VECTOR VALUED MODULAR FORMS AND DECOMPOSITIONS OF VECTOR BUNDLES

CAMERON FRANC

ABSTRACT.

Grothendieck [7] (building on Birkhoff [2]) showed that all holomorphic vector bundles on the complex projective line decompose uniquely into direct sums of line bundles. This result can be thought of as saying that there exists an analogue of a vector space basis for holomorphic vector bundles on the sphere. If you thus find yourself with such a bundle in your hands, it is natural to ask how it decomposes into line bundles.

This innocuous sounding question arises frequently in geometry, and it is a basic problem in the study of vector valued modular forms. For example, a positive solution to this problem would yield a dimension formula for spaces of congruence modular forms of weight one, which is a difficult yet basic problem that has been open for more than a century.

The history of vector valued modular forms dates back to Poincaré's work on Fuchsian functions and linear differential equations [9], [10]. In recent years, vector valued modular forms have played a main role in the mathematics spawned by the proof of the monstrous moonshine conjecture [3]. Indeed, these modular forms arise as generating series for characters of rational vertex operator algebras, and thus form an important part of their representation theory — see [6] for a survey.

Vector valued modular forms are multivalued sections of vector bundles on curves. When a bundle is pulled back to the complex upper half plane $\mathcal{H}$ via a uniformisation map, its sections can be represented as single valued functions satisfying a transformation law. A basic example is the case of a flat holomorphic bundle associated to a representation $\rho\colon \Gamma \to \mathrm{GL}_r(\mathbf{C})$ of some Fuchsian subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbf{R})$. Sections of this bundle are holomorphic functions $F\colon \mathcal{H} \to \mathbf{C}^r$ satisfying the transformation law

$$F\left(\frac{a\tau+b}{c\tau+d}\right) = \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} F(\tau) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{1}$$

More generally one often incorporates an additional automorphy factor (for example, the well-known factor $(c\tau+d)^k$) into (1), which at the level of vector bundles amounts to a twist by a line bundle. If the Fuchsian group Γ has cusps, then one typically also imposes some kind of mildness condition at the cusps (for example, meromorphy or holomorphy are common conditions) — see [4] for details.

In practice one is often interested in modular forms for a specific Fuchsian group Γ such as the modular group $\mathrm{SL}_2(\mathbf{Z})$. One might imagine that this group is too simple to be of much interest – after all, its corresponding coarse moduli space $\mathrm{SL}_2(\mathbf{Z})\backslash\mathcal{H}$ is just the field of complex numbers! But in fact, Belyi showed in [1] that every compact algebraic curve defined by equations with algebraic numbers as coefficients can be uniformized by a subgroup of $\mathrm{SL}_2(\mathbf{Z})$ of finite index. Optimistically one interprets this result as saying that the basic example of the modular group already contains a rich wealth of information of interest to both number theorists and geometers alike. Pessimistically, Belyi tells us that even if one is willing to focus on a single Fuchsian group such as the modular group, the geometry and group theory can be quite complicated.

Nevertheless, one can make nontrivial general statements about vector valued modular forms for the modular group. For example, if the forms are associated with a *unitary* representation of the modular group, then one has access to a vector valued form of the Rankin-Selberg method. In [11], Selberg pushed forward scalar valued forms on noncongruence subgroups to vector valued modular forms on the full modular group and used the Rankin-Selberg method to bound their Fourier coefficients. This was one of the first successful applications of vector valued forms to the study of scalar valued modular forms of more classical interest.

In joint work with Geoff Mason [5], we studied the decompositions of vector bundles giving rise to vector valued modular forms for the modular group. In order to state our results we must recall that the compactification of the moduli space of elliptic curves $\mathrm{SL}_2(\mathbf{Z})\backslash\mathcal{H}$ is isomorphic with the projective space $\mathbf{P}(4,6)$ with nonstandard weighting arising from the action $\lambda(x,y) = (\lambda^4 x, \lambda^6 y)$ of $\mathbf{C}^\times$ on $\mathbf{C}^2$. The Grothendieck-Birkhoff splitting principle holds for holomorphic vector bundles on $\mathbf{P}(4,6)$ [8], and line bundles on $\mathbf{P}(4,6)$ are isomorphic with the analogues of the usual bundles $\mathcal{O}(k)$ on projective space. Hence, if $\rho\colon \mathrm{SL}_2(\mathbf{Z}) \to \mathrm{GL}_r(\mathbf{C})$ denotes a rank r representation of the modular group, and if $\mathcal{V}(\rho)$ denotes the corresponding flat bundle on $\mathbf{P}(4,6)$ (more precisely, $\mathcal{V}(\rho)$ is an extension to the cusp of the flat bundle on $\mathrm{SL}_2(\mathbf{Z})\backslash\mathcal{H}$), then just as for $\mathbf{P}^1$ there is a decomposition

$$\mathcal{V}(\rho) \cong \bigoplus_{k\in\mathbf{Z}} m_k \mathcal{O}(k) \tag{2}$$

for uniquely determined integers $m_k \geq 0$ giving a partition of the rank, $\sum_{k\in\mathbf{Z}} m_k = r$. These integers m_k can be difficult to compute: for example if ρ is a unitary representation, then m_{-1} is the dimension of the space of modular forms of weight one associated with ρ, a notoriously difficult quantity to compute in general. In the nonunitary case the siutation is even worse. Setting our sights lower than obtaining general formulae for these multiplicities, in [5] we proved the following:

Theorem 1. *Suppose that* $\rho\colon \mathrm{SL}_2(\mathbf{Z}) \to \mathrm{GL}_r(\mathbf{C})$ *is irreducible and let* m_k *denote the multiplicities in the decomposition* (2). *Then the following hold:*

(a) no-gap lemma*: if* $m_{k-2} \neq 0$ *and* $m_{k+2} \neq 0$ *then* $m_k \neq 0$*;*
(b) three-term inequality*: if* ρ *is further assumed to be* unitary*, then the inequality* $m_k \leq m_{k-2} + m_{k+2}$ *holds for all integers* k.

Note that in the irreducible case, all indices k for which $m_k \neq 0$ must have the same parity. The proof of Theorem 1 in [5] uses properties of a modular differential operator that acts on sections of the bundle $\mathcal{V}(\rho)$. Computations outlined in [5] suggest that part (b) of Theorem 1 should hold without the hypothesis that ρ is unitary, although the nonunitary case remains open.

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