in $\mathbf{P}_1(\mathbf{C}_p)$.

The collection $\{\mathcal{A}_e\}_{e\in E(\mathcal{T})}$ is an admissible covering of $\mathcal{H}_p$ by connected affinoids. Moreover for edges $e \neq e'$ one has

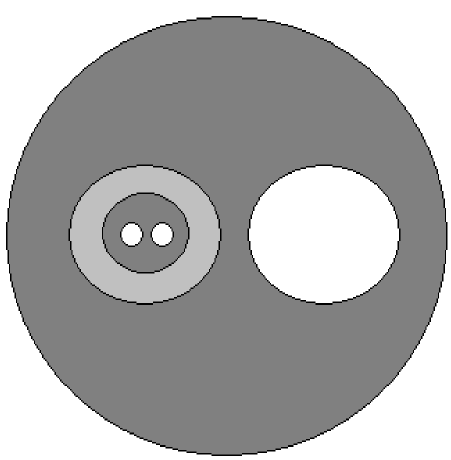
$$\mathcal{A}_e \cap \mathcal{A}_{e'} = \begin{cases} \mathcal{A}_v & \text{if } e \text{ and } e' \text{ share a vertex } v, \\ \emptyset & \text{otherwise.} \end{cases}$$

Example 3.3.3. Suppose $p = 2$. Then each affinoid $\mathcal{A}_v$ looks like a disk with two disks removed:



Note that the geometry of this picture is slightly misleading. For the standard affinoid corresponding to the lattice $\mathbf{Z}_p^2$, the disk is of radius 1 and the inner disks are also of radius 1! Note though that the standard affinoid still has plenty of points, so the "thickness" of the affinoid in the depiction above is not *such* a misrepresentation.

Recall that the affinoids $\mathcal{A}_e$ associated to edges of the Bruhat-Tits tree are obtained by gluing two affinoids as above along an annulus:



Consider the lighter gray annulus joining the two affinoids in the picture above. Points in the annulus which are also algebraic over $\mathbf{Q}_p$ lie in ramified extensions of $\mathbf{Q}_p$. Thus, the CM points of $\mathcal{H}_p$ which we consider below all happen to lie in affinoids $\mathcal{A}_v$ associated to vertices of the Bruhat-Tits tree, since they are defined over the unramified quadratic extension of $\mathbf{Q}_p$ within $\mathbf{C}_p$.

3.4 Rigid analytic functions on $\mathcal{H}_p$

The covering $\{A_e\}$ of $\mathcal{H}_p$ by affinoids introduced above leads to a convenient description of the ring $\mathcal{O} = H^0(\mathcal{H}_p, \mathcal{O}_{\mathcal{H}_p})$ of rigid analytic functions on $\mathcal{H}_p$, particularly if, like us, one is not so interested in the topological properties of the ring $\mathcal{O}$. We thus think of elements $f \in \mathcal{O}$ as families $\{f_e\}_{e\in E(\mathcal{T})}$ of rigid analytic functions f_e on $\mathcal{A}_e$, such that if e and e' share a vertex v of the Bruhat-Tits tree, then f_e and $f_{e'}$ agree on the affinoid $\mathcal{A}_v = \mathcal{A}_e \cap \mathcal{A}_{e'}$. If $f_e = 0$ then also $f_{e'}$ vanishes on $\mathcal{A}_v$, and hence must be zero on the larger affinoid $\mathcal{A}_{e'}$. Since $\mathcal{T}$ is connected, one can show that then $f_e = 0$ for all edges of $\mathcal{T}$.

Recall the following well-known result:

Proposition 3.4.1. *Let $\mathcal{A} \subseteq \mathbf{C}_p$ denote a connected affinoid subset. Then the ring of rigid analytic functions on $\mathcal{A}$ is the completion for the sup-norm of the subring of the rational functions $\mathbf{C}_p(X)$ consisting of all those with poles contained in $\mathbf{C}_p - \mathcal{A}$. Moreover, if we write*

$$\mathcal{A} = \{z \in \mathbf{C}_p \mid \ |z| \leq r, \ \ |z-\alpha_i| \geq s_i \text{ for } i = 1, \ldots, n\},$$

where $r = |x|$ and $s_i = |y_i|$ for some x and $y_i \in \mathbf{C}_p$, then every rigid analytic function f on $\mathcal{A}$ admits a unique decomposition

$$f = \sum_{j\geq 0} a_j (z/x)^j + \sum_{i=1}^{n} \sum_{j\geq 0} b_{i,j} \left(\frac{y_i}{z-\alpha_i}\right)^j,$$

where the a_j and $b_{i,j}$ are elements of $\mathbf{C}_p$ which tend to 0 as j grows.

Proof. See the first Proposition in II.1.2 of [17]. □

The next result will be useful for understanding meromorphic functions on $\mathcal{H}_p$ (that is, quotients of analytic functions on $\mathcal{H}_p$).

Proposition 3.4.2. *Let $\mathcal{A} \subset \mathbf{P}^1(\mathbf{C}_p)$ denote a connected affinoid subset with $\infty \notin \mathcal{A}$. Let $f \in \mathcal{O}_{\mathcal{A}}$ denote a rigid analytic function on $\mathcal{A}$. Then f has finitely many zeros $c_1, \ldots, c_s \in \mathcal{A}$. Each zero has a multiplicity $m_i \in \mathbf{Z}_{\geq 0}$ and f admits a decomposition*

$$f(z) = \left(\prod_{i=1}^{s} (z - c_i)^{m_i} \right) u(z)$$

with $u(z)$ a unit in $\mathcal{O}_{\mathcal{A}}$. If $u \in \mathcal{O}_{\mathcal{A}}$ then the following are equivalent:

1. *u is a unit;*
2. *u has no zeros on $\mathcal{A}$;*
3. *$\inf\{|u(z)| \mid z \in \mathcal{A}\} > 0$.*

Proof. See Theorem 2.2.9 of [16]. □

The p-adic upper half plane is a direct limit of connected affinoids. Hence $\mathcal{O}$ is a projective limit of integral domains, and is thus itself an integral domain.

Definition 3.4.3. The *function field* of $\mathcal{H}_p$, denoted $\mathcal{K}$, is the fraction field of $\mathcal{A}$.

Proposition 3.4.2 shows that an analytic function has a finite number of zeros on a connected affinoid. Thus, an analytic function $f \in \mathcal{O}$ vanishes at a discrete subset of $\mathcal{H}_p$. It follows that elements of $\mathcal{K}$ can be regarded as functions on $\mathcal{H}_p$ at all points save possibly for a discrete subset which depends on the particular meromorphic function under consideration and its representation as a ratio of analytic functions.

In complex analysis, analytic functions are determined by their values on any set with an accumulation point. One can regard the following result as an analogue of this fact. It will be crucial when we wish to interpret nearly rigid analytic modular forms as functions.

Proposition 3.4.4. *Let $F \subseteq \mathbf{C}_p$ denote the completion of the maximal unramified extension of $\mathbf{Q}_p$ inside $\mathbf{C}_p$. Then there is a natural inclusion*

$$\mathcal{O} \hookrightarrow C(\mathcal{H}_p(F), \mathbf{C}_p),$$

where $C(\mathcal{H}_p(F), \mathbf{C}_p)$ denotes the set of continuous $\mathbf{C}_p$-valued functions on $\mathcal{H}_p(F) = F - \mathbf{Q}_p$.

Proof. For $n \geq 0$ let $\mathcal{H}_n$ denote the affinoid subdomain of $\mathcal{H}_p$ which corresponds under reduction to the subtree of the Bruhat-Tits tree consisting of all edges of distance at most n from the privileged vertex. Let $\mathcal{O}_n$ denote the ring of rigid analytic functions on $\mathcal{H}_n$. Then one can show, since the residue field of F is algebraically closed, that the Tate norm of $\mathcal{O}_n$, induced from some surjective map $\mathbf{C}_p\langle T_1, \ldots, T_r\rangle \to \mathcal{O}_n$, agrees with the sup-norm computed over F-rational points of $\mathcal{H}_n$ (in particular, the sup-norm is well-defined for elements of $\mathcal{A}_n$; this is not automatic as F is not locally compact). There is thus an isometric inclusion

$$\mathcal{O}_n \hookrightarrow C_0(\mathcal{H}_n(F), \mathbf{C}_p)$$

for all n, where the left side is endowed with the Tate norm and the right side denotes the space of bounded and continuous functions from $\mathcal{H}_n(F)$ to $\mathbf{C}_p$, endowed with the sup-norm. One obtains the inclusion $\mathcal{O} \hookrightarrow C(\mathcal{H}_p(F), \mathbf{C}_p)$ by passing to the projective limit. □

Remark 3.4.5. Despite the fact that the restriction of $f \in \mathcal{O}$ to an affinoid subdomain of $\mathcal{H}_p$ defines a bounded function, f itself need not be bounded on the entire p-adic upper half plane. A bounded function on the p-adic upper half plane must be constant, just as in complex analysis. This will be crucial later, for the modular forms that we are interested in below tend to have poles at all points of $\mathbf{P}_1(\mathbf{Q}_p)$, and the corresponding residues allow one to describe modular forms in a convenient and explicit fashion.

3.5 Quotients of $\mathcal{H}_p$

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Q}_p)$ denote a subgroup. Then Γ acts on $\mathcal{H}_p$ by fractional linear transformation and we may consider the quotient topological space $\Gamma\backslash\mathcal{H}_p$. Assume that Γ is *discrete* inside $\mathrm{SL}_2(\mathbf{Q}_p)$ for the natural p-adic topology of $\mathrm{SL}_2(\mathbf{Q}_p)$ which is induced by the inclusion $\mathrm{SL}_2(\mathbf{Q}_p) \subset \mathrm{GL}_2(\mathbf{Q}_p)$. Then in this case the quotient map $\mathcal{H}_p \to \Gamma\backslash\mathcal{H}_p$ is a covering map at all but finitely many points. These so-called *elliptic points* correspond to points in $\mathcal{H}_p$ with nontrivial stabilisers in Γ.

Assume furthermore that Γ is *cocompact*, that is, that $\Gamma\backslash\mathcal{H}_p$ is compact for the quotient topology. In this case, the quotient $\Gamma\backslash\mathcal{H}_p$ carries a natural structure of rigid analytic variety which is inherited from the p-adic upper half plane. Moreover, a rigid analytic version of Serre's GAGA principle can be used to show that $\Gamma\backslash\mathcal{H}_p$ is the rigid analytification of an algebraic curve X defined over $\mathbf{Q}_p$; see the book [17] for more details.

There is a sense in which this result has a converse, but in order to state it one must understand the nature of the special fiber of the formal model of $\Gamma\backslash\mathcal{H}_p$ (for details on the connection between rigid and formal geometry, consult [3]). It can be shown that this special fiber consists of a finite number of copies of the the projective line over $\mathbf{F}_p$ which intersect at ordinary double points. This means that locally at the point of intersection, the special fiber looks like $\mathrm{Spec}(\mathbf{F}_p[X,Y]/(XY))$. Such a configuration of projective lines can be described as the *dual* of the quotient $\Gamma\backslash\mathcal{T}$ of the Bruhat-Tits tree, which is a finite connected multigraph. This suggests the following definition:

Definition 3.5.1. A curve X over $\mathbf{Q}_p$ is said to be *admissible* if it has a model over $\mathbf{Z}_p$ whose special fiber consists of a collection of projective lines which intersect at ordinary double points.

With this definition in place, we can state the following converse to the result above:

Theorem 3.5.2 (Mumford)**.** *If X is an admissible curve over $\mathbf{Q}_p$, then there exists a discrete subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Q}_p)$ such that the rigid analytification of $X_{\mathbf{C}_p}$ is isomorphic to $\Gamma\backslash\mathcal{H}_p$.*

For details about this result one can consult Mumford's original paper [31], or the book [17].

We next specialize the generalities above to the subgroups of $\mathrm{SL}_2(\mathbf{Q}_p)$ that appear in our work. Complex Shimura curves constructed from definite quaternion algebras, as in [33], are made up of genus zero curves, and so are more easily understood than their indefinite counterparts. However, definite quaternion algebras can be used to construct interesting *p-adic rigid analytic curves* at finite unramified places p. Thus, the general philosophy is that interesting Shimura curves can be constructed from quaternion algebras B over $\mathbf{Q}$ by considering an unramified place of B, be it infinite *or* finite. It is precisely the case of a definite quaternion algebra and a finite unramified prime which appears in our work.

