Pricing Insurance Contracts in Markets with Stochastic Volatility

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- Opportunities for long and short term visitors, postdocs, graduate students, etc.
- Contact the organizers: T. Hurd and MRG (McMaster), M. Rindesbacher (U of T), V. Henderson (Warwick), Y. Ait-Sahalia (Princeton).

Market Model

- We consider two factor stochastic volatility models of the form:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma(t, Y_t) S_t dW_t^1] \\
dY_t &= a(t, Y_t)dt + b(t, Y_t)[\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2]
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- Here μ and $|\rho| < 1$ are constants, a, b are deterministic functions, and W_t^1 and W_t^2 are independent one dimensional P–Brownian motions.
- In addition, we assume the existence of a risk-free bank account paying a constant interest rate $r = 0$.

Optimal Hedging and Investment

- We assume that, after selling an insurance contract B_T maturing at a future time T, the insurance company tries to solve the stochastic control problem

$$u^s(x, S, y, t) = \sup_{H \in \mathcal{A}} E\left[U\left(X_T^{H,x} - B_T \right) | X_t = x, S_t = S, Y_t = y \right],$$

where $X_T^{H,x}$ is the terminal value of a self–financing portfolio with initial wealth x and consisting of holding H_t units of the stock with the remaining value invested in the bank account.

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  where $X_T^{H,x}$ is the terminal value of a self–financing portfolio with initial wealth x and consisting of holding H_t units of the stock with the remaining value invested in the bank account.
- When $B = 0$, this reduces to the Merton problem:

$$u^0(x, y, t) = \sup_{H \in \mathcal{A}} E\left[U\left(X_T^{H,x}\right)|X_t = x, Y_t = y\right].$$

Utility based pricing

- The seller's indifference price for the claim B is the solution π^s to the equation

$$u^0(x, y, t) = u^s(x + \pi^s(x, S, y, t), S, y, t).$$

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- We can then write

$$\begin{aligned} u^s(x, S, y, t) &= -e^{-\gamma x} G(S, y, t) = -e^{-\gamma x} e^{\phi(S,y,t)} \\ u^0(x, y, t) &= -e^{-\gamma x} F(y, t) = -e^{-\gamma x} e^{\psi(y,t)} \end{aligned}$$

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- The indifference price is then given by

$$\pi^s(S, y, t) = \frac{1}{\gamma} \log \frac{G(S, y, t)}{F(y, t)} = \frac{1}{\gamma}(\phi(S, y, t) - \psi(y, t)).$$

The solution to Merton's problem

- It is well-known that the power transformation $F(y, t) = f(y, t)^{1/1-\rho^2}$ leads to the linear equation

$$f_t + \left[a - \frac{b\rho\mu}{\sigma}\right] f_y + \frac{1}{2} b^2 f_{yy} = \frac{(1-\rho^2)\mu^2}{2\sigma^2} f,$$

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subject to $f(y, T) = 1$.

- Using Feynman–Kac, we obtain

$$f(t, y) = \widetilde{E}_{t,y}\left[e^{-\int_t^T \frac{(1-\rho^2)\mu^2}{2\sigma^2(s, Y_s)} ds}\right], \qquad (1)$$

where

$$dY_s = \left[a - \frac{b\mu\rho}{\sigma}\right] ds + b\left[\rho d\widetilde{W}_s^1 + \sqrt{1-\rho^2} d\widetilde{W}_s^2\right],$$

with $d\widetilde{W}_t^1 = dW_t^1 + \widetilde{\lambda}_t^1 dt$ and $d\widetilde{W}_t^2 = dW_t^2$.

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subject to $f(y,T) = 1$.
- Using Feynman–Kac, we obtain

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with $d\widetilde{W}_t^1 = dW_t^1 + \widetilde{\lambda}_t^1 dt$ and $d\widetilde{W}_t^2 = dW_t^2$.
- Therefore, whenever σ_t^2 is the reciprocal of an affine process, the solution to Merton's problem can be calculated explicitly.

