Math Finan Econ
DOI 10.1007/s11579-012-0071-8

An analysis of the Keen model for credit expansion, asset price bubbles and financial fragility

M. R. Grasselli · B. Costa Lima

Received: 16 November 2011 / Accepted: 20 March 2012

Abstract We analyze the system of differential equations proposed by Keen to model Minsky's financial instability hypothesis. We start by describing the properties of the well-known Goodwin model for the dynamics of wages and employment. This is followed by Keen's extension to include investment financed by debt. We determine the several possible equilibria and study their local stability, discussing the economical interpretation behind each condition. We then propose a modified extension that includes the role of a Ponzi speculator and investigate its effect on the several equilibria and their stability. All models are amply illustrated with numerical examples portraying their various properties.

Keywords Minsky's financial instability hypothesis · Goodwin model · Keen model · Credit cycles · Dynamical systems

JEL Classification C02 · E10 · E22 · G01

1 Introduction

In what is arguably the best book ever written on financial crises, Kindleberger and Aliber [12] state that "most increases in the supply of credit do not lead to a mania—but nearly every mania has been associated with rapid growth in the supply of credit to a particular group of borrowers". Yet most economic models based on general equilibrium do not take into account the central role played by the availability of credit in the economy. This is explicitly acknowledged by Arrow and Hahn [2] who explain that their earlier work assumes that "at the moment an equilibrium was shown to exist, economic agents had no commitments left from the past", that is, there are no debts. Instead of relying on general equilibrium analysis, Kindleberger and Aliber find solace in the work of Hyman Minsky as a theoretical framework to make sense of the massive amount of empirical evidence reported in their book.

M. R. Grasselli (✉) · B. Costa Lima
Department of Mathematics, McMaster University, Hamilton, ON L8S 4K1, Canada,
e-mail: grasselli@math.mcmaster.ca

Published online: 03 April 2012

Minsky's Financial Instability Hypothesis, described in numerous essays [13], links the expansion of credit with the rise of asset prices and the inherent fragility of the financial system. Despite his use of a persuasive verbal style aided by convincing diagrams and incisive exploration of data, Minsky refrained from presenting his ideas as a mathematical model. This task was taken up by Keen [9], where a system of differential equations is proposed as a simplified model incorporating the basic features of Minsky's hypothesis. The purpose of the present paper is to provide a thorough mathematical analysis of the original Keen model and one an extension that incorporates speculation financed by debt [11].

We set the stage in Sect. 2 by describing the Goodwin model for dynamic evolution of wages and employment, which can be seen as a special case of the Keen model in the absence of debt. Besides fixing the notation for the subsequent sections, we make a small contribution to the literature on this model (which we briefly review in Sect. 2.3) by obtaining a Lyapunov function for this system given an arbitrary Phillips curve.

Section 3 is dedicated to the original Keen model for wages, employment and debt. We identify the equilibrium points for this three-dimensional dynamical system in terms of a general investment function κ and analyze their local stability. We find two economically meaningful equilibria, one with finite debt and strictly positive employment and another with infinite debt and zero employment, and establish that both can be locally stable.

We treat an extension of the model incorporating Ponzi speculation in Sect. 4, where we deviate from the model proposed by Keen [11] by altering the dynamics of the speculative term so that it remains positive and can be interpreted as the price of an asset that can be disposed of when it becomes worthless. We identify the equilibria for the new model and verify that Ponzi speculation has the potential to destabilize a previously stable equilibrium with finite debt. We find that two types of equilibria with infinite debt are possible, one where Ponzi speculation also becomes infinite and another where it crashes to zero, and establish mutually exclusive conditions for their local stability.

Numerical examples are provided throughout the paper to illustrate the properties of the different models. Section 5 concludes with directions for further work.

2 Goodwin model

We start with a model for wages and employment proposed by Goodwin [5]. Consider the following Leontief production function for two homogeneous factors

$$Y(t) = \min\left\{\frac{K(t)}{\nu}, a(t)L(t)\right\}. \tag{1}$$

Here Y is the total yearly output, K is the stock of capital, ν is a constant capital-to-output ratio, L is the number of employed workers, and a is the labour productivity, that is to say, the number of units of output per worker per year. All quantities are assumed to be quoted in real rather nominal terms, thereby already incorporating the effects of inflation, and are net quantities, meaning that intermediate revenues and expenditures are deducted from the final yearly output. Let the total labour force be given by

$$N(t) = N_0 e^{\beta t} \tag{2}$$

and define the employment rate by

$$\lambda(t) = \frac{L(t)}{N(t)} \tag{3}$$

Furthermore, let the labour productivity be

$$a(t) = a_0 e^{\alpha t}. \tag{4}$$

Finally, assume full capital utilization, so that

$$Y(t) = \frac{K(t)}{\nu} = a(t)L(t). \tag{5}$$

In addition to (1)–(5), Goodwin makes two key behavioural assumptions. The first is that the rate of change in real wages is a function of the employment rate. Specifically, denoting real wages per unit of labour by w, Goodwin assumes that

