Math and Music

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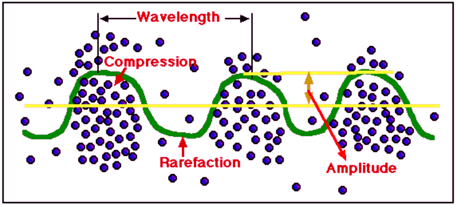
Overview

Sound



- *Sound* travels via waves of increased air pressure
- *Volume* (or *amplitude*) corresponds to the pressure level
- *Frequency* is the number f of waves that pass in a second
- *Wavelength* is the distance λ between waves
- The speed of sound in air s is about 340.29 meters/second.
- One has $s = f\lambda$.

Perception of sound

- We will describe sound by frequency and ignore amplitude
- Humans perceive sound in the range 20 Hz to $20,000$ Hz
- Higher frequency is described as being more shrill
- Sounds of frequency f and $2^n f$ sound similar to humans
- E.g. low and high E-strings on a guitar are 82.41 Hz and 329.63 Hz
- Such sounds are said to have similar *chroma*
- This breaks sound up into chromatic equivalency classes
- Notes of frequency f and $2f$ are said to be an *octave* apart

Music: continuous or discrete spectrum?

- Sound has a continuous spectrum of frequencies
- Most instruments play a discrete subset of them
- There are exceptions, like the violin
- In contrast, a guitar has frets which make it discrete
- Western music frequently selects 12 fundamental frequencies $f_1, \ldots, f_{12}$ and then allows only the discrete set of frequencies

$$\{2^n f_i \mid n \in \mathbf{Z},\ i = 1, \ldots, 12\}$$

- Thus, $f_1, \ldots, f_{12}$ are chromatic representatives for all the notes
- The fundamental frequencies are usually labeled

$$C, C\sharp, D, D\sharp, E, F, F\sharp, G, G\sharp, A, A\sharp, B$$

- This is the *chromatic scale*

Two questions:

We'll address the following two questions:

- How should we space the fundamental frequencies?
- Why pick 12 fundamental frequencies? Why not 163?

First question: how to space the fundamental frequencies?

“Obvious” idea: break $[f, 2f]$ up into 12 notes spaced equally

- (Actually, we space their logarithms evenly)
- So choose some f_1, say 261.625565 Hz, and call it middle C
- Then set $f_i = 2^{(i-1)/12} f_1$ for $i = 1, \ldots, 12$
- This is called *equal tempered tuning* and is ubiquitous in modern music

- Note: there are many other tuning systems. None are perfect.
- Some people believe that some of the dissonance of equal tempered tuning is driving us crazy!
- Equal temperament has advantages:
- it makes transposing music up or down the scale easy
- don't have to retune to change key

A remark on callibration

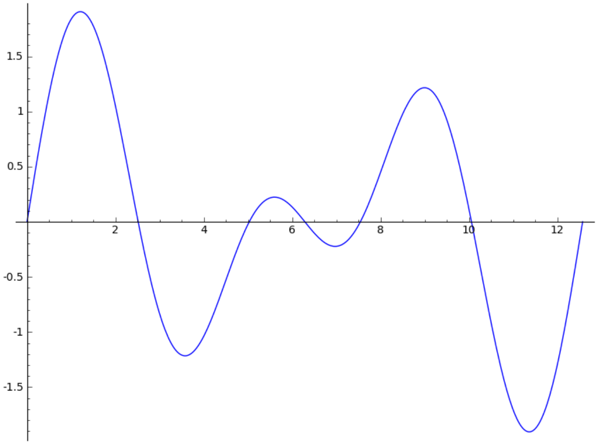
- We still need to choose a frequency f_1! This choice is basically arbitrary
- Our middle C as 261.625565 Hz derives from an international conference in London in 1939
- They unanimously adopted 440 Hz for the fundamental frequency of A
- The weird middle C is derived from this
- Apparently, Handel, Bach, Mozart, Beethoven and others had our A as about 422.5 Hz
- Their music thus sounds sharper today than when it was written!