Life insurance

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- Crucially, we assume that τ is independent of (W_1, W_2).
- In this case, we have

$$\begin{aligned} u^s(x+\pi^s) &= \sup_{H\in\mathcal{A}} E\left[-e^{-\gamma(x+\pi^s+\int_0^T H_s dS_s + B_T)}\right] \\ &= e^{-\gamma\pi^s} E\left[e^{\gamma B_T}\right] \sup_{H\in\mathcal{A}} E\left[-e^{-\gamma x - \gamma\int_0^T H_s dS_s}\right] \\ &= e^{-\gamma\pi^s} E\left[e^{\gamma B_T}\right] u^0(x,y,t) \end{aligned}$$

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- Therefore, the indifference price in this case is given by

$$\pi^s = \frac{1}{\gamma}\log E\left[e^{\gamma B_T}\right].$$

Random horizon

- To obtain a nontrivial indifference price for contracts that are independent of the financial market, we need to consider the following modified problem:

$$
\begin{aligned}
u^0(x,y,t) &= \sup_{H\in\mathcal{A}} E[U(X_{\tau\wedge T})] \\
&= \sup_{H\in\mathcal{A}} E\left[\int_0^\infty U(X_{t\wedge T})d\Phi(t)\right] \\
&= E\left[U(X_T)(1-\Phi(T)) + \int_0^T U(X_u)d\Phi(u)\right]
\end{aligned}
$$

where

$$
\Phi(t) = P[\tau \le t] = 1 - e^{-\int_0^t \lambda(s)ds}.
$$

Solution to Merton's problem - uncorrelated volatility

- Using dynamic programming, we find that the value function $u^0(x, y, t) = -e^{-\gamma} F(y, t)$ for the random horizon satisfies the HJB equation

$$
\begin{aligned}
F_t \quad + \quad & \left[a - \frac{b\rho\mu}{\sigma} \right] F_y + \frac{1}{2} b^2 F_{yy} \\
& - \left(\frac{\mu^2}{2\sigma^2} + \lambda(t) \right) F + \lambda(t) = \frac{1}{2} b^2 \rho^2 \frac{F_y^2}{F},
\end{aligned}
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subject to $F(y, T) = e^{-\int_0^T \lambda(t) dt}$.

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subject to $F(y, T) = e^{-\int_0^T \lambda(t)dt}$.

- Unfortunately, the power transformation used before does not lead to a linear equation. To proceed, we take $\rho = 0$ and obtain

$$\begin{aligned} F(y, t) &= e^{-\int_0^T \lambda(s)ds}\widetilde{E}_{t,y}\left[e^{-\int_t^T \left(\frac{\mu^2}{2\sigma^2(s, Y_s)} + \lambda(s)\right)ds}\right] \\ &+ \int_t^T \widetilde{E}_{t,y}\left[\lambda(s)e^{-\int_t^s \left(\frac{\mu^2}{2\sigma^2(u, Y_u)} + \lambda(u)\right)du}\right] ds \end{aligned}$$

Continuous life annuity - random horizon

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- It turns out that the value function $u^s(x, y, t)$ in this case satisfies the same HJB equation satisfied by $u^0(x, y, t)$, expect for an extra term of the form $\gamma G(y, t)$.

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- It turns out that the value function $u^s(x, y, t)$ in this case satisfies the same HJB equation satisfied by $u^0(x, y, t)$, expect for an extra term of the form $\gamma G(y, t)$.
- Therefore, still in the case $\rho = 0$, we have

$$\begin{aligned} G(y,t) &= e^{-\int_0^T \lambda(s)ds} \widetilde{E}_{t,y} \left[e^{-\int_t^T \left(\frac{\mu^2}{2\sigma^2(s,Y_s)} + \lambda(s) - \gamma \right) ds} \right] \\ &\quad + \int_t^T \widetilde{E}_{t,y} \left[\lambda(s) e^{-\int_t^s \left(\frac{\mu^2}{2\sigma^2(u,Y_u)} + \lambda(u) - \gamma \right) du} \right] ds \end{aligned}$$

Equity–linked contracts

- For an indifference price that genuinely depends on the underlying market, consider the an insurance contract of the form $B_T = g(S_\tau, \tau)\mathbf{1}_{\{\tau \leq T\}}$.

