Indifference Price for General Semimartingales

An Orlicz space approach

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The Problem

- Following Hodges and Neuberger (89), we define the *indifference price* $\pi(B)$ for the seller of a claim B as the the implicit solution of the equation

$$\sup_{H \in \mathcal{H}^W} E[u(x+(H\cdot S)_T)] = \sup_{H \in \mathcal{H}^W} E[u(x+\pi(B)+(H\cdot S)_T-B)]. \tag{1}$$

- The utility $u : \mathbb{R} \to \mathbb{R}$ is *strictly concave*, *increasing*, *differentiable* and satisfies the *Inada conditions*

$$\lim_{x\to-\infty} u'(x) = +\infty \quad \textit{and} \quad \lim_{x\to+\infty} u'(x) = 0. \tag{2}$$

- The initial constant endowment is $x \in \mathbb{R}$ and the fixed time horizon is $T \in (0, +\infty]$.
- The underlying process S is an $\mathbb{R}^d-$valued càdlàg semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, which is *not* assumed to be locally bounded.

The Result

- In a fairly general framework, we prove that

$$\sup_{H \in \mathcal{H}^W} E[u(x + (H \cdot S)_T - B)]$$
$$= \min_{\lambda > 0,\ Q \in \mathcal{M}^W} \left\{ \lambda x - \lambda Q(B) + E\left[\Phi\left(\lambda \frac{dQ^r}{dP} \right) \right] + \lambda \| Q^s \| \right\}.$$

- In the dual problem, $\Phi : \mathbb{R}_+ \to \mathbb{R}$ is the dual function of the utility function u:

$$\Phi(y) := \sup_{x \in \mathbb{R}} \{ u(x) - xy \}, \tag{3}$$

- $\mathcal{M}^W$ is the appropriate set of pricing measures Q, which admit the decomposition $Q = Q^r + Q^s$ into regular and singular parts.

Family Tree

- Grandfather: Bellini and Frittelli (2002), general u, locally bounded S, bounded B.
- Uncle: 6–author paper (2002), locally bounded S, exponential u, B bounded from below, exponential integrability from above.
- Father: Biagini and Frittelli (2006), general u, non-locally bounded S, $B = 0$.
- Cousins: Rouge and El Karoui (2000), Bercherer (2003), Owen (2002,2006)...
- Brothers and Sisters: joint work with M. Frittelli, S. Biagini and T. Hurd.

Admissible integrands, suitability and compatibility

- Given $W \in L^0_+$, define the W–admissible strategies as

$$\mathcal{H}^W := \{H \in L(S) \mid (H \cdot S)_t \geq -cW, \text{ for some } c > 0\}.$$

- We say that $W \geq 1$ is *suitable* for S if for each $i = 1, \dots, d$, there exists a process $H^i \in L(S^i)$ such that

$$P(\{\omega \mid \exists t \geq 0 \text{ such that } H^i_t(\omega) = 0\}) = 0 \tag{4}$$

  and

$$|(H^i \cdot X^i)_t| \leq W, \quad \forall t \in [0, T]. \tag{5}$$

- We say that $W \in L^0_+$ is *compatible* with the utility function u if

$$E[u(-\alpha W)] > -\infty \text{ for all } \alpha > 0 \tag{6}$$

  and that it is *weakly compatible* with u if

$$E[u(-\alpha W)] > -\infty \text{ for some } \alpha > 0. \tag{7}$$

Terminal values and duality

- Given a suitable and compatible random variable W, define

$$K^W = \left\{ (H \cdot S)_T \mid H \in \mathcal{H}^W \right\} \quad (8)$$

so that the primal problem (3) becomes:

$$\sup_{k \in K^W} E[u(x + k - B)]. \quad (9)$$

- We then want to define an appropriate cone C^W, related to K^W, and invoke Fenchel's duality theorem.
- For this, we need to choose a Banach spaces and its topological dual in order to define the polar set $(C^W)^0$.
- Classically, the spaces (L^∞, ba) were successfully used when dealing with locally bounded traded assets. In order to accommodate more general markets and inspired by the compatibility conditions above, we argue instead for the use of an appropriate Orlicz and its dual.

