Indifference Price of Insurance Contracts: stochastic volatility, stochastic interest rates

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Part 1: Stochastic Volatility

- We consider two factor stochastic volatility models where the financial asset satisfies:

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\begin{aligned}
dS_t &= \mu S_t dt + \sigma(Y_t) S_t dW_t^1 \\
dY_t &= a(Y_t)dt + b(Y_t)[\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2]
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- Here μ and $-1 < \rho < 1$ are constants, $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ are deterministic functions, and W_t^1 and W_t^2 are independent one dimensional P–Brownian motions.
- In addition, we assume the existence of a risk-free bank account paying a constant interest rate $r = 0$.

Optimal Hedging and Investment

- We assume that, after selling an insurance contract B_T maturing at a future time T, the insurance company tries to solve the stochastic control problem

$$u^B(x,S,y,t) = \sup_{\pi \in \mathcal{A}} E^{x,S,y,t}\left[U\left(X_T^{\pi,x,B}\right)\right],$$

where $X_t^{\pi,x,B}$ is value of a self–financing portfolio (including short position in the contract B) with initial wealth x and π_t dollars invested in the stock, with the remaining value invested in the bank account.

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  where $X_t^{\pi,x,B}$ is value of a self–financing portfolio (including short position in the contract B) with initial wealth x and π_t dollars invested in the stock, with the remaining value invested in the bank account.
- When $B = 0$, this reduces to the Merton problem:

$$u^0(x, y, t) = \sup_{H \in \pi} E^{x,y,t}\left[U\left(X_T^{\pi,x}\right)\right]$$

Utility based pricing

- The seller's indifference price for the claim B is the solution π^s to the equation

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- We can then write

$$\begin{aligned} u^B(x, S, y, t) &= -e^{-\gamma x} G(S, y, t) = -e^{-x} e^{\phi(S,y,t)} \\ u^0(x, y, t) &= -e^{\gamma x} F(y, t) = -e^{-\gamma x} e^{\psi(y,t)} \end{aligned}$$

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- The indifference price is then given by

$$P(S,y,t) = \frac{1}{\gamma} \log\left(\frac{G(S,y,t)}{F(y,t)}\right) = \frac{1}{\gamma}(\phi(S,y,t) - \psi(y,t)).$$

Equity–linked contracts

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- To obtain the equation satisfied by u^s in this case, consider the interval $[t, t+h)$ and observe that,

$$\begin{aligned} u^B(x,s,y,t) \geq\ & E[u^B(X_{t+h}, S_{t+h}, Y_{t+h}, t+h)]p(h) \\ & + E[u^0(X_{t+h} - B(S_{t+h}), Y_{t+h}, t+h)]q(h) \end{aligned}$$

where $p(h) = P(\tau > t+h | \tau > t)$ and $q(h) = 1 - p(h)$.

The HJB equation

- Using a function of the form $u^B(x, S, y, t) = -e^{-\gamma x} e^{\phi(S,y,t)}$ leads to

$$\begin{cases} \phi_t + \frac{1}{2}\sigma^2 S^2 \phi_{SS} + \rho\sigma b S \phi_{yS} + \frac{1}{2} b^2 \phi_{yy} + \left(a - \frac{\mu b \rho}{\sigma}\right)\phi_y \\ + \frac{1}{2} b^2 (1-\rho^2)\phi_y^2 + \lambda(t)\left[e^{\gamma B + \psi - \phi} - 1\right] = \frac{\mu^2}{2\sigma^2} \\ \phi(y, S, T) = 0 \end{cases}, \tag{1}$$

where, as it is well-known,

$$\psi(y,t) = \frac{1}{1-\rho^2} \log \widetilde{E}^{y,t}\left[e^{-\int_0^T \frac{(1-\rho^2)\mu^2}{2\sigma^2(Y_s)} ds}\right],$$

with $\widetilde{E}[\cdot]$ denoting an expectation with respect to the *minimal martingale measure* for this market.

Optimal hedge

- In terms of ϕ, the optimal portfolio is

$$\pi_t^B = \frac{1}{\gamma}\left[\frac{\mu}{\sigma^2(y)} + \phi_S S + \frac{b(y,t)\rho}{\sigma(y)}\phi_y\right]$$

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- Subtracting one from the other we obtain the *excess hedge*

$$\pi_t^B - \pi_t^0 = P_S(S,y,t)S_t + \frac{b(y,t)\rho}{\gamma\sigma(y)}P_y(S,y,t),$$

which has the form of a *delta* hedge plus a volatility correction.