Let N^- be an integer which is a product of an odd number of distinct primes, and let p be a prime which is coprime to N^-. Let N^+ be a positive integer which is relatively prime to pN^-. Let B denote a quaternion algebra over **Q** of discriminant exactly N^-; since N^- has an *odd* number of disctinct prime factors, B is *ramified* at infinity, or *definite*. Fix a maximal **Z**-order $R_0 \subseteq B$ and an Eichler **Z**-order $R_0(N+) \subseteq R_0$ of level N^+. Let $R = R_0[1/p]$ and $R(N^+) = R_0(N^+)[1/p]$ denote the corresponding $\mathbf{Z}[1/p]$-orders. Note that B satisfies the Eichler condition 2.1.13 for $\mathbf{Z}[1/p]$-orders, and so up to conjugation by $B^\times$, R is the unique maximal $\mathbf{Z}[1/p]$-order in B, and similarly for the Eichler order $R(N^+)$ of level N^+. Since B is unramified at p, we may and do fix an isomorphism

$$\iota\colon B_p = B \otimes_{\mathbf{Q}} \mathbf{Q}_p \xrightarrow{\sim} M_2(\mathbf{Q}_p)$$

such that $\iota(R_0 \otimes_{\mathbf{Z}} \mathbf{Z}_p) = \mathrm{GL}_2(\mathbf{Z}_p)$. Let $R(N^+)_1$ denote the collection of elements in $R(N^+)$ of reduced norm equal to one, and set $\Gamma^{(p)}_{N^+,N^-} = \iota(R(N^+)[1/p]_1)$. Note that $\Gamma = \Gamma^{(p)}_{N^+,N^-}$ is only determined up to conjugation by $B_p^\times$. Then one has the following:

Proposition 3.5.3. *The group $\Gamma^{(p)}_{N^+,N^-} \subseteq \mathrm{SL}_2(\mathbf{Q}_p)$ is a discrete and cocompact subgroup.*

3.6 CM points

In this section, as always in this thesis, $B/\mathbf{Q}$ denotes a definite quaternion algebra of discriminant N^-, and p denotes a prime which splits B.

Before discussing CM points in $\mathcal{H}_p$ for B, we turn to the case of CM points in the complex upper half plane. Let $K/\mathbf{Q}$ denote a quadratic imaginary extension which is embedded inside **C**. In the introduction we referred to points in $\mathcal{H} \cap K$ as CM points. We would like to make an analogous definition for the p-adic upper half plane, but of course it won't be quite as simple as in the complex case. The identity

$$\mathcal{H} \cong \mathrm{Hom}_{\mathbf{Alg_R}}(\mathbf{C}, M_2(\mathbf{R}))$$

is what suggests the way forward.

How can we interpret $\mathcal{H} \cap K$ using the description $\mathrm{Hom}(\mathbf{C}, M_2(\mathbf{R}))$ for $\mathcal{H}$? Note that if $K \to M_2(\mathbf{Q})$ is an embedding of **Q**-algebras, then since $K/\mathbf{Q}$ is quadratic imaginary, we obtain an embedding of $\mathbf{R}$-algebras:

$$\mathbf{C} \cong K \otimes_{\mathbf{Q}} \mathbf{R} \to M_2(\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{R} \cong M_2(\mathbf{R}).$$

The identification on the left uses the fact that we have already embedded K inside $\mathbf{C}$. The rightmost identification is canonical. In this way one can identify $\mathcal{H} \cap K$ with $\mathrm{Hom}_{\mathbf{Alg_Q}}(K, M_2(\mathbf{Q}))$. It is this set which can be adapted to our setting.

Let $K_p/\mathbf{Q}_p$ denote the quadratic unramified extension of $\mathbf{Q}_p$ within $\mathbf{C}_p$. This will play the role of $\mathbf{C}$ in our analogy. Above we needed to use an identification $K \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathbf{C}$. Thus, in our case we'd like an identity

$$K \otimes_{\mathbf{Q}} \mathbf{Q}_p \cong K_p.$$

One can make such an identification precisely when p is inert in K.

Remark 3.6.1. If p is ramified in K then of course $K \otimes_{\mathbf{Q}} \mathbf{Q}_p$ is a ramified extension of $\mathbf{Q}_p$, but there is no canonical quadratic ramified extension of $\mathbf{Q}_p$. One might consider questions analogous to those treated in this thesis for ramified CM points simply by working with whichever ramified quadratic extension occurs, but the analogy with $\mathbf{C}$ is slightly less enticing in this case. If p splits in K then $K \otimes_{\mathbf{Q}} \mathbf{Q}_p \cong \mathbf{Q}_p^2$ is not a field. Embeddings $K \to B$ where p splits correspond to points in $\mathbf{P}_1(\mathbf{Q}_p)$, and thus don't define points of $\mathcal{H}_p$. In this case one can define certain interesting "geodesic cycles" on $\Gamma \backslash \mathcal{H}_p$ which are infinite paths through the Bruhat-Tits tree which join the two pixed points in $\mathbf{P}_1(\mathbf{Q}_p)$.

In the setting of this thesis we have replaced $M_2(\mathbf{Q})$ by a definite quaternion algebra $B/\mathbf{Q}$ which is split at p via a map

$$\iota \colon B \otimes_{\mathbf{Q}} \mathbf{Q}_p \xrightarrow{\sim} M_2(\mathbf{Q}_p).$$

This splitting plays the role of the *canonical* identification $M_2(\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{R} \cong M_2(\mathbf{R})$ above. Similarly, we make the following definition.

Definition 3.6.2. Let $K/\mathbf{Q}$ denote a quadratic imaginary extension. Then the set of CM points in $\mathcal{H}_p$ for K is denoted

$$\mathrm{CM}(K) = \mathrm{Hom}_{\mathbf{Alg_Q}}(K, B).$$

When p is inert in K and $\phi \in \mathrm{CM}(K)$, then using ι one obtains

$$K_p \cong K \otimes_{\mathbf{Q}} \mathbf{Q}_p \stackrel{\phi\otimes 1}{\to} B \otimes_{\mathbf{Q}} \mathbf{Q}_p \stackrel{\iota}{\to} M_2(\mathbf{Q}_p),$$

and Lemma 3.1.7 shows that this defines a point of

$$\mathcal{H}_p(K_p) \cong \mathrm{Hom}_{\mathbf{Alg}_{\mathbf{Q}_p}}(K_p, M_2(\mathbf{Q}_p)).$$

We summarize and supplement these observations slightly in the following:

Lemma 3.6.3. *Let $K/\mathbf{Q}$ denote a quadratic imaginary field in which p is inert, and let ι denote a splitting of B at p. Then, up to the choice of ι, there is a canonical identification of $\mathrm{CM}(K)$ with a subset of $\mathcal{H}_p(K_p)$. Moreover, $\mathrm{CM}(K)$ is nonempty if and only if all the primes dividing N^- are inert or ramified in K. In particular, if the discriminant of K is relatively prime to N^-, then $\mathrm{CM}(K)$ is nonempty if and only if all the primes dividing N^- are inert in K.*

Proof. This is a summary of the observations above, save for the remarks about $\mathrm{CM}(K)$ being nonempty. These follow from Proposition 2.1.15. □

3.7 The Cerednik-Drinfeld theorem

Let B be a definite quaternion algebra over $\mathbf{Q}$ as in the preceding section 3.1. Let $\mathcal{B}$ denote a quaternion algebra over $\mathbf{Q}$ of discriminant pN^-; this means that $\mathcal{B}$ is ramified precisely at the finite primes of ramification for B, as well as p, and is unramified at infinity. The algebra $\mathcal{B}$ is uniquely determined up to isomorphism by these ramification conditions. We say that the algebra B is obtained from $\mathcal{B}$ by *interchanging invariants* at p and ∞. Fix a maximal order $\mathcal{R} \subseteq \mathcal{B}$, let N^+ be a positive integer which is relatively prime to pN^-, and let $\mathcal{R}(N^+) \subseteq \mathcal{R}$ denote an Eichler order of level N^+, which is unique up to conjugation since $\mathcal{B}$ is indefinite, and thus satisfies the Eichler condition.

Let $X_{N^+,pN^-}/\mathbf{Q}$ denote the Shimura curve corresponding to $\mathcal{R}(N^+)$. It is the coarse solution to the moduli problem defined by the functor $\mathcal{F}_{\mathbf{Q}}$ which associates to every scheme S over $\mathbf{Q}$ the collection of S-isomorphism classes of triples (A, i, C) where:

1. A is an abelian scheme over S of relative dimension 2;

2. $\iota\colon \mathcal{R} \to \mathrm{End}_S(A)$ is an inclusion which defines an action of $\mathcal{R}$ on A;

3. C is a subgroup scheme of A which is locally isomorphic to $\mathbf{Z}/N^+\mathbf{Z}$, and which is stable and locally cyclic for the action of $\mathcal{R}(N^+)$. Such a subgroup is called an N^+*-level structure*.

Cerednik and Drinfeld proved the following remarkable rigid analytic uniformisation result for the Shimura curves X_{N^+,pN^-}:

Theorem 3.7.1 (Cerednik-Drinfeld)**.** *If $K_p/\mathbf{Q}_p$ denotes the quadratic unramified extension of $\mathbf{Q}_p$, then there exists a rigid analytic K_p-isomorphism:*

$$\mathcal{U}_p\colon \left(X_{N^+,pN^-} \times_{\mathrm{Spec}(\mathbf{Q})} \mathrm{Spec}(K_p)\right)^{\mathrm{rig}} \xrightarrow{\sim} \mathcal{H}_p \,/\Gamma^{(p)}_{N^+,N^-},$$

where X_{N^+,pN^-} is the Shimura curve defined above, the group $\Gamma^{(p)}_{N^+,N^-}$ is defined as in Proposition 3.5.3, and the superscript rig *denotes rigid analytification of the base-changed curve.*

This result was first proved by Cerednik in [8]. Drinfeld then gave a more conceptual proof [14] by giving a moduli interpretation for the space $\Gamma^{(p)}_{N^+,N^-} \backslash \mathcal{H}_p$. Note that this result is different from Mumford's result 3.5.2, in that it gives a *precise* uniformisation of the particular Shimura curve X_{N^+,pN^-} in terms of an arithmetically defined quotient of the p-adic upper half plane.

The following theorem describes the moduli interpretation for the CM-points of a $\mathbf{Z}[1/p]$-order $\mathcal{O}$ in a quadratic imaginary field $K/\mathbf{Q}$:

Theorem 3.7.2. *Let $\tau \in \mathrm{CM}(\mathcal{O})$ with $K = \mathrm{Frac}(\mathcal{O})$. Then under the Cerednik-Drinfeld uniformisation, τ corresponds with an abelian scheme A over K_p which is isomorphic to a product $A \cong E \times E$ where E is a supersingular elliptic curve over K_p with complex multiplication by $\mathcal{O} \cap \mathcal{O}_K$, where $\mathcal{O}_K$ denotes the maximal $\mathbf{Z}$-order of K.*

Proof. See [1]. $\square$

Chapter 4

Rigid analytic modular forms

In this chapter we fix a prime p and let $\mathcal{H}_p$ denote the p-adic upper half plane as in Chapter 3. Similarly, let $\mathcal{T}$ denote the Bruhat-Tits tree for $\mathrm{GL}_2(\mathbf{Q}_p)$ as in Section 3.2. Let $\Gamma = \Gamma^{(p)}_{N^+,N^-}$ be as defined in 3.5.3.

4.1 Definition

Let $\mathcal{O}$ denote the ring of rigid analytic functions on $\mathcal{H}_p$. Define a right action of $\mathrm{GL}_2(\mathbf{Q}_p)$ on $\mathcal{O}$ in the following way: for each even integer $k \geq 2$, put:

$$(f|_k\gamma)(z) = \frac{\det \gamma^{k/2}}{(az+b)^k} f\left(\frac{az+b}{cz+d}\right), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q}_p).$$

We let $\mathcal{O}(k)$ denote the space of functions $\mathcal{O}$ endowed with the right *weight* k action of $\mathrm{GL}_2(\mathbf{Q}_p)$.

Definition 4.1.1. The space of *rigid analytic modular forms* for Γ of weight k is the space of Γ-invariant functions in $\mathcal{O}(k)$. We denote this space $S_k(\Gamma)$.

Let $\Omega = H^0(\Omega^1_{\mathcal{H}_p})$ denote the $\mathcal{O}$-module of global sections of the sheaf of rigid analytic one-forms on $\mathcal{H}_p$.

Proposition 4.1.2. *Let $k \geq 2$ be an even integer. The map $\mathcal{O}(k) \to \Omega^{\otimes k/2}$ defined by mapping $f \mapsto f(dz)^{k/2}$ is a $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant isomorphism. Thus,*

$$S_k(\Gamma) \cong (\Omega^{\otimes k/2})^\Gamma.$$

This will be useful later when we wish to formulate an algebraicity condition for a rigid analytic modular form analogous to the algebraicity of the Fourier coefficients of a classical modular form.

4.2 Harmonic cocycles

The material in this section will only be used in our discussion of computations which the author performed while preparing this thesis. Anybody uninterested in such computations, which are contained in the appendix, may safely skip this section.

Let $\mathcal{T}$ denote the Bruhat-Tits tree for $\mathrm{GL}_2(\mathbf{Q}_p)$, as defined in Section 3.2.

Definition 4.2.1. Let M be a $\mathbf{C}_p$-module with a left action of $\mathrm{GL}_2(\mathbf{Q}_p)$. Then an M-valued *harmonic cocycle* on $\mathcal{T}$ is a map $c\colon E(\mathcal{T}) \to M$ which satisfies the following properties:

1. if $e \in E(\mathcal{T})$ is an oriented edge with opposite edge e', then $c(e') = -c(e)$;
2. if $v \in V(\mathcal{T})$ is a vertex and $e_0, \dots, e_p$ are the p edges which leave from v, one has $c(e_0) + \cdots + c(e_p) = 0$.