$$\dot{\mathrm{w}} = \Phi(\lambda)\mathrm{w} \tag{6}$$

where $\Phi(\lambda)$ is an increasing function known as the Phillips curve. The second key assumption is known as Say's law and states that all wages are consumed and all profits are reinvested, so that the change in capital is given by

$$\dot{K} = (Y - \mathrm{w}L) - \delta K = (1-\omega)Y - \delta K \tag{7}$$

where δ is a constant depreciation rate and ω is the wage share of the economy defined by

$$\omega(t) := \frac{\mathrm{w}(t)L(t)}{a(t)L(t)} = \frac{\mathrm{w}(t)}{a(t)}. \tag{8}$$

It then follows from (5) and (7) that the growth rate for the economy in this model is given by

$$\frac{\dot{Y}}{Y} = \frac{1-\omega}{\nu} - \delta := g(\omega). \tag{9}$$

Using (4), (6) and (8), we conclude that the wage share evolves according to

$$\frac{\dot{\omega}}{\omega} = \frac{\dot{\mathrm{w}}}{\mathrm{w}} - \frac{\dot{a}}{a} = \Phi(\lambda) - \alpha. \tag{10}$$

Similarly, it follows from (2), (4), (3) and (9) that the dynamics for the employment rate is

$$\frac{\dot{\lambda}}{\lambda} = \frac{\dot{Y}}{Y} - \frac{\dot{a}}{a} - \frac{\dot{N}}{N} = \frac{1-\omega}{\nu} - \alpha - \beta - \delta. \tag{11}$$

Combining (10) and (11) we arrive at the following two-dimensional system of differential equations:

$$\begin{aligned} \dot{\omega} &= \omega[\Phi(\lambda) - \alpha] \\ \dot{\lambda} &= \lambda\left[\frac{1-\omega}{\nu} - \alpha - \beta - \delta\right] \end{aligned} \tag{12}$$

2.1 Properties

In Goodwin's original article [5] the Phillips curve is taken to be the linear relationship

$$\Phi(\lambda) = -\phi_0 + \phi_1\lambda, \tag{13}$$

for positive constants ϕ_0, ϕ_1, so that the system (12) reduces to the Lotka–Volterra equations describing the dynamics of a predator ω and a prey λ. Provided

$$\frac{1}{\nu} - \alpha - \beta - \delta > 0, \tag{14}$$

it is well known (see for example [8]) that the trivial equilibrium $(\overline{\omega}_0, \overline{\lambda}_0) = (0, 0)$ is a saddle point, whereas the only non-trivial equilibrium

$$(\overline{\omega}_1, \overline{\lambda}_1) = \left(1 - \nu(\alpha + \beta + \delta), \frac{\alpha + \phi_0}{\phi_1}\right) \tag{15}$$

is non-hyperbolic. Moreover, solution curves with initial conditions in the positive quadrant are periodic orbits centred at $(\overline{\omega}_1, \overline{\lambda}_1)$.

One obvious drawback of the model is that is does not constrain the variables ω and λ to remain in the unit square, as should be the case given their economic interpretation. In the next section we will drop Say's law as an assumption and replace (7) with a more general investment function allowing for external financing in the form of debt. Accordingly, the wage share of economic output can exceed unity, so there is no need to impose a constraint on ω. The employment rate λ, however, still needs to satisfy $0 \leq \lambda(t) \leq 1$ for all times. As shown in [4] this can be achieved by taking the Phillips curve to be a continuously differentiable function Φ on $(0, 1)$ satisfying

$$\Phi'(\lambda) > 0 \text{ on } (0, 1) \tag{16}$$

$$\Phi(0) < \alpha \tag{17}$$

$$\lim_{\lambda \to 1^-} \Phi(\lambda) = \infty. \tag{18}$$

It can then be verified again that $(\overline{\omega}_0, \overline{\lambda}_0) = (0, 0)$ is a saddle point and that the non-trivial equilibrium

$$(\overline{\omega}_1, \overline{\lambda}_1) = \big(1 - \nu(\alpha + \beta + \delta), \Phi^{-1}(\alpha)\big) \tag{19}$$

is non-hyperbolic. Using separation of variables and integrating the equation for $d\lambda/d\omega$, we find that the solution passing through the initial condition (ω_0, λ_0) satisfies the equation

$$\left(\frac{1}{\nu} - \alpha - \beta - \delta\right) \log \frac{\omega}{\omega_0} - \frac{1}{\nu}(\omega - \omega_0) = -\alpha \log \frac{\lambda}{\lambda_0} + \int_{\lambda_0}^{\lambda} \frac{\Phi(s)}{s} ds. \tag{20}$$