Second question: why 12 fundamental frequencies?

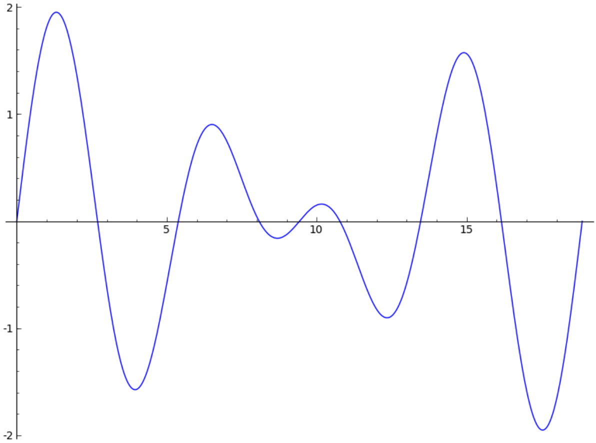
Simple rational multiples of frequencies sound consonant

- We already noted the octave f, $2f$
- Next simplest is the perfect fifth f, $(3/2)f$
- Consider a note f_1 in our tuning system and the note $f_2 = 2^{7/12} f_1$ which is seven semitones above f_1
- E.g., C and G and frequencies $261.625565\ldots$ and $391.99543\ldots$ Hz
- $2^{7/12} = 1.49830\ldots$ is nearly $3/2$, so that $f_2/f_1 \simeq 3/2$

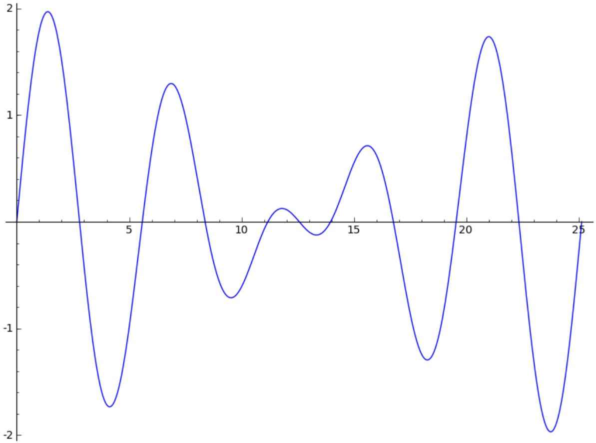
One period of a perfect fifth (3/2)



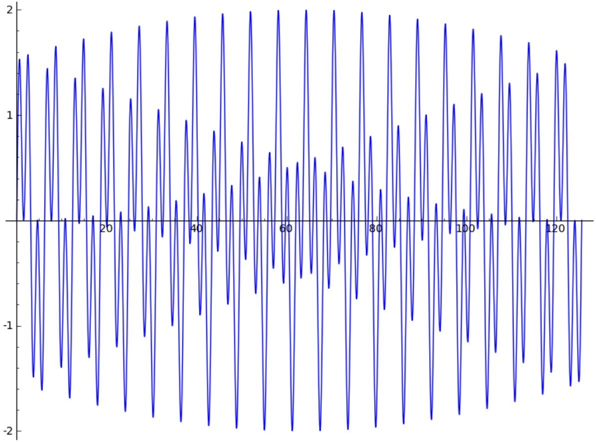
One period of a perfect fourth (4/3)



One period of a major third (5/4)



One period of a 61/20



What about 14 notes?

If we increase the number of notes, can we always get a better approximation to $3/2$?