Let $C_h(M)$ denote the collection of all M-valued harmonic cocycles.

The space $C_h(M)$ inherits the structure of a $\mathbf{C}_p$-module equipped with a *right* action of $\mathrm{GL}_2(\mathbf{Q}_p)$:

$$(c \cdot \gamma)(e) = \gamma^{-1} \cdot c(\gamma e).$$

An apt choice for M allows us to realize the space of rigid analytic modular forms of weight k as the corresponding Γ-invariant harmonic cocycles. We introduce this representation now.

Fix an even integer $k \geq 2$ and put $n = k - 2$.

Definition 4.2.2. Let $P_n \subset \mathbf{C}_p[u, v]$ denote the subspace of homogeneous polynomials in two variables of degree n. Let $\mathrm{GL}_2(\mathbf{Q}_p)$ act on P_n on the right via the formula

$$(p \cdot \gamma)(u, v) = p(au + bv, cu + dv)$$

for $p(u, v) \in P_n$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q}_p)$. Let V_n be the $\mathbf{C}_p$-linear dual endowed with the dual left action of $\mathrm{GL}_2(\mathbf{Q}_p)$. Write $C_h(k) = C_h(V_n)$.

We would like to prove that $S_k(\Gamma) \cong C_h(k)^\Gamma$. In [34], Schneider uses p-adic residues of a rigid analytic 1-form on $\mathcal{H}_p$ to define a map

$$I_k\colon S_k(\Gamma) \to C_h(k)^\Gamma.$$

In [43], Teitelbaum defines the inverse map. We recall these constructions now.

If $f \in S_k(\Gamma)$ and $f(z)dz$ is the corresponding holomorphic differential on $\mathcal{H}_p$ (which is not Γ-invariant if f is nonzero), then for each oriented edge $e \in E(\mathcal{T})$ we define

$$c_f(e) = \sum_{i=0}^{n} \mathrm{Res}_e(z^i f(z)dz)\binom{k-2}{i}\delta_i,$$

where $\delta_i \in V_n$ is dual to $u^i v^{n-i}$ in P_n. Here Res_e denotes the annular residue of $z^i f(z)dz$ along the oriented anulus W_e associated to the oriented edge e. In short, one developes $z^i f(z)$ into a Laurent series on W_e in terms of an appropriate parameter w, and then defines the residue to be the coefficients of the w^{-1} term. For details consult 2.3 of [16].

Conversely, suppose that $c \in C_h(k)^\Gamma$ is a Γ-invariant harmonic cocycle. Then we can use c to define a p-adic measure on $\mathbf{P}_1(\mathbf{Q}_p)$ in the following way: if $e \in E(\mathcal{T})$ is an oriented edge, then let $U_e \subseteq \mathbf{P}_1(\mathbf{Q}_p)$ denote the compact open subset consisting of all ends of $\mathcal{H}_p$ which pass through e. The map $e \mapsto U_e$ describes a bijection between the compact open balls of $\mathbf{P}_1(\mathbf{Q}_p)$ and the oriented edges of the Bruhat-Tits tree. Define a measure μ_c on $\mathbf{P}_1(\mathbf{Q}_p)$ by putting $\mu_c(U_e) = c(e)$. For the definition of a boundary measure, and indeed, for more on the construction about to be described, consult section 2 of [43].

Let A_n denote the space of $\mathbf{C}_p$-valued functions on $\mathbf{P}_1(\mathbf{Q}_p)$ which are locally analytic except possibly for a pole at ∞ of order at most n. Then by approximating functions $f \in A_n$ locally by polynomials of degree $\leq n$, one can integrate such functions against the measure μ_c. For a precise statement see Proposition 9 of [43].

Thus, for all $z \in \mathcal{H}_p$, one can define a function

$$f_c(z) = \int_{\mathbf{P}_1(\mathbf{Q}_p)} \frac{1}{(z-t)} d\mu_c(t),$$

since $1/(z-t)$ is continuous and in fact locally analytic on $\mathbf{P}_1(\mathbf{Q}_p)$. The assignment

$c \mapsto f_c$ yields a map

$$P_k \colon C_h(k)^\Gamma \to S_k(\Gamma).$$

Theorem 4.2.3. *The maps I_k and P_k are mutually inverse, so that*

$$S_k(\Gamma) \cong C_h(k)^\Gamma.$$

Proof. See Corollary 11 of [43]. □

This result is very useful, as the space $C_h(k)^\Gamma$ is readily computable. Each cocycle $c \in C_h(k)^\Gamma$ is determined by the finitely many values $c(e)$ where e runs over a set of representatives for the quotient $\Gamma\backslash\mathcal{T}$ of the Bruhat-Tits tree. The author and Marc Masdeu have implemented a package in *Sage* which computes quotients $\Gamma\backslash\mathcal{T}$ and the corresponding spaces of modular forms $C_h(k)^\Gamma$. This work will be published in a forthcoming paper. Once one has the space of harmonic cocycles at hand, then one can use the work of Greenberg [20], [21] and Darmon-Pollack [12] to efficiently compute p-adic integrals. In the appendix to this thesis we describe how we used these ideas to evaluate rigid analytic modular forms at CM points.

4.3 Nearly rigid analytic modular forms

Let $K/\mathbf{Q}_p$ be an extension of $\mathbf{Q}_p$ contained inside $\mathbf{C}_p$. We write $\mathcal{O}$ for the ring of rigid analytic functions on $\mathcal{H}_p$. Let $F/\mathbf{Q}_p$ denote the completion of the maximal unramified extension of $\mathbf{Q}_p$ inside $\mathbf{C}_p$, and let $\sigma\colon F \to F$ denote the Frobenius automorphism. Thus σ is a pro-generator for $\mathrm{Gal}(F/\mathbf{Q}_p)$ and satisfies $\sigma(z) \equiv z^p \pmod p$ for all $z \in F$. We will always write $\sigma(z) = \overline{z}$ below, save for in the statement of the following crucial lemma:

Lemma 4.3.1. *The restriction of Frobenius to the unramified points of the p-adic upper half plane, $\sigma\colon \mathcal{H}_p(F) \to \mathcal{H}_p(F)$, does not agree with any meromorphic function f on $\mathcal{H}_p$ on the domain of definition of f within $\mathcal{H}_p(F)$.*

Proof. If f is a meromorphic function on $\mathcal{H}_p$, then it defines a meromorphic function on the standard affinoid

$$\mathcal{A} = \{z \in \mathbf{C}_p \mid \ |z| \le 1,\ \ |z-a| \ge 1 \text{ for } a = 0, \ldots, p-1\}.$$

Since this is a connected affinoid, Proposition 3.4.2 shows that f admits a decomposition $f(z) = r(z)u(z)$ where $r(z) \in \mathbf{C}_p(z)$ is a rational function and where $u(z)$ is a unit in $\mathcal{O}_{\mathcal{A}}$, the ring of rigid analytic functions on $\mathcal{A}$. It thus suffices to prove the following: for every nonzero polynomial $q(z) \in \mathbf{C}_p[z]$, the product $q(z)\overline{z}$ does not agree with any analytic function $f \in \mathcal{O}_{\mathcal{A}}$ on $\mathcal{A}(F) = \mathcal{A} \cap F$.

We prove this by induction on the degree of $q(z)$. The case $\deg q = 0$ follows by taking an expression for f as in Proposition 3.4.1, and by using the fact that $z \mapsto \overline{z}$ is $\mathbf{Q}_p$-linear. So suppose that $\deg q = N > 1$ and that we have $f \in \mathcal{O}_{\mathcal{A}}$ such that

$$q(z)\overline{z} = f(z) \quad \text{for all} \quad z \in \mathcal{A}(F). \tag{4.1}$$

Then note that this implies that for all $\alpha \in \mathbf{Q}_p^{\times}$,

$$\alpha q(\alpha z)\overline{z} = f(\alpha z). \tag{4.2}$$

Hence, if we multiply 4.1 by α^{N+1} and subtract 4.2, we obtain

$$(\alpha^{N+1} q(z) - \alpha q(\alpha z))\overline{z} = \alpha^{N+1} f(z) - f(\alpha z)$$

for all $z \in \mathcal{A}(F)$. Since the polynomial on the left has degree strictly smaller than N, by induction it must vanish. So we deduce

$$\alpha^N q(z) = q(\alpha z) \quad \text{and} \quad \alpha^{N+1} f(z) = f(\alpha z)$$

for all $\alpha \in \mathbf{Q}_p^{\times}$ and all $z \in \mathcal{A}(F)$. Since q is a polynomial and f is an analytic function on $\mathcal{H}_p$, it follows that $q(z) = az^N$ and $f(z) = bz^N$ for some scalars a, $b \in \mathbf{C}_p$ (see Lemma 4.3.4 below, for example). But then our assumption becomes

$$az^N\overline{z} = bz^{N+1}$$

for all $z \in \mathcal{A}(F)$. Hence we must have $a = b = 0$, contradicting $q(z) \neq 0$. □

Remark 4.3.2. We regard the Frobenius of $\mathcal{H}_p(F)$ as an analogue of complex conjugation. Note that on all the unramified points, this analogy is not perfect: for example, there is an amibguity as to whether one works with arithmetic or geometric Frobenius. This ambiguity disappears when one restricts to the points $\mathcal{H}_P(K_p)$, where $K_p/\mathbf{Q}_p$ denotes the quadratic unramified extension in $\mathbf{Q}_p$. Here Frobenius has order two and is canonical, just like complex conjugation is the unique non-identity element of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$.

The previous result is analogous to the fact that complex conjugation is not holomorphic.

Note that the function $z - \overline{z}$ is invertible on $\mathcal{H}_p(F)$, and so $1/(z-\overline{z})$ is a continuous function on $\mathcal{H}_p(F)$. We may thus define a map

$$\mathcal{O}[X] \to C(\mathcal{H}_p(F), \mathbf{C}_p),$$

where $C(\mathcal{H}_p(F), \mathbf{C}_p)$ denotes the ring of continuous functions from $\mathcal{H}_p(F)$ to $\mathbf{C}_p$, by sending X to $1/(z-\overline{z})$. The field F has enough points to distinguish elements in the image of this map:

Proposition 4.3.3. *The assignment $X \mapsto 1/(z-\overline{z})$ yields an injective ring homomorphism:*

$$\mathcal{O}[X] \hookrightarrow C(\mathcal{H}_p(F), \mathbf{C}_p).$$

Proof. Suppose that

$$\sum_{i=0}^{N} \frac{f_i(z)}{(z-\overline{z})^i} = 0$$

for some rigid analytic functions $f_i \in \mathcal{O}$ and all $z \in \mathcal{H}_p(F)$. This is equivalent with

$$\begin{aligned} 0 &= \sum_{i=0}^{N} f_i(z)(z-\overline{z})^{N-i} \\ &= \sum_{i=0}^{N} \sum_{j=0}^{N-i} \binom{N-i}{j} f_i(z) z^{N-i-j} \overline{z}^j \\ &= \sum_{j=0}^{N} \left(\sum_{i=0}^{N-j} \binom{N-i}{j} f_i(z) z^{N-i-j} \right) \overline{z}^j. \end{aligned}$$

Note that if we put $g_j(z) = \sum_{i=0}^{N-j} \binom{N-i}{j} f_i(z) z^{N-i-j}$ then g_j is rigid analytic, and $g_j = 0$ for $j = 0, \ldots, N$ if and only if $f_i = 0$ for $i = 0, \ldots, N$. Thus, it suffices to prove the following claim: if $\sum_{i=0}^{N} f_i(z)\overline{z}^i = 0$ for some $f_i \in \mathcal{O}$ and all $z \in \mathcal{H}_p(F)$ then $f_i = 0$ for all i.

We proceed by induction on N, and prove in fact the stronger statement that there is no relation $\sum f_i(z)\overline{z}^i = 0$ for *meromorphic* functions f_i on $\mathcal{H}_p$.

Suppose we have such a relation for $N \geq 1$. Cancel the factor of f_N, so that we have

$$\overline{z}^N + g_{N-1}(z)\overline{z}^{N-1} + \cdots + g_0(z) = 0$$

for meromorphic functions $g_i = f_i/f_N$ on $\mathcal{H}_p$ and all $z \in \mathcal{H}_p(F)$ at which the meromorphic functions are defined. We will show that the g_i must vanish. The case $N = 0$ is clear and $N = 1$ follows from Lemma 4.3.1.