Fast-mean reversion asymptotics

- Let us now take

$$dY_t = \alpha(m - Y_t)dt + \beta(\rho dW_t + \sqrt{1-\rho^2}dZ_t)$$

and consider the regime $\frac{1}{\alpha} = \varepsilon << 1$, with $\beta = \sqrt{2\nu}/\sqrt{\varepsilon}$ where ν^2 is a fixed variance for the invariant distribution of Y_t.

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- We then look for expansion of the form

$$\phi^{\varepsilon} = \phi^{(0)}(y, S, t) + \sqrt{\varepsilon}\phi^{(1)}(y, S, t) + \varepsilon\phi^{(2)}(y, S, t) + ...$$

Operators

- The previous PDE can be rewritten in compact notation as

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)\phi + NL^{\phi} = \frac{\mu^2}{2\sigma^2} \qquad (2)$$

where $NL^{\phi} = \lambda(t)\left[e^{\gamma B+\psi-\phi} - 1\right] + \frac{\mu^2}{\varepsilon}(1-\rho^2)\phi_y^2$.

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- Here

$$\mathcal{L}_0 = \nu^2\frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}$$

$$\mathcal{L}_1 = \sqrt{2}\rho\nu\left(\sigma(y)S\frac{\partial^2}{\partial y\partial S} - \frac{\mu}{\sigma(y)}\frac{\partial}{\partial y}\right)$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2(y)S^2\frac{\partial^2}{\partial S^2}$$

Main result

- The insurer's indifference price satisfy:

$$|P(y,S,t) - P^{(0)}(S,t) - \widetilde{P1}(y,S,t)| = \mathcal{O}(\varepsilon) \qquad (3)$$

where

$$\widetilde{P1}(y,S,t) = -(T-t)(V_3 S^3 P^{(0)}_{SSS} + V_2 S^2 P^{(0)}_{SS})$$

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- Here $P^{(0)}$ satisfies

$$\begin{cases} P^{(0)}_t + \frac{1}{2}\sigma_\star^2 P^{(0)}_{SS} + \frac{\lambda(t)}{\gamma}\left[e^{\gamma(g-P^{(0)})} - 1\right] = 0 \\ P^{(0)}(S,T) = 0 \end{cases} \qquad (4)$$

where $\sigma_\star^2 = \langle \sigma^2 \rangle$.

Example

- Consider the contract

$$g(S) = \begin{cases} 4, & \text{if } 0 \leq S \leq 50 \\ 0.8S, & \text{if } 5 \leq S \leq 20 \\ 16, & \text{if } 20 \leq S \leq 100 \end{cases} \tag{5}$$

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$$\lambda_x(t) = \frac{1}{b} e^{\frac{x+t-m}{\beta}}$$

with $\beta = 8.75$ and $m = 92.63$.

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- The other model parameters are:

$$\alpha = 200, \quad m = \log 0.1, \quad \nu = \frac{1}{\sqrt{2}}, \quad \rho = -0.2, \quad \mu = 0.2.$$

Price correction

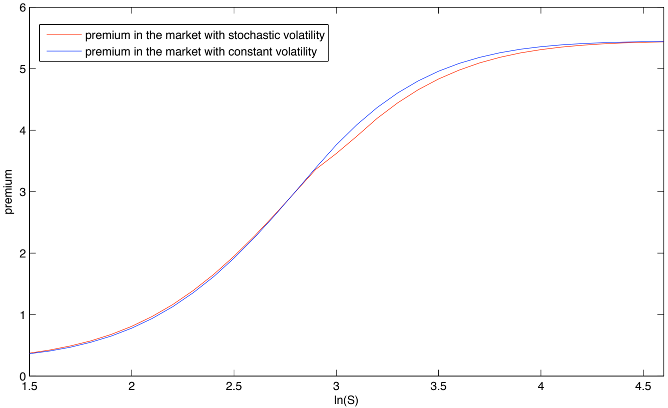

Figure: Premium for the equity linked contract in a market with constant volatility $\sigma = 0.165$ and in the market with stochastic volatility for $T - t = 15$ years and $\gamma = 0.3$.