- Not always! If we take 14 notes then we see the following:

a	$2^{a/14}$	a	$2^{a/14}$
0	1.00000000000000	7	1.41421356237310
1	1.05075663865322	8	1.48599428913695
2	1.10408951367381	9	1.56141836431142
3	1.16012938616016	10	1.64067071201528
4	1.21901365420448	11	1.72394564249396
5	1.28088668976427	12	1.81144732852781
6	1.34590019263236	13	1.90339030602124

- The value $2^{8/14}$ is not as close to $3/2$ as

$$2^{7/12} = 1.498307\ldots$$

- 12 generates such a nice perfect fifth because it appears in a convergent of the continued fraction expansion of $\log(3)/\log(2)$
- These convergents give "really good" approximations to $\log(3)/\log(2)$ relative to the size of the denominator and numerator of the rational approximation
- the definition of "really good" can be made precise

Continued fractions

$$\frac{\log(3)}{\log(2)} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\vdots}}}}}$$

- Convergents of $\log(3)/\log(2)$ are 1, 2, 3/2, 8/5, 19/12, 65/41, ...
- If we used 41 notes in an octave, then going up 24 notes would generate something even closer to a perfect fifth
- Many humans would have difficulty distinguishing the 41 notes

Moral:

12 **notes per octave yields a nice balance between distinguishability and good approximation to the perfect fifth**

(Of course, humans stumbled on 12 notes “accidentally” because they sound nice. Only after the fact did humans realize that the choice of 12 notes is supported by mathematics as a good choice)

Other ratios

- the perfect fourth and major third are also important frequency ratios
- they correspond to the simple ratios $4/3$ and $5/4$
- going up 5 notes in our tuning system nearly generates a perfect fourth since $2^{5/12} = 1.33483\ldots \simeq 4/3$
- going up 4 notes in our tuning system nearly generates a major third since $2^{4/12} = 1.25992\ldots \simeq 5/4$

Overview

Definition

Recall the following:

Definition

A group is a pair (G, m) where G is a set and m is a map $m\colon G \times G \to G$ satisfying the following axioms:

- (Identity) There exists $e \in G$ such that $m(e, g) = m(g, e) = g$ for all $g \in G$;
- (Inverses) For each $g \in G$, there exists $h \in G$ such that $m(g, h) = m(h, g) = e$;
- (Associativity) For all g, h and $k \in G$, $m(g, m(h, k)) = m(m(g, h), k)$.

Think of m as a multiplication on G and write $m(g, h) = gh$

Group actions

Definition

If X is a set and G is a group, then a group action of G on X is a map $\rho\colon G \times X \to X$ such that

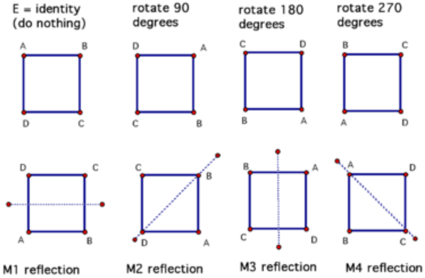
- (Identity) $\rho(e, x) = x$ for all $x \in X$;
- (Multiplication) $\rho(gh, x) = \rho(g, \rho(h, x))$ for all g, $h \in G$ and $x \in X$.

We usually write $\rho(g, x)$ as $g(x)$ or simply gx.

Example 1

The dihedral groups:

- D_n = group of symmetries of a regular planar n-gon
- e.g. D_4 is the symmetries of the square



- D_n consists of rotations and reflections only

Example 1 cont'd.

- Let X be a regular n-gon
- Then D_n acts on X as follows:
- if $P \in X$ is a point and $\sigma \in D_n$ then $\sigma(P)$ is the result of moving P according to σ
- So if σ is a reflection, then $\sigma(P)$ is the reflection of P
- If σ is a rotation, then $\sigma(P)$ is P rotated

Example 2

- Recall: S_n is the group of premutations of n things
- Let X_n be a set of n things
- then S_n acts on X_n by permuting the elements of X_n
- (this is *our very definition* of S_n!)

Definitions

Let $\rho\colon G \times X \to X$ be a group action.

Definition

G acts sharply transitively on X if given x and $y \in X$, there exists a unique $g \in G$ taking x to y, that is, such that $gx = y$.

Let $k \geq 1$ be an integer. Then G acts sharply k-transitively on X if for every $x_1, \ldots, x_k$ distinct elements of X and for all $y_1, \ldots, y_k$ distinct elements of X, there exists a *single* and *unique* $g \in G$ such that $gx_i = y_i$ for all i.