Orlicz spaces

- Consider the Young function $\widehat{u} : \mathbb{R} \to [0, +\infty)$ associated with the utility function u, defined as

$$\widehat{u}(x) = -u(-|x|) - u'(0)|x| + u(0).$$

- Its corresponding Orlicz space is

$$L^{\widehat{u}}(P) = \{f \in L^0(P) \mid E[\widehat{u}(\alpha f)] < +\infty \text{ for some } \alpha > 0\},$$

equipped with the Luxemburg norm

$$\|f\|_{\widehat{u}} = \inf \left\{ c > 0 \mid E\left[\widehat{u}\left(\frac{f}{c}\right)\right] \leq 1 \right\}.$$

- We then have $L^\infty \subseteq L^{\widehat{u}}(P) \subseteq L^1(P)$
- Next we consider the closed subspace

$$M^{\widehat{u}}(P) = \{f \in L^{\widehat{u}}(P) \mid E[\widehat{u}(\alpha f)] < +\infty \text{ for all } \alpha > 0\}.$$

- In general $M^{\widehat{u}} \subset L^{\widehat{u}}$ (strict inclusion).

Compatibility revisited

- The Young function $\widehat{u}$ carries information about the utility on large losses, in the sense for $\alpha > 0$ we have that

$$E[\widehat{u}(\alpha f)] < +\infty \qquad \Longleftrightarrow \qquad E[u(-\alpha|f|)] > -\infty. \quad (10)$$

- We can then see that a positive random variable W is compatible (resp. weakly compatible) with the utility function u if and only if $W \in M^{\widehat{u}}$ (resp. $W \in L^{\widehat{u}}$).

Complementary spaces

- The convex conjugate of $\widehat{u}$, called the *complementary* Young function in the theory Orlicz spaces, is denoted here by $\widehat{\Phi}$, since it admits the representation
$$\widehat{\Phi}(y) := \sup_x \{x|y| - \widehat{u}(x)\} = \Phi(|y| + \beta) - \Phi(\beta),$$
where $\beta = u'(0) > 0$.
- We consider the Orlicz space $L^{\widehat{\Phi}}$ endowed with the Orlicz norm
$$|||g|||_{\widehat{\Phi}} = \sup\{|E[fg]|, f \in L^{\widehat{u}}, E[\widehat{u}(f)] \leq 1\}.$$
- It then follows that $(M^{\widehat{u}})^* = L^{\widehat{\Phi}}$ in the sense that if $z \in (M^{\widehat{u}})^*$ is a continuous linear functional on $M^{\widehat{u}}$, then there exists a unique $g \in L^{\widehat{\Phi}}$ such that
$$z(f) = \int_\Omega fgdP, \qquad f \in M^{\widehat{u}},$$
with $\|z\|_{(M^{\widehat{u}})^*} := \sup_{\|f\|_{\widehat{u}} \leq 1} |z(f)| = |||g|||_{\widehat{\Phi}}$.

The dual of $L^{\widehat{u}}$

- It follows from the properties of the pair $(\widehat{u}, \widehat{\Phi})$ that each element $z \in (L^{\widehat{u}})^*$ can be uniquely expressed as

$$z = z^r + z^s,$$

where the *regular* part z^r is given by

$$z^r(f) = \int_\Omega fgdP, \qquad f \in L^{\widehat{u}},$$

for a unique $g \in L^{\widehat{\Phi}}$, and the *singular* part z^s satisfies

$$z^s(f) = 0, \qquad \forall f \in M^{\widehat{u}}. \tag{11}$$

- That is, $(L^{\widehat{u}})^* = (M^{\widehat{u}})^* \oplus (M^{\widehat{u}})^\perp$.

Positive singular functionals

Consider the concave integral functional

$$\begin{aligned} I_u : L^{\widehat{u}} &\rightarrow [-\infty, \infty) \\ f &\mapsto E[u(f)] \end{aligned}$$

with effective domain

$$\mathcal{D}(P) = \left\{ f \in L^{\widehat{u}}(P) \mid E[u(f)] > -\infty \right\}. \tag{12}$$

One consequence of choosing the correct Orlicz spaces is that the norm of a *non negative* singular element $0 \leq z \in (M^{\widehat{u}})^{\perp}$ satisfies

$$\|z\|_{(L^{\widehat{u}}(P))^*} := \sup_{\|f\|_{\widehat{u}} \leq 1} |z(f)| = \sup_{f \in \mathcal{D}(P)} z(-f),$$