Risk aversion

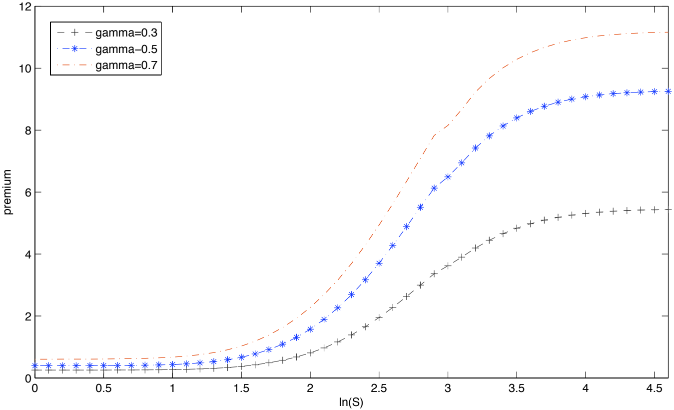

Figure: Dependence with risk aversion

Hedge

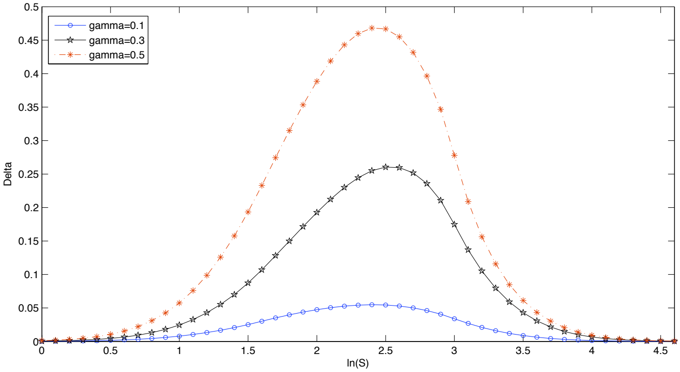

Figure: Hedge ratio for different risk aversion parameters

Part 2: Stochastic Interest Rates

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- We model the short rate as

$$\begin{cases} dr_s = (a_0(s)r_s + b_0(s))ds + \sqrt{c(s)r_s + d(s)}dZ_s \\ r_t = r \end{cases},$$

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where $Z_t = \rho W_t^1 + \sqrt{1-\rho^2} dW_t^2$.

- It then follows that the price of a zero–coupon bond with maturity T_1 is given by

$$F_{tT_1} = e^{A(t,T_1) - C(t,T_1)r_t},$$

for deterministic functions $A(\cdot,\cdot)$ and $C(\cdot,\cdot)$.

Portfolio choice

- In this context, the insurance company can invest π_t dollars in the stock S_t and η_t dollars in the bond F_{tT_1}, with the remaining of its wealth in a bank account paying the interest rate r_t.

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- We assume the market for bonds of different maturities has a market price of risk of the form

$$q(r_s, s) = \frac{(a_0(s) - a(s))r_s + (b_0(s) - b(s))}{\sqrt{c(s)r_s + d(s)}} \tag{6}$$

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- Under this assumption, one can show that the dynamics of the discounted bond price is

$$\frac{d(e^{-\int_0^s r_u du} F_{sT_1})}{e^{-\int_0^s r_u du} F_{sT_1}} = -C(s, T_1)\Big[(\Delta a(s) r_s + \Delta b(s))dt + \sqrt{c(s)r_s + d(s)}dZ_s\Big]$$

Path–dependent claims

- We consider path–dependent claims of the form $B_t = B(S_t, r_t, v_t)$, where

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$$\begin{cases} dX_s = \pi_s \frac{dS_s}{S_s} + \eta_s \frac{d(e^{-\int_0^s r_u du} F_{sT_1})}{e^{-\int_0^s r_u du} F_{sT_1}} \\ dX_s = [\pi_s(\mu - r) - \eta_s C(s, T_1)(\Delta a(s) r_s + \Delta b(s))]ds \\ +\pi_s \sigma dW^1 - \eta_s C(s, T_1)\sqrt{c(s) r_s + d(s)} dZ_s \\ X_\tau = X_{\tau-} - B(S_\tau, r_\tau, V_\tau), \quad \tau < T \\ X_t = x \end{cases}$$

The solution to Merton's Problem

- The Merton problem for the insurance company is now

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- Using the same reasoning as before for the function $u^0(x, r, t) = -e^{-\gamma x} e^{\psi(r,t)}$ we arrive at the following PDE:

$$\psi_t + (ar + b)\psi_r + \frac{1}{2}\psi_{rr}(cr + d) - \left[\frac{1}{2}\left(\frac{\mu - r - \sigma\rho q}{\sqrt{1 - \rho^2}\sigma}\right)^2 + \frac{q^2}{2}\right] = 0,$$

subject to $\psi(r, T) = 0$.