Suppose that $N \geq 2$ and note that since $z \mapsto \overline{z}$ is $\mathbf{Q}_p$-linear, the previous displayed formula also implies

$$a^N\overline{z}^N + \sum_{i=0}^{N-1} g_i(az)a^i\overline{z}^i = 0$$

for all $a \in \mathbf{Q}_p^\times$ and all $z \in \mathcal{H}_p(F)$ at which the meromorphic functions $g_i(az)$ are defined. It follows that

$$\sum_{i=0}^{N-1}(a^i g_i(az) - a^N g_i(z))\overline{z}^i = 0$$

for all $z \in \mathcal{H}_p(F)$ at which the meromorphic coefficients above are defined. By induction we conclude that $g_i(az) = a^{N-i}g_i(z)$ for all $a \in \mathbf{Q}_p^\times$. We deduce, by Lemma 4.3.4 below, that $g_i(z) = b_i z^{N-i}$ for some $b_i \in \mathbf{C}_p$. Thus, we have

$$\overline{z}^N + b_{N-1}z\overline{z}^{N-1} + \cdots + b_0 z^N = 0$$

for all $z \in \mathcal{H}_p(F)$. If all the b_i are zero, then this is a contradiction. Otherwise, if necessary, we may rescale this equation so that the b_i have absolute value ≤ 1 with at least one $|b_i| = 1$. Then if we apply the above to $z \in \mathcal{H}_p(F)$ with $|z| \leq 1$ and reduce mod p, we obtain a nontrivial identity

$$b_N z^{pN} + b_{N-1}z^{pN-(p-1)} + b_{N-2}^{pN-2(p-1)} + \cdots + b_0 z^N \equiv 0 \pmod{p}$$

for all $|z| \leq 1$, where the first b_N arises due to rescaling. Since the residue field of F is an algebraic closure of $\mathbf{F}_p$, it follows that all of the b_i must vanish mod p, contradicting that we normalized them so that at least one was a p-unit. This contradiction concludes the proof. $\square$

Lemma 4.3.4. *Let f be a meromorphic function on $\mathcal{H}_p$. If $f(az) = a^k f(z)$ for some* *integer $k \geq 0$ and all $a \in \mathbf{Q}_p^\times$, then $f(z) = bz^k$ for some $b \in \mathbf{C}_p$.*

Proof. By differentiating the equality $f(az) = a^k f(z)$ repeatedly, one reduces to the case $k = 0$. Thus suppose $f(az) = f(z)$ for all $a \in \mathbf{Q}_p^\times$. Since the poles of a meromorphic function on $\mathcal{H}_p$ are discrete, this equality implies that f must in fact be analytic. Consider its expansion on the standard affinoid

$$A = \{z \in \mathbf{C}_p \mid |z| \leq 1,\ |z - x| \geq 1 \text{ for } x = 0, \ldots, p-1\}.$$

On A the function f may be expressed uniquely as

$$f(z) = \sum_{n \geq 0} a_n z^n + \sum_{x=0}^{p-1} \sum_{n \geq 1} \frac{b_n(x)}{(z-x)^n}$$

for elements a_n and $b_n(x)$ in $\mathbf{C}_p$ which tend to 0 as n grows. Let $\alpha \in \mathbf{Z}_p$ and note that since $f(z/(1+p\alpha)) = f(z)$ by hypothesis, we deduce that

$$\begin{aligned}
f(z) &= \sum_{n \geq 0} \frac{a_n}{(1+p\alpha)^n} z^n + \sum_{x=0}^{p-1} \sum_{n \geq 1} \frac{b_n(x)(1+p\alpha)^n}{(z - x - p\alpha x)^n} \\
&= \sum_{n \geq 0} \frac{a_n}{(1+p\alpha)^n} z^n + \sum_{x=0}^{p-1} \sum_{n \geq 1} \frac{b_n(x)(1+p\alpha)^n}{(z-x)^n (1 - p\alpha x/(z-x))^n} \\
&= \sum_{n \geq 0} \frac{a_n}{(1+p\alpha)^n} z^n + \sum_{x=0}^{p-1} \sum_{n \geq 1} \sum_{m \geq 0} \binom{m+n-1}{m} b_n(x)(1+p\alpha)^n \frac{(p\alpha x)^m}{(z-x)^{n+m}} \\
&= \sum_{n \geq 0} \frac{a_n}{(1+p\alpha)^n} z^n + \\
&\quad \sum_{x=0}^{p-1} \sum_{n \geq 1} \left(\sum_{m \geq 0}^{n-1} \binom{n-1}{m} b_{n-m}(x)(1+p\alpha)^{n-m}(p\alpha x)^m \right) \frac{1}{(z-x)^n}.
\end{aligned}$$

By the uniqueness of the expansion of f on A we deduce that $a_n = a_n/(1+p\alpha)^n$ for all n and $\alpha \in \mathbf{Z}_p$. Choosing α such that $1 + p\alpha$ is not an nth root of unity implies $a_n = 0$ for all $n \geq 1$. Similarly, one deduces that for each x and n,

$$b_n(x) = \sum_{m \geq 0}^{n-1} \binom{n-1}{m} b_{n-m}(x)(1+p\alpha)^{n-m}(p\alpha x)^m.$$

One argues by induction on n that $b_n(x) = 0$ for all $n \geq 1$ and all $x = 0, \ldots, p-1$.

Thus, f must be constant on A, and hence constant on $\mathcal{H}_p$ by analytic continuation. $\square$

Proposition 4.3.3 suggests the following definition.

Definition 4.3.5. The image of the map $\mathcal{O}[X] \to C(\mathcal{H}_p(F), \mathbf{C}_p)$ defined by $X \mapsto 1/(z - \overline{z})$ is denoted $\mathcal{N}$ and called the *ring of nearly rigid analytic functions* on $\mathcal{H}_p(F)$.

We will write elements in $\mathcal{N}$ as polynomial expressions

$$f(z) = \sum_i \frac{f_i(z)}{(z - \overline{z})^i}$$

with the f_i rigid analytic functions on $\mathcal{H}_p$. Note that by what we just proved, if we view $f(z)$ as a continuous function on $\mathcal{H}_p(F)$, then the coefficients $f_i(z)$ are uniquely determined by $f(z)$.

Definition 4.3.6. Define the differential operator $d/d\overline{z}$ on $\mathcal{N}$ by setting, for $f(z) \in \mathcal{O}$ and $j \geq 0$:

$$\frac{d}{d\overline{z}} \left(\frac{f(z)}{(z - \overline{z})^j} \right) = \frac{j f(z)}{(z - \overline{z})^{j+1}},$$

and extending linearly. Then Maass's *lowering operator* is the operator defined by

$$\varepsilon = (z - \overline{z})^2 \frac{d}{d\overline{z}}.$$

Similarly, for each integer $k \geq 0$ define the *Shimura-Maass differential operator* of weight k, denoted δ_k, on $\mathcal{N}$ by setting

$$\delta_k \left(\frac{f(z)}{(z - \overline{z})^j} \right) = \frac{f'(z)}{(z - \overline{z})^j} + \frac{(k - j) f(z)}{(z - \overline{z})^{j+1}}$$

and extending linearly.

The weight k slash operation of $\mathrm{GL}_2(\mathbf{Q}_p)$ on rigid analytic functions extends to $\mathcal{N}$ in a natural way.

Definition 4.3.7. Let $\Gamma \subseteq \mathrm{GL}_2(\mathbf{Q}_p)$ denote a discrete and cocompact subgroup. For each integer $k \geq 0$, we let $\mathcal{N}_k(\Gamma) \subseteq \mathcal{N}$ denote the set of nearly analytic functions which are invariant under the weight k action of Γ on $\mathcal{N}$. We call $\mathcal{N}_k(\Gamma)$ the space of *nearly rigid analytic modular forms* of weight k for Γ. Note that $S_k(\Gamma) \subseteq \mathcal{N}_k(\Gamma)$, so that every rigid analytic modular form is also nearly rigid analytic.

For each integer $r \geq 0$ let $\mathcal{N}^r \subseteq \mathcal{N}$ denote the subspace consisting of forms

$$f(z) = \sum_{j=0}^{r} \frac{f_j(z)}{(z - \overline{z})^j}.$$

Set $\mathcal{N}_k^r(\Gamma) = \mathcal{N}_k(\Gamma) \cap \mathcal{N}^r$.

The following lemma is proved via a straightforward but tedious computation:

Lemma 4.3.8. *Let $f \in \mathcal{N}$ and let $\gamma \in \mathrm{GL}_2(\mathbf{Q}_p)$. Then*

$$\delta_k(f|_k\gamma) = (\delta_k f)|_{k+2}\gamma,$$
$$\varepsilon(f|_k\gamma) = \varepsilon(f)|_{k-2}\gamma.$$

Let $\Gamma \subseteq \mathrm{GL}_2(\mathbf{Q}_p)$ denote a discrete and cocompact subgroup. The previous lemma shows that δ_k maps $\mathcal{N}_k(\Gamma)$ to $\mathcal{N}_{k+2}(\Gamma)$, while ε maps $\mathcal{N}_{k+2}(\Gamma)$ to $\mathcal{N}_k(\Gamma)$. Just as $S_k(\Gamma) \cap S_{k'}(\Gamma) = \{0\}$ if $k \neq k'$, the same is true for $\mathcal{N}_k(\Gamma)$. Let $\mathcal{N}(\Gamma) = \bigoplus_{k\geq 0} \mathcal{N}_k(\Gamma)$. Then $\delta = \bigoplus_{k \geq 0} \delta_k$ and ε define graded derivations of $\mathcal{N}(\Gamma)$, of weight 2 and -2, respectively. The filtrations $\mathcal{N}_k^\bullet(\Gamma)$ of $\mathcal{N}_k(\Gamma)$ induce an increasing filtration of $\mathcal{N}$: we write $\mathcal{N}^r = \bigoplus_{k\geq 0} \mathcal{N}_k^r(\Gamma)$. Then δ maps $\mathcal{N}^r$ to $\mathcal{N}^{r+1}$.

Our next lemma follows immediately from the definition of ε:

Lemma 4.3.9. *If $f(z) = \sum_{j=0}^r f_j(z)(z - \overline{z})^{-j} \in \mathcal{N}$ then $\varepsilon^r f = (r!) f_r$.*

Proposition 4.3.10. *Let $\Gamma \subseteq \mathrm{GL}_2(\mathbf{Q}_p)$ be a discrete and cocompact subgroup. Let $f \in \mathcal{N}_k(\Gamma)$ and write $f(z) = \sum_{j=0}^r f_j(z)(z-\overline{z})^{-j}$. Then $f_r(z) \in S_{k-2r}(\Gamma)$. In particular, $\mathcal{N}_{2r}(\Gamma) = \mathcal{N}_{2r}^r(\Gamma)$.*

Proof. The previous two lemmas show that for all $\gamma \in \Gamma$,

$$f_r|_{k-2r}\gamma = \frac{1}{r!}\varepsilon^r(f|_k\gamma) = \frac{1}{r!}\varepsilon^r(f) = f_r.$$

Thus $f_r \in S_{k-2r}(\Gamma)$, and this space is trivial if $k - 2r < 0$. Repeatedly applying this observation to elements in $\mathcal{N}_{2r}(\Gamma)$ proves the last part of the proposition. $\square$

One can use the above observations to prove the following:

Theorem 4.3.11. *Let* $\Gamma \subseteq \mathrm{GL}_2(\mathbf{Q}_p)$ *denote a discrete and cocompact subgroup. Let* $k \geq 2$ *and* $r \geq 0$ *be integers. Then there is an isomorphism of* $\mathbf{C}_p$*-vector spaces*

$$\bigoplus_{j=0}^{r} S_{k+2r-2j}(\Gamma) \xrightarrow{\sim} \mathcal{N}^r_{k+2r}(\Gamma)$$

which maps $(h_j) \mapsto \sum_{j=0}^{r} \delta^j_{k+2r-2j} h_j$*. In particular, for* $k = 2r$ *and* $r \geq 1$*, one has*

$$\mathcal{N}_k(\Gamma) \cong \bigoplus_{j=0}^{r} S_{k-2j}(\Gamma)$$

as vector spaces over $\mathbf{C}_p$*.*

Proof. The proof of Theorem 1 in §10.1 of [23] carries over from the complex case treated there to our rigid analytic setting. The argument uses that if $f = \sum_{j=0}^{r} f_j(z)(z-\overline{z})^{-j} \in \mathcal{N}^r_{k+2r}(\Gamma)$, then $f_r \in S_k(\Gamma)$, and one considers the expression $\delta^r_k f_r \in \mathcal{N}^r_{k+2r}(\Gamma)$. If we write $h_r = (k!/(k+r)!) f_r$, then Proposition (4.3.3) shows that h_r is uniquely determined by f, and a routine but messy computation shows that $f - \delta^r_k h_r \in \mathcal{N}^{r-1}_{k+2r}(\Gamma)$. One continues inductively. $\square$

Chapter 5

Proof of main theorem

5.1 Notation and statement of theorem

Let p, N^-, N^+ and B be as defined in 3.1, and let $\Gamma = \Gamma^{(p)}_{N^+,N^-} \subseteq \mathrm{SL}_2(\mathbf{Q}_p)$ be as defined in 3.5.3. Let K_p denote the quadratic unramified extension of $\mathbf{Q}_p$. Write $X = X_{N^+,pN^-}$, which is a Shimura curve defined over $\mathbf{Q}$, and let

$$X_{K_p} = X \times_{\mathrm{Spec}(\mathbf{Q})} \mathrm{Spec}(K_p)$$

denote the base change. Let

$$\mathcal{U}_p \colon X^{\mathrm{rig}}_{K_p} \xrightarrow{\sim} \Gamma \backslash \mathcal{H}_p,$$

denote the Cerednik-Drinfeld uniformisation as in 3.7.1. Recall that rigid analytic modular forms for Γ of *even* weight k correspond via pullback under $\mathcal{U}_P$ to global sections of $(\Omega^1_{K_p})^{\otimes k/2}$, where $\Omega^1_{K_p}$ denotes the sheaf of regular differentials on the curve X_{K_p}.