Examples

- S_n always acts n-transitively!
- Even permutations A_{n+2} act n-transitively on $\{1, \ldots, n\}$
- (the $n + 2$ is not a typo)
- We will see below that this almost lists all examples of k-transitive actions for large k.

Group actions give group homomorphisms

- Let X be a set and let S_X be the *group* of all bijective maps $X \to X$
- The group law on S_X is composition
- Let G be a group that acts on X
- There is a natural group homomorphism $G \to S_X$ defined as follows:
- If $g \in G$ then define $\sigma_g \colon X \to X$ by $\sigma_g(x) = gx$
- Exercise: the map $g \mapsto \sigma_g$ defines a group homomorphism $G \to S_X$
- If $G \to S_X$ is injective, G is said to act *faithfully* on X

An amazing fact

Theorem (Camille Jordan)

Let $k \geq 4$. The only finite groups that act faithfully and sharply k-transitively are:

1. S_k;
2. A_{k+2};
3. *if $k = 4$, the Mathieu group M_{11} of order 7920 acts sharply 4-transitively on a set of 11 elements;*
4. *if $k = 5$, the Mathieu group M_{12} of order 95040 acts sharply 5-transitively on a set of 12 elements*

Note: The Mathieu groups are certain remarkable *simple* groups (no nontrivial normal subgroups).

Overview

Mathematical representation of notes

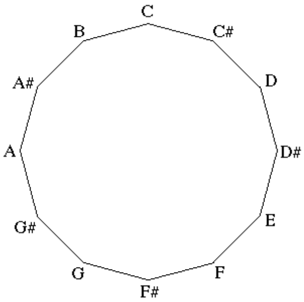
Recall:

- Notes f and $2f$ sound the same
- Traditionally the notes are labeled

$$\cdots, C, C\sharp, D, D\sharp, E, F, F\sharp, G, G\sharp, A, A\sharp, B, C, \cdots$$

- If we identify notes differing by an octave, we can model notes by $\mathbf{Z}/12\mathbf{Z}$
- We'll thus sometimes write $0 = C$, $1 = C\sharp$, $2 = D$, etc

Transpositions and inversions



- Picture notes as the vertices of a regular 12-gon
- In this way D_{12} acts on the notes
- The rotations are called transpositions
- The reflections are called inversions

Chords

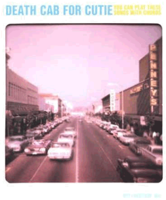


Figure: Why is this picture here?

- Chords are collections of notes played simultaneously
- We'll focus on the three-note major triads and minor triads
- There is one such triad for each note
- Thus, there are 12 major triads and 12 minor triads

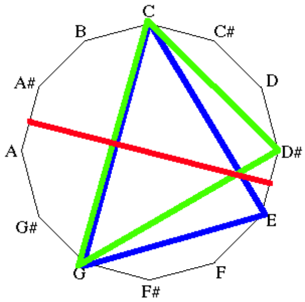
Major triads

- A major triad is determined by a root note
- A major triad also contains the notes which are 4 and 7 semitones above the root
- E.g. the C-major triad is $\{C, E, G\} = \{0, 4, 7\}$, $C\sharp$-major is $\{1, 5, 7\}$, etc
- The root and the second note form a major third
- The root and the third note form a perfect fifth
- Thus major triads sound particularly nice

Minor triads

- Minor triads are obtained from the major triads by inversion
- Concretely, a minor triad is also determined by a root
- The other notes are 3 and 7 semitones above the root
- E.g. the C-minor triad is $\{C, D\sharp, G\} = \{0, 3, 7\}$

C-major $\rightarrow$ C-minor



- The blue C-major is reflected through the red line to the green C-minor
- This reflection exchanges the root and the perfect fifth
- The major and minor thirds are also exchanged