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subject to $\psi(r,T)=0$.

- Using Feynmann-Kac we obtain that

$$\psi(r,t) = -\int_t^T \widehat{E}^{t,r}\left[\left(\frac{\mu-r-\sigma\rho q}{2\sqrt{1-\rho^2}\sigma}\right)^2+\frac{q^2}{2}\right],$$

where $\widehat{E}[\cdot]$ denotes expectation with respect to the (unique) martingale measure for bond prices defined by the market price of risk q.

The value function with the claim

- Similarly, the hedging problem for the insurance company is now
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- For a function of the form $u^B(x,S,y,t) = -e^{-\gamma x}e^{\phi(S,r,v,t)}$, we obtain that ϕ satisfies the PDE

$$\begin{cases} \phi_t + (ar+b)\phi_r + \frac{1}{2}(cr+d)\phi_{rr} + \rho\sigma\sqrt{cr+d}S\phi_{Sr} + \frac{1}{2}\sigma^2 S^2\phi_{SS} \\ +f(S,r,t)\phi_v - \left[\frac{1}{2}\left(\frac{\mu-r-\sigma\rho q}{\sqrt{1-\rho^2}\sigma}\right)^2 + \frac{q^2}{2}\right] - \lambda(t)\left(1-e^{\gamma B+\psi-\phi}\right) = \end{cases} \qquad (8)$$

subject to $\phi(S,r,v,T)=0$.

Optimal hedge

- In terms of ϕ, the optimizers for (7) are

$$\pi_t^B = \frac{1}{\gamma}\left[\frac{\mu - r - q\rho\sigma}{(1 - rho^2)\sigma^2} + \phi_S S\right]$$

$$\eta_t^B = \frac{1}{\gamma C\sqrt{cr+d}}\left[\frac{\rho\sigma(\mu - r) - q\sigma^2}{(1-\rho^2)\sigma^2} - \sqrt{cr+d}\,\phi_r\right]$$

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- Subtracting one from the other we obtain the *excess hedge*

$$\begin{aligned}\pi_t^B - \pi_t^0 &= P_S(S, r, v, t)S_t \\ \eta_t^B - \eta_t^0 &= -\frac{1}{C}P_r(S, r, v, t)\end{aligned}$$

The pricing equation and integral representation

- Therefore, P satisfies the following nonlinear PDE:

$$\begin{cases} P_t + (ar+b)P_r + \dfrac{1}{2}(cr+d)P_{rr} + \rho\sigma\sqrt{cr+d}\,SP_{Sr} + \dfrac{1}{2}\sigma^2 S^2 P_{SS} \\ +f(S,r,t)P_v - \frac{\lambda(t)}{\gamma}\left(1 - e^{\gamma B - \gamma P}\right) = 0 \\ P(S,r,T) = 0 \end{cases}$$

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$$\begin{cases} P_t + (ar+b)P_r + \frac{1}{2}(cr+d)P_{rr} + \rho\sigma\sqrt{cr+d}SP_{Sr} + \frac{1}{2}\sigma^2 S^2 P_{SS} \\ +f(S,r,t)P_v - \frac{\lambda(t)}{\gamma}\left(1 - e^{\gamma B - \gamma P}\right) = 0 \\ P(S,r,T) = 0 \end{cases}$$

- This leads to an integral representation of the premium as follows:

$$\begin{aligned} P(S,r,V,t) = \frac{1}{\gamma}\sup_{y>0}\Bigg[& E^{Q}_{t,S,r,V}\left[\int_t^T g(S,V,r,t)e^{-\int_t^s \frac{y_s\lambda_s}{\gamma}du} y_s\lambda_s ds\right] \\ & - E^{Q}_{t,S,r,V}\left[\int_t^T \left(\frac{1}{y_s} - \frac{1}{\gamma}\left(1 - \ln\frac{y_s}{\gamma}\right)\right) y_s\lambda_s e^{-\int_t^s \frac{y_s\lambda_s}{\gamma}du} ds\right]\Bigg] \end{aligned} \quad (9)$$