Definition 5.1.1. Let k be an even integer. We say that a rigid analytic modular form for Γ is *algebraic* if it corresponds to a global section of $(\Omega^1_{K_p})^{\otimes k/2}$, via the map $f(\tau) \mapsto f(\tau)(d\tau)^{k/2}$, which is defined over $\overline{\mathbf{Q}} \cap K_p$. We write $S_k(\Gamma^{(p)}_{N^+,N^-}, \overline{\mathbf{Q}})$ for the space of algebraic rigid analytic modular forms.

Remark 5.1.2. Note that this definition differs from the case of complex modular curves, where one identifies $S_k(\Gamma_0(N), \overline{\mathbf{Q}})$ and $\Omega^1(X_0(N)/\overline{\mathbf{Q}})^{k/2}$, in that there one maps $f(\tau) \mapsto f(\tau)(2\pi i d\tau)^{k/2}$. Thus, in our formulation, any sort of analogue for the factor of $2\pi i$ in the definition of the complex Shimura-Maass operator is incorporated here into the condition of algebraicity formualted above.

Suppose instead that $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Q}_p)$ is a congruence subgroup. Then if we take $\Gamma^{(p)}_{N^+,N^-} \subseteq \Gamma$ for some N^+, we have that $S_k(\Gamma) \subseteq S_k(\Gamma^{(p)}_{N^+,N^-})$. Define

$$S_k(\Gamma, \overline{\mathbf{Q}}) = S_k(\Gamma) \cap S_k(\Gamma^{(p)}_{N^+,N^-}, \overline{\mathbf{Q}}).$$

Lemma 5.1.3. *The space $S_k(\Gamma, \overline{\mathbf{Q}})$ is well-defined independently of N^+. That is, if M and M' are two positive integers coprime to pN^- such that Γ contains both $\Gamma^{(p)}_{M,N^-}$ and $\Gamma^{(p)}_{M',N^-}$, then*

$$S_k(\Gamma) \cap S_k(\Gamma^{(p)}_{M,N^-}, \overline{\mathbf{Q}}) = S_k(\Gamma) \cap S_k(\Gamma^{(p)}_{M',N^-}, \overline{\mathbf{Q}}).$$

Definition 5.1.4. Let Γ be a congruence subgroup for B. Then a modular form $f \in S_k(\Gamma)$ is said to be *algebraic* if it lies in $S_k(\Gamma, \overline{\mathbf{Q}})$.

If Γ and Γ' are two congruence subgroups for B, then so is $\Gamma \cap \Gamma'$. The spaces $S_k(\Gamma)$ as Γ varies thus define a filtered injective system and it makes sense to write

$$S_k = \bigcup_{\Gamma \subseteq \mathrm{SL}_2(\mathbf{Q}_p)} S_k(\Gamma),$$

where the union is taken over all congruence subgroups Γ. Define $S_k(\overline{\mathbf{Q}})$ similarly. Let $S = \bigoplus_k S_k$ and $S(\overline{\mathbf{Q}}) = \bigoplus_k S_k(\overline{\mathbf{Q}})$. Let $A_k(\overline{\mathbf{Q}})$ denote the kth graded piece of $\mathrm{Frac}(S(\overline{\mathbf{Q}}))$, where the degree of a quotient f/g is defined in the obvious way as $\deg(f) - \deg(g)$. Note that $S_k(\overline{\mathbf{Q}}) \subseteq A_k(\overline{\mathbf{Q}})$.

The main theorem of our thesis is the following analogue of Main Theorem 1 in [39]:

Theorem 5.1.5. *Let $f \in S_k(\overline{\mathbf{Q}})$ and $g \in S_{k+2r}(\overline{\mathbf{Q}})$. Let $K/\mathbf{Q}$ denote a quadratic imaginary extension. Assume that all the primes dividing pN^- are inert in K, so that there are CM-points in $\mathcal{H}_p$ for K. Let $\tau \in \mathrm{CM}(K)$ denote a CM-point such that $g(\tau) \neq 0$. Then*

$$\frac{\delta^r_k(f)(\tau)}{g(\tau)} \in \overline{\mathbf{Q}}.$$

5.2 Proof of Theorem 5.1.5

We begin with some lemmas.

Lemma 5.2.1. *For all $k \geq 0$, $A_{2k}(\overline{\mathbf{Q}}) \neq \{0\}$. If $f \in A_{2k}(\overline{\mathbf{Q}})$ and $\alpha \in B^{\times}$, then $f|_{2k}\alpha \in A_{2k}(\overline{\mathbf{Q}})$.*

Proof. For the first claim, use Cerednik-Drinfeld and the analogous statement for meromorphic modular forms on the corresponding Shimura curve (or alternatively, use Riemann-Roch).

For the second claim, note that if $f \in A_{2k}(\overline{\mathbf{Q}})$ is modular for some congruence subgroup Γ, then $f|_{2k}\alpha$ is modular for $\Gamma' = \alpha^{-1}\Gamma\alpha$. This conjugate is another congruence subgroup, and thus $f|_{2k}\alpha \in S_{2k}(\Gamma')$. Next note that since $\alpha \in B^{\times}$, the map

$$\Gamma \backslash \mathcal{H}_p \to \Gamma' \backslash \mathcal{H}_p$$

given by $\tau \mapsto \alpha^{-1} \cdot \tau$, corresponds via Cerednik-Drinfeld, to a map of Shimura curves is defined over $\mathbf{Q}$. It thus preserves algebraic modular forms and so $f|_{2k}\alpha \in S_{2k}(\Gamma', \overline{\mathbf{Q}}) \subseteq A_{2k}(\overline{\mathbf{Q}})$. $\square$

Lemma 5.2.2. *With τ as in the statement of Theorem 5.1.5, for each $k \in \mathbf{Z}_{>0}$ there exists*

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B_1 \subset \mathrm{SL}_2(\mathbf{Q}_p)$$

such that $\alpha \cdot \tau = \tau$ and $(c\tau + d)^{2k} \neq 1$. (Recall that B_1 denotes the elements in B of reduced norm equal to 1).

Proof. The proof is analogous to the proof of (1.12) in Shimura's paper [39]. Note that the proof of this claim is found on page 496 of Shimura's paper, in the paragraph containing lined formula (1.17). $\square$

We turn now to the proof of Theorem 5.1.5. We will prove the following slightly stronger result:

> for all $f \in A_k(\overline{\mathbf{Q}})$ and $g \in A_{k+2r}(\overline{\mathbf{Q}})$ such that $g(\tau) \neq 0$, we have $(\delta_k^r f)(\tau)/g(\tau) \in \overline{\mathbf{Q}}$.

Since $S_k(\overline{\mathbf{Q}})$ is contained in $A_k(\overline{\mathbf{Q}})$ for all k, this clearly implies Theorem 5.1.5.

Our proof proceeds by induction on $r \geq 0$. In the case $r = 0$, the quotient f/g defines an element of the function field of X_{K_p} which is defined over $\overline{\mathbf{Q}}$: that is, f/g agrees with an element of the function field of the Shimura curve $X_{\overline{\mathbf{Q}}}$. In Theorem 5.3 of [1] it is proved that $\tau \in \mathcal{H}_p \cap K$ corresponds to a CM-point for K on $X/\mathbf{Q}$. It thus follows from the classical theory developed by Shimura that $f(\tau)/g(\tau) \in \overline{\mathbf{Q}}$. This proves the case $r = 0$.

Now suppose $r \geq 1$. Following Shimura, we may assume $f(\tau) \neq 0$. To see this, note that the set

$$\{\gamma \cdot \tau \in \mathcal{H}_p \mid \gamma \in B^\times \subset \mathrm{GL}_2(\mathbf{Q}_p)\}$$

is dense in $\mathcal{H}_p$ for the p-adic topology. Thus, since if $f = 0$ there is nothing to show, we may find $\beta \in B^\times$ with $f(\beta \cdot \tau) \neq 0$. If $f(\tau) = 0$ then we can instead consider $f' = f + f|_k\beta$, which is modular of weight k for the congruence subgroup $\Gamma \cap \beta^{-1}\Gamma\beta$ by the second claim of Lemma 5.2.1. Since $f = f' - f|_k\beta$, and neither of the forms on the right vanish at τ, it indeed suffices to treat the case where $f(\tau) \neq 0$.

Take

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B_1$$

such that $\alpha \cdot \tau = \tau$ and $\alpha \neq \pm 1$. Set $h = (f|_k\alpha)/f$, so that $h \in A_0(\overline{\mathbf{Q}})$. Since α fixes τ we see that $h(\tau) = \lambda^{-k}$ where $\lambda = c\tau + d$. Suppose $r = 1$. Apply δ_k to $f|_k\alpha = fh$, so that the Leibniz rule, combined with the identity $\delta_k(f|_k\alpha) = (\delta_k f)|_{k+2}\alpha$, yields

$$(\delta_k f)|_{k+2}\alpha = (\delta_k f)h + f(\delta_0 h). \tag{5.1}$$

Evaluate this identity at τ to obtain

$$(\delta_k f)(\tau)(\lambda^{-2} - 1)\lambda^{-k} = f(\tau)(\delta_0 h)(\tau). \tag{5.2}$$

Note that since h transforms as of weight 0, we have $f \cdot \delta_0(h) \in A_{k+2}(\overline{\mathbf{Q}})$. Hence $g^{-1} f \delta_0(h) \in A_0(\overline{\mathbf{Q}})$ and so

$$\frac{f(\tau)(\delta_0 h)(\tau)}{g(\tau)} \in \overline{\mathbf{Q}}$$

by the previous case. Thus, from this and from equation 5.2 one deduces that $\delta_k(f)(\tau)/g(\tau) \in \overline{\mathbf{Q}}$.

For general $r \geq 2$, use Lemma 5.2.2 to obtain α fixing τ and such that $(c\tau +$

$d)^{2r} \neq 1$. Apply δ_k^r to fh, where h is defined as above, to obtain

$$\delta_k^r(fh) = \sum_{i=0}^{r} \binom{r}{i} (\delta_k^i f)(\delta_0^{r-i} h).$$

Since $fh = f|_k\alpha$ and $\delta_k^r(f|_k\alpha) = (\delta_k^r f)|_{k+2r}\alpha$ we have

$$\delta_k^r(f)|_{k+2r}\alpha = \sum_{i=0}^{r} \binom{r}{i} (\delta_k^i f)(\delta_0^{r-i} h). \tag{5.3}$$

As above, evaluate this at τ and use $\alpha \cdot \tau = \tau$ to deduce

$$\delta_k^r(f)(\tau)(\lambda^{-2r} - 1)\lambda^{-k} = \sum_{i=1}^{r} \binom{r}{i} (\delta_k^i f)(\tau)(\delta_0^{r-i} h)(\tau) \tag{5.4}$$

Now here is the final trick: for each i in the range $0 \leq i \leq r$, as above choose $g_i \in S_{2i}(\overline{\mathbf{Q}})$ such that $g_i(\tau) \neq 0$. Then we have

$$(g^{-1}\delta_k^r f)(\tau)(\lambda^{-2r} - 1)\lambda^{-k} = \sum_{i=1}^{r} \binom{r}{i} (g_i g^{-1} \delta_k^{r-i} f)(\tau)(g_i^{-1}\delta_0^i h)(\tau). \tag{5.5}$$

For the terms $i > 0$, if we apply our inductive hypothesis to $\delta_k^{r-i} f/(gg_i^{-1})$, then we deduce that $(g_i g^{-1}\delta_k^{r-i}(f))(\tau) \in \overline{\mathbf{Q}}$. Since h is of weight 0, its derivative $\delta_0 h$ is meromorphic of weight 2, and thus

$$\delta_0^i(h) = \delta_2^{i-1}(\delta_0 h).$$

Hence our inductive hypothesis also implies $(g_i^{-1}\delta_0^i h)(\tau) \in \overline{\mathbf{Q}}$ for $i = 1, \ldots, r$. Thus, by equation 5.5, we deduce that $(\delta_k^r f)(\tau)/g(\tau) \in \overline{\mathbf{Q}}$ as claimed. This concludes the proof of Theorem 5.1.5.

5.3 Restatement of Theorem 5.1.5

The main theorem 5.1.5 can be rephrased as follows:

Theorem 5.3.1. *Let $k \geq 0$ be an even integer and let $f \in S_k(\overline{\mathbf{Q}})$. Let $K/\mathbf{Q}$ denote a quadratic imaginary extension. Assume that all the primes dividing pN^- are inert in K, so that there are CM-points in $\mathcal{H}_p$ for K. Then there exists a constant $\Omega_K \in \mathbf{C}_p^\times$*

depending only on $K/\mathbf{Q}$ *such that if* $\tau \in \mathrm{CM}(K)$ *denotes a CM-point, then*

$$\frac{\delta_k^r(f)(\tau)}{\Omega_k^{k+2r}} \in \overline{\mathbf{Q}}$$

for all $r \geq 0$.