The T/I-group

- Let T denote the set of all major and minor consonant triads
- Then T has 24 elements
- D_{12} acts on T through transposition and inversion
- This action is faithful and transitive
- This gives an embedding

$$D_{12} \hookrightarrow S_{24}$$

- The image inside S_{24} is called the T/I-group
- (the name is due to music theorists)

The PLR-group

- Another group of musical interest acts on the consonant triads
- It's easier to define if we write triads as ascending tuples (f_1, f_2, f_3) with f_1 the root
- Also, we now write notes as elements of $\mathbf{Z}/12\mathbf{Z}$
- For each $n \in \mathbf{Z}/12\mathbf{Z}$ let $I_n \colon \mathbf{Z}/12\mathbf{Z} \to \mathbf{Z}/12\mathbf{Z}$ denote the inversion $I_n(x) = -x + n \pmod{12}$
- Of course each I_n acts on triads $(f_1, f_2, f_3) \in T$ componentwise

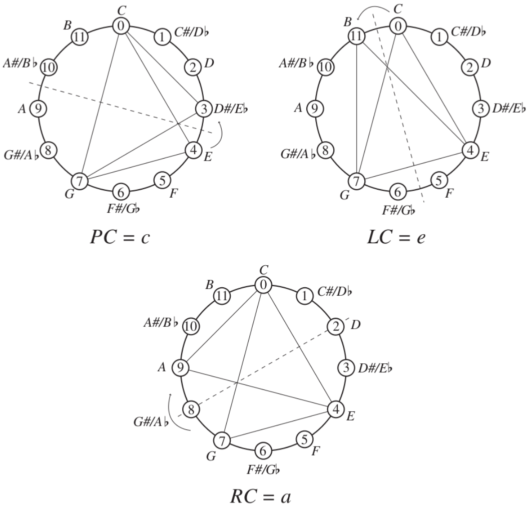
The *PLR*-group

- Define:

$$\begin{aligned} P\colon T \to T \quad &\text{by} \quad P(f_1, f_2, f_3) = I_{f_1+f_3}(f_1, f_2, f_3) \\ L\colon T \to T \quad &\text{by} \quad L(f_1, f_2, f_3) = I_{f_2+f_3}(f_1, f_2, f_3) \\ R\colon T \to T \quad &\text{by} \quad R(f_1, f_2, f_3) = I_{f_1+f_2}(f_1, f_2, f_3) \end{aligned}$$

- Note: These are not componentwise actions!
- They are contextual because the action rule changes as the input changes
- This defines elements P, L and R in S_{24}
- The *PLR*-group is the group in S_{24} generated by P, L and R

Another description of P, L and R



Some history

Figure: Music theorist Hugo Riemann, 1849-1919

Although P, L and R were used in music through 1500 onward, H. Riemann was the first to identify this important group structure

- P stands for parallel
- L stands for leading tone exchange
- R stands for relative

First cool fact

Theorem

The PLR-group is a group of order 24 generated by the transpositions L and R. It acts sharply transitively on the set T of consonant triads.

- If you take three random elements in S_{24} and consider the group they generate, it's unlikely you'll get D_{12}
- *Full disclosure:* We didn't actually check this claim, but it sounds eminently reasonable to us!

Example: Beethoven's *Ninth Symphony*

- Start with C major and alternately apply R and L and you get:
- C, a, F, d, $B\flat$, g, $E\flat$, c, $A\flat$, f, $D\flat$, $b\flat$, $G\flat$, $e\flat$, B, $g\sharp$, E, $c\sharp$, A, $f\sharp$, D, b, G, e, C
- This shows that L and R generate a group of order at least 24
- The first 19 chords above occur in order in measures 143-176 of the second movement of Beethoven's *Ninth Symphony*!
- (Play Beethoven)

A last bit of group theory

- Let G be a group and H_1 and H_2 subgroups of G
- Recall that the centralizer of H_1 in G is the set of elements which commute with H_1:

$$C_G(H_1) = \{x \in G \mid xh = hx \text{ for all } h \in H_1\}.$$

- H_1 and H_2 are said to be dual in G if H_1 is the centralizer of H_2 and vice-versa

Second cool fact

Theorem

The actions of the T/I-group and the PLR group on the set of consonant triads realize these groups as dual subgroups of the symmetric group S_{24}.