Dual Variables

- Given a loss variable $W \in \mathbb{S} \cap L^{\widehat{u}}$ we define the cone

$$C^W = (K^W - L^0_+) \cap L^{\widehat{u}}.$$

- Define the polar cone

$$(C^W)^0 := \left\{ z \in (L^{\widehat{u}})^* \mid z(f) \leq 0, \quad \forall f \in C^W \right\}, \tag{13}$$

which satisfies $(C^W)^0 \subseteq (L^{\widehat{u}})^*_+$, since $(-L^{\widehat{u}}_+) \subseteq C^W$.

- The subset of normalized functionals in $(C^W)^0$ is defined by

$$\mathcal{M}^W := \{Q \in (C^W)^0 \mid Q(\mathbf{1}_\Omega) = 1\}. \tag{14}$$

- It was shown in BF06 that

$$\mathcal{M}^W \cap L^1(P) = \mathbb{M}_\sigma \cap L^{\widehat{\Phi}}. \tag{15}$$

Utility optimization with a claim

- Consider a claim $B \in \mathcal{F}_T$ satisfying

$$E[u(-(1+\varepsilon)B)] > -\infty, \quad E[u(\varepsilon B)] > -\infty \text{ for some } \varepsilon > 0. \tag{16}$$

- We are interested in the concave functional $I_u^B : L^{\widehat{u}} \to [-\infty, \infty)$ defined by

$$I_u^B(f) := E[u(x - B + f)].$$

- We then want to find the concave conjugate $J_u^B(z) := -(I_{-u}^B)^*(-z)$, where

$$(I_{-u}^B)^*(z) := \sup_{f \in L^{\widehat{u}}} \left\{ z(f) - I_{(-u)}^B(f) \right\}, \quad z \in (L^{\widehat{u}})^*.$$

- It follows from Kozek (79) that

$$J_u^B(z) = -E[\Phi(z^r)] - \|z^s\| + z(B - x).$$

Technical Steps

- Lemma
  If $W \in \mathbb{S} \cap L^{\widehat{u}}$ and $B \in \mathcal{F}_T$ satisfies (16), then

$$\sup_{k \in K^W} E[u(x + k - B)] = \sup_{f \in C^W} E[u(x + f - B)] \qquad (17)$$

- Lemma
  Suppose that $B \in \mathcal{F}_T$ satisfies (16). Then the concave functional I_u^B on $L^{\widehat{u}}$ is norm–continuous on the interior of its effective domain, which is non–empty. Moreover, there exists a norm continuity point of I_u^B that belongs to C^W.

Main result

Theorem

Suppose that $B \in \mathcal{F}_T$ *satisfies (16) and that there exists* $W \in \mathbb{S} \cap L^{\widehat{u}}$ *that satisfies*

$$\sup_{H \in \mathcal{H}^W} E[u(x + (H \cdot S)_T - B] < u(\infty). \tag{18}$$

Then $\mathcal{M}^W$ *is not empty and*

$$\begin{aligned} \sup_{H \in \mathcal{H}^W} \quad & E_P[u(x + (H \cdot S)_T - B)] \\ = \quad & \min_{\lambda > 0,\ Q \in \mathcal{M}^W} \left\{ \lambda x - \lambda Q(B) + E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] + \lambda \|Q^s\| \right\}, \end{aligned}$$

where $Q(B) = E_{Q^r}[B] + Q^s(B)$*. If* $B \in M^{\widehat{u}}$*, then* $Q^s(B) = 0$*. Moreover, if* $W \in M^{\widehat{u}}$ *and* $B \in M^{\widehat{u}}$ *then* $\mathcal{M}^W$ *can be replaced by* $\mathbb{M}_\sigma \cap L^{\widehat{\Phi}}$ *and no singular term appears.*