To obtain Theorem 5.3.1 from Theorem 5.1.5, let $g \in A_2(\overline{\mathbf{Q}})$ be nonzero; such a form exists by Lemma 5.2.1. As in the proof of 5.3.1, if $g(\tau) = 0$ then we may consider instead $g' = g + g|_2\beta \in A_0(\overline{\mathbf{Q}})$ for some $\beta \in B^\times$ such that $g'(\tau) \neq 0$. We may thus assume $g(\tau) \neq 0$. Set $\Omega_K = g(\tau)$. Then Theorem 5.3.1 says exactly that $\delta_k^r(f)(\tau)/\Omega_K^{k+2r} \in \overline{\mathbf{Q}}$ for all $f \in S_k(\overline{\mathbf{Q}})$.

It remains to show that the same period Ω_K accounts for the transcendental part of the CM-values of nearly rigid analytic modular forms at CM-points for K different from the one $\tau \in \mathrm{CM}(K)$ used to define Ω_K. If τ' is another CM-point for K, then we can find $\alpha \in B^\times$ such that $\alpha \cdot \tau = \tau'$. Set $h = f|_k\alpha$, which is another element of $S_k(\overline{\mathbf{Q}})$. Note that

$$(\delta_k^r h)(\tau) = ((\delta_k^r f)|_{k+2r}\alpha)(\tau) = j(\alpha,\tau)^{k+2r}(\delta_k^r f)(\tau'),$$

and hence

$$\frac{(\delta_k^r h)(\tau)}{\Omega_K^{k+2r}} \in \overline{\mathbf{Q}} \quad \Longleftrightarrow \quad \frac{(\delta_k^r f)(\tau')}{\Omega_K^{k+2r}} \in \overline{\mathbf{Q}}.$$

Since the left side holds by the previous paragraph, we deduce that Ω_K is a suitable period for all CM-points for K.

Chapter 6

Future directions

6.1 Geometric proof

6.1.1 Katz's interpretation of Shimura's result

In this section we give a geometric definition of the complex Shimura-Maass differential operator δ_k. Let $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ and consider the space $\mathcal{H} \times \mathbf{C}$. Both Γ and $\mathbf{Z}^2$ act on $\mathcal{H} \times \mathbf{C}$, on the left and right, respectively:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, (c\tau + d)^{-1} z \right),$$
$$(\tau, z) \cdot (m, n) = (\tau, z + m\tau + n).$$

Let $\pi\colon \mathcal{H} \times \mathbf{C} \to \mathcal{H}$ be the projection, and note that π is Γ-equivariant. It is also $\mathbf{Z}^2$-equivariant when $\mathcal{H}$ is endowed with the trivial action of $\mathbf{Z}^2$. If we write $\mathcal{E} = \Gamma\backslash(\mathcal{H} \times \mathbf{C})/\mathbf{Z}^2$, then π induces a surjective map:

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"] \\ Y(1) \end{tikzcd}$$

where $Y(1)$ denotes the open modular curve. We call $\mathcal{E}$ the *universal elliptic curve* over $Y(1)$, despite the fact that it is not exactly a universal solution to a moduli problem.

Write $\omega = \pi_*(\Omega^1_{\mathcal{E}/Y(1)})$ and put $\omega^k = \mathrm{Sym}^k(\omega)$ for each $k \geq 1$. Let $f(\tau)$ be a holomorphic modular form of weight k for Γ. The modularity of f implies that $f(\tau)(2\pi i dz)^k$ is a Γ-invariant global section of ω^k. It turns out that the map $f \mapsto f(\tau)(2\pi i dz)^k$ gives an inclusion of complex vector spaces:

$$S_k(\Gamma) \to H^0(Y(1), \omega^k).$$

The universal elliptic curve $\mathcal{E}$ is a complex analytic curve over $Y(1)$, which is in fact algebraic over $\mathbf{Q}$, and we will be interested in the Hodge filtration for this curve:

$$0 \to \omega \to \mathcal{H}^1_{dR}(\mathcal{E}/Y(1)) \to (R^1\pi_*)(\mathcal{O}_{\mathcal{E}}) \to 0.$$

Here $\mathcal{H}^1_{dR}(\mathcal{E}/Y(1))$ denotes the relative de Rham cohomology for $\mathcal{E}/Y(1)$. It is a locally free $\mathcal{O}_{Y(1)}$-module of rank 2 on $Y(1)$:

Lemma 6.1.1. *The differentials $2\pi i dz$ and $2\pi i d\overline{z}$ define a basis for $\mathcal{H}^1_{dR}(\mathcal{E}/Y(1))_\tau$ for each $\tau \in \mathcal{H}$.*

Proof. We represent points of $Y(1)$ by lifts $\tau \in \mathcal{H}$. For such τ let $E_\tau = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$. The stalk of $\mathcal{H}^1_{dR}(\mathcal{E}/Y(1))$ at $\tau \in \mathcal{H}$ is isomorphic with $H^1_{dR}(E_\tau)$. It will thus suffice to show that the stalks of $2\pi i dz$ and $2\pi i d\overline{z}$ make up a $\mathbf{C}$-basis of $H^1_{dR}(E_\tau)$ for each $\tau \in \mathcal{H}$. Recall that Poincare duality:

$$H^1_{dR}(E_\tau) \times H_1(E_\tau) \to \mathbf{C},$$

which maps:

$$(\omega, \gamma) \mapsto \int_\gamma \omega$$

is a perfect pairing. This shows that each de Rham cohomology class is determined by its periods over γ_1, γ_2, where these are generators for $H_1(E_\tau) \simeq \mathbf{Z}^2$. Take γ_1 to be the image of the line from 0 to 1 in $\mathbf{C}$ under the projection $\mathbf{C} \to E_\tau$. Similarly let γ_2 be the image of the line from 0 to τ. It then follows that:

$$\int_{\gamma_1} 2\pi i dz = 2\pi i \int_0^1 dz = 2\pi i \quad \text{and} \quad \int_{\gamma_2} 2\pi i dz = 2\pi i \int_0^\tau dz = 2\pi i \tau.$$

Similarly:

$$\int_{\gamma_1} 2\pi i d\overline{z} = 2\pi i \int_0^1 d\overline{z} = 2\pi i \quad \text{and} \quad \int_{\gamma_2} 2\pi i d\overline{z} = 2\pi i \int_0^\tau d\overline{z} = 2\pi i \overline{\tau}.$$

Since $\tau \in \mathcal{H}$, the period vectors $(2\pi i, 2\pi i\tau)$ and $(2\pi i, 2\pi i\overline{\tau})$ are linearly independent over $\mathbf{C}$. It follows that the classes of $2\pi i dz$ and $2\pi i d\overline{z}$ make up a $\mathbf{C}$-basis for $H^1_{dR}(E_\tau)$.

Note that this result is just "Hodge theory for $E(\mathbf{C})$"; see Chapter 1 of [7]. $\square$

We may thus define a complex splitting of the Hodge filtration:

$$s\colon \mathcal{H}^1_{dR}(\mathcal{E}/Y(1)) \to \omega$$

which is described locally on sections as:

$$f(\tau)dz + g(\tau)d\overline{z} \mapsto f(\tau)dz.$$

This splitting, along with some geometric machinery, can be used to define the Shimura-Maass differential operator.

In our first step towards such a definition, we use the fact that a class in $H^1_{dR}(E_\tau)$ is determined by its periods to give an ad-hoc definition of the Gauss-Manin connection. Consider a de Rham cohomology class $\eta \in \mathcal{H}^1_{dR}(\mathcal{E}/Y(1))$ over some open subset $U \subset Y(1)$. For each $\tau \in U$ let η_τ denote the stalk of η at τ, so that $\eta_\tau \in H^1_{dR}(E_\tau)$. Then with notation as in the proof of the previous lemma, we consider the period maps:

$$p_i(\tau) = \int_{\gamma_i} \eta_\tau,$$

for $i = 1, 2$. One can show that these are smooth functions on U. There is thus a de Rham cohomology class $\nabla_\tau(\eta) \in \mathcal{H}^1_{dR}(\mathcal{E}/Y(1))$ defined over U, such that the stalk of $\nabla_\tau(\eta)$ at $\tau \in U$ is the differential with periods given by the vector:

$$\left(\frac{d}{d\tau}p_1(\tau), \frac{d}{d\tau}p_2(\tau)\right).$$

The Gauss-Manin connection is a sheaf map:

$$\nabla\colon \mathcal{H}^1_{dR}(\mathcal{E}/Y(1)) \to \mathcal{H}^1_{dR}(\mathcal{E}/Y(1)) \otimes \Omega^1_{Y(1)}$$

described on sections by the formula:

$$\nabla(\eta) = \nabla_\tau(\eta) \otimes d\tau.$$

We saw above that $2\pi i dz$ corresponds to the period vector $(2\pi i, 2\pi i\tau)$ and $2\pi i d\overline{z}$ corresponds to $(2\pi i, 2\pi i\overline{\tau})$. Note that these are indeed smooth functions of τ on $\mathcal{H}$. It follows that $\nabla_\tau(2\pi i dz)$ corresponds to the period vector $(d/d\tau)(2\pi i, 2\pi i\tau) = (0, 2\pi i)$. We have:

$$(0, 2\pi i) = 2\pi i(\tau - \overline{\tau})^{-1}\left((1,\tau) - (1,\overline{\tau})\right),$$

so that:

$$\nabla_\tau(2\pi i dz) = \frac{2\pi i dz - 2\pi i d\overline{z}}{\tau - \overline{\tau}}.$$

Similarly, $\nabla_\tau(2\pi i d\overline{z})$ is the cohomology class corresponding to the period vector $(d/d\tau)(2\pi i, 2\pi i\overline{\tau}) = (0,0)$, so that $\nabla_\tau(2\pi i d\overline{z}) = 0$.

Next we recall the analytic definition of the Serre-Poincare pairing:

$$\langle , \rangle \colon \mathcal{H}^1_{dR}(\mathcal{E}/Y(1)) \times \mathcal{H}^1_{dR}(\mathcal{E}/Y(1)) \to \mathcal{C}^\infty_{Y(1)},$$

where $\mathcal{C}^\infty_{Y(1)}$ denotes the sheaf of smooth functions on $Y(1)$. If η, η' are sections of $\mathcal{H}^1_{dR}(\mathcal{E}/Y(1))$ on U, then for each $\tau \in U$, the stalk of $\langle \eta, \eta' \rangle$ is defined by the formula:

$$\langle \eta, \eta' \rangle_\tau = \frac{1}{2\pi i}\int_{E_\tau} \eta \wedge \eta'.$$

In this formula we take representatives for η and η' and restrict them to the fibre $\pi^{-1}(\tau) \simeq E_\tau$. This is well-defined, and gives a smooth function on U. The Serre-Poincare pairing is perfect, alternating and bilinear. We will use it to define the Kodaira-Spencer map:

$$\kappa \colon \omega^2 \to \Omega^1_{Y(1)}$$

which is described on sections as:

$$\kappa(\eta \otimes \eta') = \langle \nabla_\tau(\eta), \eta' \rangle d\tau.$$

We admit the following theorem without proof:

Theorem 6.1.2. *The Kodaira-Spencer map is well-defined. In our particular situation, i.e. with respect to the relative curve $\mathcal{E}/Y(1)$, it is an isomorphism.*

Proof. The Kodaira-Spencer map is discussed in many texts on complex algebraic geometry; for example, consult [46]. □

The result of the following computation will be required below:

$$\begin{aligned}\kappa((2\pi i dz)^2) &= \langle \nabla_\tau(2\pi i dz), 2\pi i dz\rangle d\tau \\ &= (2\pi i)^2 \left\langle \frac{dz - d\overline{z}}{\tau - \overline{\tau}}, dz \right\rangle d\tau \\ &= (2\pi i)^2 \frac{\langle dz, d\overline{z}\rangle}{\tau - \overline{\tau}} d\tau \qquad \text{since } \langle , \rangle \text{ is alternating,} \\ &= (2\pi i)\left(\int_{E_\tau} dz \wedge d\overline{z}\right) \frac{d\tau}{\tau - \overline{\tau}}.\end{aligned}$$

Now note that $dz \wedge d\overline{z} = -2i dx \wedge dy$, so that:

$$\int_{E_\tau} dz \wedge d\overline{z} = -2i\,\mathrm{Area}(E_\tau) = -(\tau - \overline{\tau}).$$

We thus have $\kappa((2\pi i dz)^2) = -2\pi i d\tau$ and hence $\kappa^{-1}(d\tau) = -(2\pi i)^{-1}(2\pi i dz)^2$. We point out that if the reader accepts that κ is well-defined, then this computation verifies that κ is an isomorphism.