It is remarkable that the operations of transposition and inversion, and P, L and R arose organically in music by the 1500s, and it took us about four centuries to realize the mathematical relationship between these musical symmetries!

Some idle thoughts

- Before moving beyond group theory we discuss ideas from Week 234 of John Baez's *This week's finds in mathematical physics*
- Recall that M_{12} is an exceptional simple group that acts 5-transitively on a set of 12 elements
- Can this group be related to musical symmetries in an interesting way, similar to the PLR and T/I groups?
- Baez notes that if one could choose 6 note chords with the properties: (i) every 5 notes determine a unique such chord and (ii) the chords sound nice, then one could relate M_{12} to music in an interesting way
- There are lots of ways to choose chords satisfying (i)
- Most such choices will sound very dissonant though!
- Is there a nice choice? What if one works with 41 notes per octave?

Overview

- Inspired by the work of Hugo Riemann, music theorests (e.g. David Lewin, Brian Hyer, Richard Cohn, Henry Klumpenhouwer) began in the 1980s to use techniques in group theory to analyze music.
- recently various limitations of this approach have been noted
- We'll end by discussing some relevant ideas of Dmitri Tymoczko for extending the neo-riemannian theory using ideas from geometry

A problem with group actions?

- A group action is a rule that says "given $x \in X$, take x over to the new element $gx \in X$"
- There's not always a unique way to get from point A to point B.

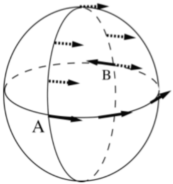


Figure 2. Transporting an arrow along a sphere produces path-dependent results. Here, sliding an east-pointing arrow halfway around the earth along the equator produces an east-pointing arrow. Sliding it halfway around the earth along a perpendicular path produces a west-pointing arrow.

Figure: From Tymoczko's *Generalizing Musical Intervals*

A musical example

- A guitar can play two notes simultaneously, represented by an unordered pair
- All such combinations can be represented on a mobius strip:

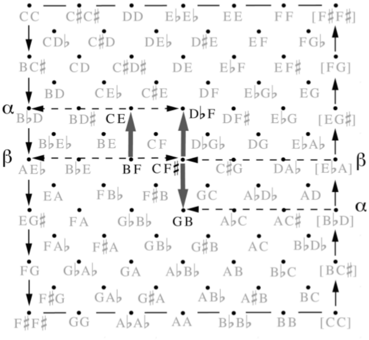
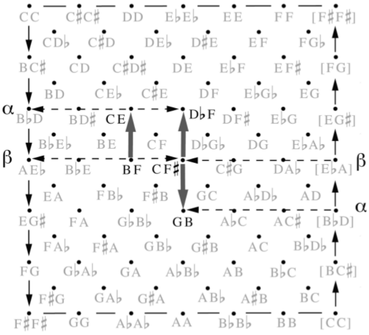


Figure: Also from Tymoczko's *Generalizing Musical Intervals*

A musical example

- Tymoczko asks: how can we represent the movements between pairs, represented by arrows, via a group action?
- As on the sphere, there is not a unique way to move between these arrows:



- rather than try to adjust this space to fit some group structure to it, we should just leave the space as is
- Tymoczko: *"Our twisted geometry is faithfully reflecting genuine musical relationships"*
- in this case, the fact that we can map $\{B, F\}$ to $\{C, F\sharp\}$ two ways:

$$B \mapsto C, \quad F \mapsto F\sharp,$$

and

$$B \mapsto F\sharp, \quad F \mapsto C.$$

- Tymoczko and others have identified many geometric spaces as above that arise naturally in music, and such that musical transformations between the relevant objects cannot be easily modelled by a natural group action
- His view is that in such cases one should reach for geometrical tools, rather than group theoretical ones, to help analyze music
- This is an exciting and modern perspective!

Thanks for listening!