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- In this case, inserting $u^s(x, S, y, t) = -e^{-\gamma x} e^{\phi(S,y,t)}$ into the corresponding HJB leads to

$$\phi_t + \left(a - \frac{\mu b \rho}{\sigma(y)}\right)\phi_y + \frac{1}{2}\sigma^2(y)S^2\phi_{SS} + \rho\sigma(y)bS\phi_{Sy} + \frac{1}{2}b^2\phi_{yy} + \frac{1}{2}b^2(1-\rho^2)\phi_y^2 = \frac{\mu^2}{2\sigma^2(y)} + \lambda(t)\left[1 - e^{\gamma g(S,t)+\psi-\phi}\right],$$

subject to $\phi(S, y, T) = 1$.

Fast-mean reversion asymptotics

- Let us now take

$$dY_t = \alpha(m - Y_t)dt + \beta(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)$$

and consider the regime $\alpha = 1/\varepsilon$, $0 < \varepsilon << 1$, with $\beta = \sqrt{2}\nu/\sqrt{\varepsilon}$, where ν^2 is a fixed variance for the invariant distribution of Y_t.

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- We then look for expansion of the form

$$\phi(S,y,t) = \phi^{(0)}(S,y,t) + \sqrt{\varepsilon}\phi^{(1)}(S,y,t) + \varepsilon\phi^{(2)}(S,y,t) + \cdots$$

Operators

- The previous PDE can be rewritten in compact notation as

$$\mathcal{L}^\varepsilon \phi + \frac{\nu^2}{\varepsilon}(1-\rho^2)\phi_y^2 = \frac{\mu^2}{2\sigma^2(y)} + \lambda(t)\left[1 - e^{\gamma g(S,t)+\psi-\phi}\right],$$

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- Here

$$\begin{aligned}
\mathcal{L}^0 &= \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y} \\
\mathcal{L}^1 &= \sqrt{2}\rho\nu\left(\sigma(y)S\frac{\partial^2}{\partial S\partial y} - \frac{\mu}{\sigma(y)}\frac{\partial}{\partial y}\right) \\
\mathcal{L}^2 &= \frac{\partial}{\partial t} + \frac{1}{2}\sigma(y)^2 S^2 \frac{\partial^2}{\partial S^2}
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\end{aligned}$$

- Collecting terms $\mathcal{O}(\varepsilon^{-1})$, $\mathcal{O}(\varepsilon^{-1/2})$ and $\mathcal{O}(1)$ lead us to formal expressions for $\phi^{(0)}$, $\phi^{(1)}$, etc.

Zeroth and First Order terms

- **Proposition:** The first two terms in the expansion for ϕ are

$$\begin{aligned} \phi^{(0)} &= \gamma\pi^{(0)}(S,t) - \frac{\mu^2}{2\sigma_*^2}(T-t) \\ \phi^{(1)} &= -\gamma(T-t)\left[c_1 S^3 \pi^{(0)}_{SSS}(S,t) + c_2 S^2 \pi^{(0)}_{SS}(S,t) + \frac{\mu^3 c_3}{\gamma}\right] \end{aligned}$$

and satisfy

$$|\phi(S,y,t) - (\phi^{(0)}(S,t) + \sqrt{\varepsilon}\phi(S,t)| = \mathcal{O}(\varepsilon).$$

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  and satisfy

$$|\phi(S,y,t) - (\phi^{(0)}(S,t) + \sqrt{\varepsilon}\phi(S,t)| = \mathcal{O}(\varepsilon).$$

- Here $\pi^{(0)}$ is the indifference price for the same contract under a constant volatility $\bar{\sigma}^2 = \langle\sigma^2\rangle$, where $\langle\cdot\rangle$ denotes the mean with respect to the invariant distribution of Y_t.