All of the ingredients are now in place to define:

$$\delta_k \colon M_k(\Gamma) \to M_{k+2}(\Gamma).$$

Given a holomorphic modular form $f(\tau)$ of weight k for Γ, consider $f(\tau)(2\pi i dz)^k$ as a differential in ω^k. The differentials in ω^k inject via the Hodge filtration into $\mathrm{Sym}^k(\mathcal{H}^1_{dR}(\mathcal{E}/\mathcal{H}))$. We can thus apply the Gauss-Manin connection to obtain:

$$\begin{aligned}\nabla(f(\tau)(2\pi i dz)^k) &= d(f(\tau))(2\pi i dz)^k + f(\tau)\nabla((2\pi i dz)^k) \\ &= f'(\tau)(2\pi i dz)^k d\tau + k f(\tau)(2\pi i dz)^{k-1}\nabla(2\pi i dz) \\ &= f'(\tau)(2\pi i dz)^k d\tau + 2\pi i k f(\tau)(2\pi i dz)^{k-1}\left(\frac{dz - d\overline{z}}{\tau - \overline{\tau}}\right) d\tau \\ &= f'(\tau)(2\pi i dz)^k d\tau + \frac{k f(\tau)(2\pi i dz)^k d\tau}{\tau - \overline{\tau}} - \frac{k f(\tau)(2\pi i dz)^{k-1}(2\pi i d\overline{z})d\tau}{\tau - \overline{\tau}}.\end{aligned}$$

This lives in $\mathrm{Sym}^k(\mathcal{H}^1_{dR}(\mathcal{E}/\mathcal{H})) \otimes \Omega^1_{\mathcal{H}}$. Apply the splitting s of the Hodge filtration, which takes $d\overline{z}$ to zero, to this expression and obtain:

$$(s \circ \nabla)(f(\tau)(2\pi i dz)^k) = f'(\tau)(2\pi i dz)^k d\tau + \frac{k f(\tau)(2\pi i dz)^k d\tau}{\tau - \overline{\tau}}.$$

Now apply κ^{-1} to obtain:

$$\frac{-1}{2\pi i}\left(\frac{df}{d\tau}(\tau)+\frac{kf(\tau)}{\tau-\overline{\tau}}\right)(2\pi i dz)^{k+2}.$$

The coefficient in front of $(2\pi i dz)^{k+2}$ above is (the negative of) $\delta_k(f)$ as defined previously. It is modular of weight k for Γ since global sections of ω^{k+2} are *invariant* under the action of Γ when regarded as functions on $\mathcal{H}$.

6.1.2 Katz's two-variable p-adic L-function

The following appears as the first paragraph in V.V. Sokurov's *MathSciNet* review of Katz's paper [27]:

> Recently a group of mathematicians (Kubota, Leopoldt, Serre, Manin, Mazur, Swinnerton-Dyer, Katz and many others) have discovered a new land of mathematics a little way off a beaten path put to rights by A. Weil [47]. The paper under review may be of great help to those who wish to visit that new land.

In this quotation, Sokurov refers to the theory of p-adic L-functions, and the concomitant phenomenon of p-adic interpolation. In [27], Katz constructs a two-variable p-adic L-function for a quadratic imaginary extension $K/\mathbf{Q}$ in which p splits. Later, in [28], Katz extended this contstruction to CM fields K/F with F an arbitrary totally real field, but with a restriction on the CM type of K, which agrees with p being split in K when $F=\mathbf{Q}$. In this more general case, Katz interpolates certain values of Hilbert modular Eisenstein series (it is worth remarking here that Shimura's paper [39] also discusses the algebraicity properties of Hilbert modular forms).

On page 205 of [28], Katz remarks the following:

> In an earlier paper [27], we treated the case of a quadratic imaginary extension of $\mathbf{Q}$. In that case, the abelian varieties involved are simply elliptic curves, and we were able to prove the theorems by dipping into the wealth of classical material available for elliptic curves ... We were for a long time blinded by these riches to the simple cohomological mechanism which in some sense underlies them.

The last sentence above refers to the following technique pioneered in [28] (we confine ourselves to the case $F = \mathbf{Q}$ for simplicity): Katz uses the geometric interpretation of the Shimura-Maass operator discussed in the previous subsection, which involved a choice of splitting of Hodge filtration of the relative cohomology of a universal elliptic curve, as well as a p-differential operator, Serre's θ-operator, whose geometric definition uses the unit-root splitting of the Hodge filtration. When one looks at the corresponding fibers of the cohomology over a CM curve, these two splittings must induce the natural splitting of the fiber (which is simply the first algebraic de Rham cohomology of the CM elliptic curve under consideration), into its eigenspaces for the CM action on the cohomology of the fiber. Since the CM action is defined over $\overline{\mathbf{Q}}$, Katz is able to deduce the algebraicity result of Shimura. Then, since all three of his splittings of the cohomology agree, he is also able to deduce p-adic properties of values of modular forms at the CM point by considering the unit-root splitting.

One might ask if there is a similar geometric description of the results contained in this thesis. It could help to ellucidate the results above, and perhaps also help to solve the open problems discussed in the sections which follow. We discuss how such a rigid analtyic analogue might be formulated in the following subsection.

6.1.3 A rigid analogue

We would like a geometric description of the space of rigid analytic modular forms. Let P_n denote the n-th symmetric power of the standard *right* representation for $\mathrm{GL}_2(\mathbf{Q}_p)$. We work with the following concrete model for P_n: let P_n denote the $\mathbf{Q}_p$-vector space of homogeneous polynomials of degree n in the variables u and v. This space has a right action of $\mathrm{GL}_2(\mathbf{Q}_p)$: if $p(u, v) \in P_n$ and

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q}_p)$$

then $(P \cdot \gamma)(u, v) = P(au + bv, cu + dv)$.

Write $\mathcal{O} = \mathcal{O}_{\mathcal{H}_p}$ for the sheaf of rigid analytic functions on the p-adic upper half plane. Write $\mathcal{O}(n)$ for the rigid analytic sheaf $\mathcal{O}$ endowed with the right weight n action of $\mathrm{GL}_2(\mathbf{Q}_p)$ defined via the same formula as above. Let $\mathcal{P}_n = \mathcal{O}(n) \otimes_{\mathbf{Q}_p} P_n$ denote the coherent sheaf of $\mathcal{O}_X$-modules on $\mathcal{H}_p$ endowed with the diagonal action of $\mathrm{GL}_2(\mathbf{Q}_p)$. This sheaf comes equipped with a natural integrable connection via the differential $d\colon \mathcal{O}_X \to \Omega^1_X$. We denote this connection by ∇. Note that $\mathcal{P}_n = \mathrm{Symm}^n \mathcal{P}_1$.

Finally we must introduce a subsheaf of $\mathcal{V}_n$ corresponding to modular forms of weight k. Recall that in Chapter 1 this sheaf was the canonical sheaf on a modular curve. If one attempts to proceed similarly using the Cerdnik-Drinfeld theorem, then one will obtain a coherente sheaf on X that is too large. This is because in this case, the canonical sheaf arises from a universal surface over a curve, and so it does not give rise to a line bundle on X, but rather a sheaf which is locally free of rank 2. In any case, it is not hard to define the correct line bundle on X whose global sections recover $S_k(\Gamma)$.

Let ω denote the subsheaf of $\mathcal{V}_1$ defined by the condition that the pull-back of its sections to $\mathcal{H}_p$ are of the form $f(z)(u - zv)$, where $f(z)$ is a rigid analytic function on the corresponding admissible subset.

Lemma 6.1.3. *The sheaf ω is locally free of rank 1 and degree $g-1$, where g is the genus of X. Moreover $\omega^2 \cong \Omega^1_{X/\mathbf{Q}_p}$ and $H^0(X, \omega^k) = S_k(\Gamma)$.*

The short exact sequence

$$0 \to \omega \to \mathcal{V}_1 \to \omega^{-1} \to 0$$

yields a descending filtration on $\mathcal{V}_n$ with $\mathrm{Fil}^i\mathcal{V}_n = 0$ for $i > n$ and $\mathrm{Fil}^i\mathcal{V}_n = \mathcal{V}_n$ for $i \leq 0$, and such that $\mathrm{Fil}^i\mathcal{V}_n/\mathrm{Fil}^{i+1}\mathcal{V}_n \cong \omega^{2i-n}$ for $0 \leq i \leq n$. In particular we have $\mathrm{Fil}^n\mathcal{V}_n = \omega^n$. It is hoped that by using this aparatus, and some kind of non-rigid analytic splitting of the filtration on the sheaves cV_n, one might be able to give a more conceptual definition for our Shimura-Maass operator. This might lead to a better understanding of the topics discussed in the following sections. It might also allow one to imitate a Katz style construction of an l-adic L-function for some prime $l \neq p$, by interpolating p-adically defined CM values of a rigid modular form and its Shimura-Maass derivatives.

6.2 Algebraicity of rigid analytic modular forms

Definition 5.1.1 gives a condition for a modular form f to be algebraic. We described how one can associate a measure μ_f on $\mathbf{P}_1(\mathbf{Q}_p)$ to a modular form in Section 4.2. If the values of this measure are taken to be analogues for the Fourier expansion of a classical modular form, then one might guess that the space of measures μ_f whose values are all defined over $\overline{\mathbf{Q}}$ is an important $\overline{\mathbf{Q}}$-subspace of $S_k(\Gamma)$. Numerical computations strongly suggest that this space does not coincide with the space of algebraic rigid analytic modular forms defined via the Cerednik-Drinfeld isomorphism. Thus, it seems that the most naive substitute for the q-expansion principle, expressed in terms of the boundary measure associated to a rigid analytic modular form, fails to hold. This prompts the following question:

Question 6.2.1. *Can one formulate a natural condition on the values of a boundary measure μ_f associated to a rigid analytic modular form f, for some quaternionic group defined as in Section 2.1.4, which characterizes the algebraicity condition of Definition 5.1.1?*

We can make this question more precise: for simplicity restrict to weight 2. Suppose that $f \in S_2(\Gamma)$ is an eigenform for the Hecke algebra. Then since the Cerednik-Drinfeld isomorphism is Hecke-equivariant, some multiple of the pullback of $f(z)dz$ under Cerednik-Drinfeld will be defined over $\overline{\mathbf{Q}}$. Suppose that the $\mathbf{C}_p$-valued measure μ_f associated to f takes integer values. Then by the previous remark, some multiple $\alpha_f \mu_f$ corresponds to an algebraic modular form, where $\alpha_f \in \mathbf{C}_p^\times$. One can ask: what is the Cerednik-Drinfeld period α_f associated to f?

Numerical computations suggest that the period α_f honestly depends on the eigenform $f \in S_2(\Gamma)$. For example, if one computes values $f(\tau)/g(\tau)$ where f and $g \in S_2(\Gamma)$ are distinct eigenforms, and $\tau \in \mathcal{H}_p$ is a CM-point, then by our main thereom this value should be algebraic once f and g are normalized to be algebraic. In the Appendix we describe computations of this sort evaluated to high p-adic precision. LLL-lattice reduction techniques fail to recognize the values $f(\tau)/g(\tau)$ as algebraic numbers. This suggests that the relevant periods α_f and α_g which rescale f and g to be algebraic forms are not equal, for otherwise they would cancel one another in the computation of the ratio.

One might hope that a well-known p-adic invariant attached to f describes the Cerednik-Drinfeld period. We have tested the $\mathcal{L}$-invariant attached to f, as well as the value of E_2 at the semistable elliptic curve associated to f via Cerednik-Drinfeld and Jacquet-Langlands. Neither of these appear to be equal to the mysterious Cerednik-Drinfeld period.

6.3 A rigid analytic Chowla-Selberg formula

Let l be an odd prime number and let $K = \mathbf{Q}(\sqrt{-l})$. Let h denote the class number of K and let ε denote the quadratic character

$$\varepsilon(c) = \left(\frac{-l}{c}\right).$$

Let $\tau_1, \ldots, \tau_h \in \mathcal{H}$ denote distinct $\mathrm{SL}_2(\mathbf{Z})$-representatives for the CM-points associated to the maximal order $\mathcal{O}_K$ of K. Let Δ denote Ramanujan's discriminant function, which is a cusp form of weight 12 for $\mathrm{SL}_2(\mathbf{Z})$ defined over $\mathbf{Q}$:

$$\Delta(q) = q \prod_{q \geq 1} (1 - q^n)^{24}.$$

Then Chowla and Selberg proved the following remarkable result:

Theorem 6.3.1 (Chowla-Selberg)**.** *With notation as above, and with*

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt,$$

one has:

$$\prod_{i=1}^{h} |\Delta(\tau_i)| \, Im(\tau_i)^6 = \left(\frac{1}{2\pi\sqrt{l}}\right)^{6h} \prod_{0<c<l} \Gamma\left(\frac{c}{l}\right)^{6\varepsilon(c)}.$$

Proof. See [36]. □

If one combines this result with Shimura's Theorem 1.1.4, one deduces an explicit formula for a possible choice of the CM-period Ω_K:

Corollary 6.3.2. *The statement of Shimura's theorem holds with the particular choice of CM-period:*

$$\Omega_K = \left(\frac{1}{2\pi}\right)^{1/2} \left(\prod_{0<c<l} \Gamma\left(\frac{c}{l}\right)^{\varepsilon(c)/2h} \right).$$

Proof. Take a $(12h)$th root of the Chowla-Selberg formula, and note that one may ignore the terms $\mathrm{Im}(\tau_i)$ and $\sqrt{l}$ since they are algebraic. □

Remark 6.3.3. The formula above is a little ambiguous because we have not described how to take the roots which appear within it. However, any choice of roots is admissible, as Ω_K is only well-defined up to multiplication by nonzero algebraic numbers.