Indifference price representation

- Corollary

Under the same hypotheses of the Theorem we have:

$$\begin{aligned}
\pi_\gamma(B) &= \min_{Q\in\mathcal{M}^W}\left[\frac{1}{\gamma}H(Q^r|P)+\|Q^s\|\right] && (19)\\
&\quad -\min_{Q\in\mathcal{M}^W}\left[\frac{1}{\gamma}H(Q^r|P)+\|Q^s\|-Q(B)\right] \\
&= \max_{Q\in\mathcal{M}^W(P)}\left[Q(B)-\frac{1}{\gamma}\mathbb{H}(Q,P)\right], && (20)
\end{aligned}$$

where the penalty term is given by:

$$\mathbb{H}(Q,P) := H(Q^r|P)+\gamma\|Q^s\| - \min_{Q\in\mathcal{M}^W}\left[H(Q^r|P)+\gamma\|G(Q^s\|\right].$$

Exponential Utility

- For an exponential utility function $u(x) = -e^{-\gamma x}$, $\gamma > 0$, we have

$$\begin{aligned}
\Phi(y) &= \frac{y}{\gamma} \log \frac{y}{\gamma} - \frac{y}{\gamma} \\
\widehat{u}(x) &= e^{\gamma |x|} - \gamma |x| - 1 \\
\widehat{\Phi}(y) &= \left(\frac{|y|}{\gamma} + 1 \right) \log \left(\frac{|y|}{\gamma} + 1 \right) - \frac{|y|}{\gamma}.
\end{aligned}$$

- Moreover, the boundedness condition on B is equivalent to

$$E\left[e^{(\gamma+\epsilon)B}\right] < \infty, \quad E\left[e^{-\varepsilon B}\right] < \infty, \text{ for some } \varepsilon > 0. \quad (21)$$

Change of Measure

- Since this implies that $E[e^{\gamma B}] < \infty$, we can define a new probability measure P_B by:

$$\frac{dP_B}{dP} = c_B e^{\gamma B}, \quad c_B = \left(E[e^{\gamma B}]\right)^{-1}.$$

- [Pistone/Rogantin 99] Let f and g be the densities of two probability measures equivalent to P and suppose that they are connected by a one-dimensional exponential model, that is, $f = c(t)exp(tB)$, $g = c(s)exp(sB)$, where $c(\cdot)$ is a normalization factor, $B \in L^{\widehat{u}}(P)$ and t, s belong to an open interval $I \ni 0$ with the property that that $\theta \in I \Rightarrow E[exp(\theta B)] < +\infty$. Then

$$L^{\widehat{u}}(f.P) = L^{\widehat{u}}(g.P)$$

as Banach spaces.

Dependence on P_B

- Denote

$$G_P(Q^s) := \sup_{f \in \mathcal{D}(P)} Q^s(-f), \tag{22}$$

- Lemma

Let $B \in \mathcal{F}_T$ satisfy (21) Then:

1. $\mathcal{D}(P_B) = \mathcal{D}(P) + \gamma B$ *and* $G_{P_B}(Q^s) = G_P(Q^s) - \gamma Q^s(B)$. *Hence,*

$$G_P(Q^s) < \infty \quad \Longleftrightarrow \quad G_{P_B}(Q^s) < \infty.$$

   In particular, we have $G_{P_B}(Q^s) = G_P(Q^s)$ *for any* $B \in M^{\widehat{u}}(P)$.
2. *If* $Q \ll P \sim P_B$, *then*

$$H(Q|P) < \infty \quad \Longleftrightarrow H(Q|P_B) < \infty$$

3. $\mathbb{S} \cap L^{\widehat{u}}(P) = \mathbb{S} \cap L^{\widehat{u}}(P_B)$ *and* $C^W(P) = C^W(P_B)$.
4. $\mathcal{M}^W(P) = \mathcal{M}^W(P_B)$.

Bounds on the singular part

Lemma

Suppose that $B \in \mathcal{F}_T$ satisfies (21). Then, for any $Q \in \mathcal{M}^W$, we have

$$\frac{\varepsilon}{\gamma + \varepsilon} G_P(Q^s) \leq G_{P_B}(Q^s) \leq \frac{\gamma + \varepsilon}{\varepsilon} G_P(Q^s). \tag{23}$$

As a consequence, the singular contribution $-Q^s(B)$ in the main Theorem satisfies the following bounds:

$$-\frac{1}{\gamma + \varepsilon} \|Q^s\| \leq -Q^s(B) \leq \frac{1}{\varepsilon} \|Q^s\|. \tag{24}$$

Further work

- Properties of the indifference price
- Levy market example
- Solution of the primal problem
- Learn how to properly pronounce *Arigato*.