It follows that

$$V(\omega, \lambda) = (\omega + (\nu(\alpha + \beta + \gamma) - 1) \log \omega) + \nu \left(\int_{\lambda_0}^{\lambda} \frac{\Phi(s)}{s} - \alpha \log \lambda \right) \tag{21}$$

is a Lyapunov function associated to the system. In fact, $V(\omega_0, \lambda_0)$ is a constant of motion, since

$$\frac{dV}{dt} = \nabla V \cdot (\dot{\omega}, \dot{\lambda}) = 0, \tag{22}$$

so that solutions starting at $(\omega_0, \lambda_0) \in (0, \infty) \times (0, 1)$ remain bounded and satisfy $0 < \lambda < 1$ because of condition (18). Moreover, conditions (16) and (17) guarantee that the right-hand side of (20) has exactly one critical point at $\overline{\lambda}_1 = \Phi^{-1}(\alpha)$ in $(0, 1)$, so that any line of the form $\omega = p$ intersects it at most twice, which shows that the solution curves above do not spiral and are therefore closed bounded orbits around the equilibrium $(\overline{\omega}_1, \overline{\lambda}_1)$.

Interestingly, the growth rate for the economy at the equilibrium point $(\overline{\omega}_1, \overline{\lambda}_1)$ is given by

$$g(\overline{\omega}_1) = \frac{1 - \overline{\omega}_1}{\nu} - \delta = \alpha + \beta, \tag{23}$$

which is the sum of the population and productivity growth rates.

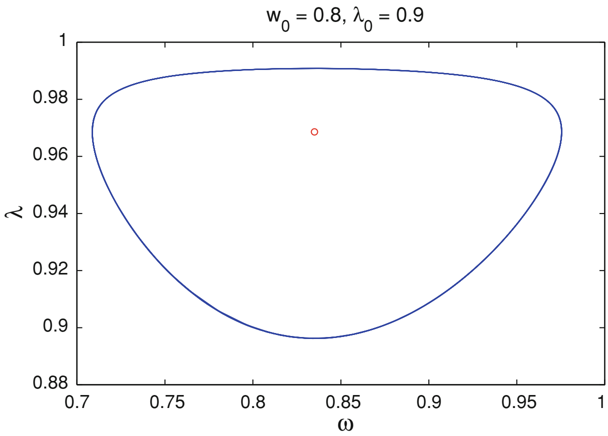

Fig. 1 Employment rate versus wage share in the Goodwin model

2.2 Example

We choose the fundamental economic constants to be

$$\alpha = 0.025, \quad \beta = 0.02, \quad \delta = 0.01, \quad \nu = 3 \tag{24}$$

and, following [9], take the Phillips curve to be

$$\Phi(\lambda) = \frac{\phi_1}{(1-\lambda)^2} - \phi_0, \tag{25}$$

with constants

$$\phi_0 = \frac{0.04}{1-0.04^2} \qquad \phi_1 = \frac{0.04^3}{1-0.04^2} \tag{26}$$

so that $\Phi(0.96) = 0$ and $\Phi(0) = -0.04$, and Eqs. (16)–(18) are satisfied. It is then easy to see that the trajectories are the closed orbits given by

$$\left(\frac{1}{\nu} - \alpha - \beta - \delta\right) \log \frac{\omega}{\omega_0} - \frac{1}{\nu}(\omega - \omega_0)$$
$$= (\phi_1 - \phi_0 - \alpha) \log \frac{\lambda}{\lambda_0} - \phi_1 \log \frac{1-\lambda}{1-\lambda_0} + \phi_1 \left(\frac{\lambda - \lambda_0}{(1-\lambda)(1-\lambda_0)}\right)$$

around the equilibrium point

$$(\overline{\omega}_1, \overline{\lambda}_1) = (0.8350, 0.9686) \tag{27}$$

as shown in the phase portrait in Fig. 1 for specific initial conditions (ω_0, λ_0). The cyclical behaviour of the model can be seen in Fig. 2, where we also plot the total output Y as a function of time, showing a clear growth trend with rate

$$g(\overline{\omega}_1, \overline{\lambda}_1) = 0.045 \tag{28}$$

but subject to the underlying fluctuations in wages and employment.