It would be interesting to find a similar formula for the rigid analytic Chowla-Selberg period of Theorem 5.3.1. In [22], Gross proves the Chowla-Selberg formula using tools from algebraic geometry. More precisely, he considers the moduli space Y of abelian varieties of dimension n which are equipped with a fixed polarization and complex multiplication by a fixed quadratic imaginary field $K = \mathbf{Q}(\sqrt{-d})$. Gross considers the nth relative de Rham cohomology $\mathcal{H}^n_{\mathrm{dR}}(\mathcal{A}/Y)$ of the universal abelian surface $\mathcal{A}/Y$, and proves that it has a nonvanishing global section ω which is horizontal for the Gauss-Manin connection and which satisfies the following: along a certain abelian variety corresponding to a factor of the Jacobian of the Fermat curve $x^d + y^d = 1$, one can explicitely compute the specialization of ω in terms of Γ-values and a power of $2\pi i$. Along a fiber corresponding to a product of CM-elliptic curves E, Gross relates the specialization of ω to the periods of the canonical differential of E, as well as a power of $2\pi i$. Since the section ω is horizontal for Gauss-Manin, its periods must be constant, and one thereby deduces a relation between the periods of E and a product of Γ-values as in the Chowla-Selberg period (up to an algebraic factor).

The author hopes that a geometric interpretation of this thesis, as discussed in Section 6.1, might be combined with ideas of Gross [22] and Ogus [32] to yield a description of the Chowla-Selberg period of Theorem 5.3.1. The author has not been able to test any conjectures, as we are stymied by the problem of computing an algebraic rigid analytic modular form, as discussed in Section 6.2.

Appendix A: Computations

A.1 Method of computation

The author and Marc Masdeu have implemented methods for computing with rigid analytic modular forms of arbitrary weight arising from rational quaternion algebras in *Sage*. We briefly explain how these algorithms work; for more details consult the forthcoming paper [15]. Note also that the particular example discussed below appears in Chapter 6 of [11], although Shimura-Maass derivatives are not computed there.

Let $B/\mathbf{Q}$ be a definite quaternion algebra which is unramified at a finite prime p, and suppose that $\Gamma \subseteq B^\times$ is a congruence subgroup. We will regard Γ as a cocompact subgroup in $\mathrm{GL}_2(\mathbf{Q}_p)$ via a splitting $\iota\colon B_p \cong \mathrm{GL}_2(\mathbf{Q}_p)$ of B at p. The explicit descriptions of forms and CM-points in what follows are all expressed relative to this fixed splitting ι.

Recall that the edges of the Bruhat-Tits tree $\mathcal{T}$ for $\mathrm{GL}_2(\mathbf{Q}_p)$ correspond one-to-one with the compact open balls of $\mathbf{P}_1(\mathbf{Q}_p)$; the ball corresponding to an edge consists all the ends which pass through the oriented edge. Given a form f on Γ, we store the moments

$$\int_{a+p^n\mathbf{Z}_p} (x-a)^k d\mu_f(x)$$

for a collection of balls $a + p^n\mathbf{Z}_p$ which correspond to a "fundamental domain" for $\Gamma\backslash\mathcal{T}$. Locally analytic functions can then be integrated by taking linear combinations of these moments; for details see [12]. In particular, values of f can be computed from Teitelbaum's Poisson-Kernel:

$$f(\tau) = \int_{\mathbf{P}_1(\mathbf{Q}_p)} \frac{d\mu_f(x)}{(x-\tau)}.$$

Note that we store these moments as p-adic number expressed relative to some finite precision, and that we only store the first k-moments. If one knows the moments to least k (p-adic) digits, then this equates to knowledge of $f(\tau)$ up to approximately k digits of p-adic accuracy. Note that the moments can be precomputed once and for all, and then many locally analytic functions can be integrated against the corresponding measure $d\mu_f$.

Note that, if f is of weight k, then formally differentiating under the integral sign actually gives a valid formula for the value of the Shimura-Maass derivative. Hence,

$$\frac{df}{d\tau}(\tau) = \int_{\mathbf{P}_1(\mathbf{Q}_p)} \frac{d\mu_f(x)}{(x-\tau)^2},$$

and so

$$\delta_k f(\tau) = \int_{\mathbf{P}_1(\mathbf{Q}_p)} \frac{d\mu_f(x)}{(x-\tau)^2} + \frac{k}{\tau - \overline{\tau}} \int_{\mathbf{P}_1(\mathbf{Q}_p)} \frac{d\mu_f(x)}{(x-\tau)}$$

for all $\tau \in \mathbf{C}_p$ lying in an unramified extension of $\mathbf{Q}_p$. Similarly for the iterated Shimura-Maass derivatives.

A.2 Example

The rational Hamilton quaternions are the definite quaternion algebra $B/\mathbf{Q}$ such that the generators i, j and k satisfy

$$i^2 = j^2 = k^2 = -1.$$

The algebra B has discriminant 2, and so is split at all odd primes. We will consider the case $p = 7$ below.

We will work with the maximal $\mathbf{Z}$-order $R_0 \subseteq B$ which is generated over $\mathbf{Z}$ by the elements 1, i, j and

$$\omega = \frac{1+i+j+k}{2}.$$

Alternatively R_0 can be described as

$$R_0 = \left\{ a + bi + cj + dk \;\middle|\; a,b,c,d \in \mathbf{Z} \text{ or } a,b,c,d \in \frac{1}{2} + \mathbf{Z} \right\}.$$

Let $K_7 = \mathbf{Q}_7[g]/(g^2 + 6g + 3)$ denote an explicit model for the quadratic unramified extension of $\mathbf{Q}_7$. We chose this particular model since it is the one used by *Sage* in our computation. We represent elements in K_7 as series $\sum_{i\geq 0}(a_i g + b_i)7^i$ for a_i and b_i integers in the interval $[0,6]$.

Let $\rho = \lim_n 2^{7^n}$ denote the Teichmuller lift of 2 in $\mathbf{Q}_7$, which is an explicit cube root of unity. Define a splitting $\iota\colon B_7 \to M_2(\mathbf{Q}_7)$ by mapping:

$$\iota(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \iota(j) = \begin{pmatrix} \rho & \rho+1 \\ \rho+1 & -\rho \end{pmatrix}, \quad \iota(k) = \begin{pmatrix} \rho+1 & -\rho \\ -\rho & -\rho-1 \end{pmatrix}.$$

Note that ι identifies $R_0 \otimes_{\mathbf{Z}} \mathbf{Z}_7$ with $M_2(\mathbf{Z}_7)$.

Let $R = R_0[1/7]$, let R_1 denote the set of elements of R of reduced norm equal to 1, and let $\Gamma = \iota(R_1)$. Then one can check that $\Gamma\backslash\mathcal{T}$, where $\mathcal{T}$ is the Bruhat-Tits tree for $\mathrm{GL}_2(\mathbf{Q}_7)$, consists of two vertices joined by two edges. The space $S_2(\Gamma)$ of weight 2 rigid analytic modular forms for Γ is 1-dimensional. A cocycle c_f corresponding to a nonzero form $f \in S_2(\Gamma)$ is given by the function taking value 1 on one of the edges of $\Gamma\backslash\mathcal{T}$, and -1 on the other edge. This is an eigenform, and a multiple of it is necessarily algebraic in the sense of Definition 5.1.1. The eigenline spanned by this form corresponds, via Jacquet-Langlands and Cerednik-Drinfeld, with the line spanned by the unique normalized cusp form of weight 2 on $\Gamma_0(14)$.

Suppose that $K/\mathbf{Q}$ is a quadratic imaginary extension of class number ≥ 2 such that both 2 and 7 are inert in K. Then K embeds into B and there are CM-points for K; in fact, due to our assumption on the class number, there are at least two Γ-equivalence classes of CM-points. Let τ_1 and τ_2 denote representatives in $\mathcal{H}_p$ for two such distinct classes. We do not know if the form f we have selected is algebraic, but some multiple of it must be algebraic. Thus, the Main Theorem 5.1.5 implies that if $f(\tau_2) \neq 0$ then the value

$$\frac{f(\tau_1)}{f(\tau_2)}$$

is algebraic, regardless of our scaling of f within its eigenline. Both the Chowla-Selberg and Cerednik-Drinfeld periods cancel in this ratio. The same is true for the Shimura-Maass derivatives.

If $K = \mathbf{Q}(\sqrt{-51})$ then both 2 and 7 are inert in K and the class number of $K/\mathbf{Q}$

is 2. Consider the two quaternions

$$q_1 = \frac{1-i+j-7k}{2}, \quad q_2 = \frac{1-i-j-7k}{2}$$

in R_0. These both have reduced trace equal to 1 and reduced norm equal to 13. Since $\mathcal{O}_K$ is generated over **Z** by 1 and

$$\tau = \frac{1+\sqrt{-51}}{2},$$

which has reduced trace equal to 1 and reduced norm equal to 13, we obtain two embeddings $K \hookrightarrow B$ by mapping τ to q_1 and q_2, respectively. Embedding these quaternions into $M_2(\mathbf{Q}_7)$ via ι and finding the fixed points of the corresponding matrix acting on $\mathcal{H}_p$ produces the following two CM-points as elements of K_p, which are not equivalent under Γ:

$$\tau_1 = (6g+1) + (4g)7 + (6g+6)7^2 + (4)7^3 + (g+4)7^4 + (4)7^5 + (6g+2)7^6 + \cdots$$

and

$$\tau_2 = (2g)+(5g+1)7+(4g+2)7^2+(3g+6)7^3+(6g+3)7^4+(6g+4)7^5+(g+1)7^6+\cdots.$$

We computed these values to over 100 digits of 7-adic accuracy. Then, using the overconvergent methods of Greenberg [20], we computed the moments of the measure μ_f associated to f. Then we used the ideas of Darmon-Pollack [12] to compute the value

$$\frac{f(\tau_1)}{f(\tau_2)} = (4g+2)+(4g+5)7+(4g+2)7^2+(5g+1)7^3+(5g+4)7^4+(2)7^5+(g+1)7^6+\cdots$$

to 100 digits, using the moments of μ_f. We found that this value agrees with a root of the polynomial

$$p(X) = X^4 - X^3 + \frac{113}{36}X^2 - \frac{1}{2}X + \frac{169}{81}$$

to 100 digits of accuracy. An analogous computation also allowed us to compute the corresponding values of the first two Shimura-Maass derivatives of f. We found the following values:

r	Minimal polynomial of $(\delta_2^r f)(\tau_1)/(\delta_2^r f)(\tau_2)$
0	$X^4 - X^3 + \frac{113}{36}X^2 - \frac{1}{2}X + \frac{169}{81}$
1	$X^4 - \frac{95}{18}X^3 + \frac{5627}{432}X^2 - \frac{37465}{2916}X + \frac{28561}{6561}$
2	$X^4 + \frac{83}{12}X^3 + \frac{633593}{46656}X^2 - \frac{29401}{8748}X + \frac{4826809}{531441}$

Since we only performed the computation with 100 digits of accuracy, we were not able to recognize the ratio of the third derivatives or higher as algebraic numbers.

Notice that the polynomials above are all irreducible quartics over $\mathbf{Q}$. This is due to the fact that $N^+ = 1$ in our example, so that we were working with the norm 1 elements in a maximal $\mathbf{Z}[1/p]$-order of B. Moreover the embeddings of $K \to B$ corresponding to our CM points were defined via optimal embeddings of the ring of integers of K into the corresponding maximal order of B. Hence, by the theory of complex multiplication, the values computed above are defined over the Hilbert class field H of $K = \mathbf{Q}(\sqrt{-51})$, which is obtained by adjoining $\sqrt{-3}$ to K. Indeed, the values computed above do lie H. Since H is a degree 4 extension of $\mathbf{Q}$, this explains why the degrees of the polynomials above are 4. Note also that we chose K to have class number 2 so that there would be two inequivalent CM points. If we worked with K of class number 1 and optimal embeddings of its ring of integers, then in computing the ratios above we would have only been picking up an automorphy factor. Our computation with class number ≥ 2 is thus more interesting.

We finish by remarking that it would be interesting to study how the height of the value

$$\frac{(\delta_k^r f)(\tau)}{\Omega_K^{k+2r}} \in \overline{\mathbf{Q}}$$

grows as r increases. This would be interesting from both theoretical and computational vantage points. For example, if one is interested in recognizing such p-adic values as algebraic numbers in a completely algorithmic fashion, then one requires a priori estimates on the height of the value. In this way one knows before beginning the computation of the lined quanitity above how many p-adic digits of precision are necessary to recognize the output as an algebraic number.

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