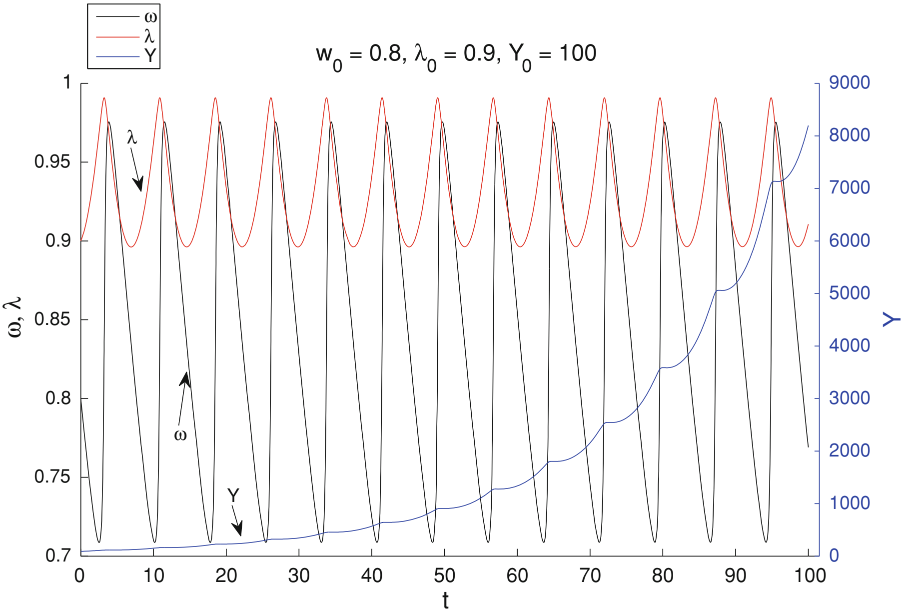

Fig. 2 Employment rate, wages, and output over time in the Goodwin model

2.3 Criticisms and extensions

Several extensions of the basic Goodwin model have demonstrated its structural instability, in the sense that small perturbations of the vector field in (12) change the qualitative properties of its solution. For example, Desai [3] shows that expressing wages in nominal rather than real terms (as it is common in the literature related to the Phillips curve) turns the non-hyperbolic equilibrium into a stable sink with trajectories spiralling towards it; whereas relaxing the assumption of a constant capital to output ratio ν leads to two non-trivial equilibrium points. In a different direction, van der Ploeg [16] shows that introducing some degree of substitutability between labour and capital in the form of a more general constant elasticity of substitution (CES) production function also leads to a locally stable equilibrium, whereas Goodwin himself [6] showed that allowing labour productivity to depend pro-cyclically on capital leads to unbounded oscillations, and the relative strength of both effects were analyzed by Aguiar-Conraria [1].

These and other extensions are reviewed by Veneziani and Mohun [17], where it is argued that instead of being a shortcoming, the structural instability of the Goodwin model can be used to analyze the "factors that determine the fragility of the basic mechanism". They go on to say that under this interpretation, the "perturbed models describe different economic and institutional environments with different implications for the outcome of distributive conflict". Under this interpretation, the proposed extensions also help to explain the poor fitting of the original model to data as reported by Solow [14] and Harvie [7], with structural changes being responsible for the observed long run phase portraits for (λ, ω), which shows an overall tendency towards cycles, but nothing resembling the closed trajectories implied by the Goodwin model.

In the remainder of this paper we will describe and analyze yet another extension of the Goodwin model, which maintains the simplicity of all the basic assumptions of the original model but relaxes it in one crucial way: Say's law is not required to hold. In other words, new investment does not need to equal profits but can be financed by debt. As we shall see, this leads to a much richer set of possible outcomes, likely to tell a more compelling economic story and provide a better description of observed data.

3 Keen model

The extension of the basic Goodwin model proposed by Keen [9] consists of introducing a banking sector to finance new investments. Denoting by D the amount of debt in real terms, the net profit after paying wages and interest on debt is

$$(1-\omega-rd)Y \tag{29}$$

where r is a constant real interest rate and $d = D/Y$ is the debt ratio in the economy. If capitalists reinvested all this net profit and nothing more, debt levels would remain constant over time. The key insight provided by Minsky [13] is that current cash-flows validate past liabilities and form the basis for future ones. In other words, high net profits lead to more borrowing whereas low net profits (possibly negative) lead to a deleveraging of the economy. Keen [9] formalizes this insigh by taking the change in capital stock to be

$$\dot{K} = \kappa(1-\omega-rd)Y - \delta K \tag{30}$$

where the rate of new investment is a nonlinear increasing function κ of the net profit share $\pi = (1-\omega-rd)$ and δ is a constant depreciation rate as before. Accordingly, total output evolves as

$$\frac{\dot{Y}}{Y} = \frac{\kappa(1-\omega-rd)}{\nu} - \delta := g(\omega, d) \tag{31}$$

and the employment rate dynamics becomes

$$\frac{\dot{\lambda}}{\lambda} = \frac{\kappa(1-\omega-rd)}{\nu} - \alpha - \beta - \delta, \tag{32}$$

whereas the time evolution for the wage share remains (8).

The new dynamic variable in this model is the amount of debt, which changes based on the difference between new investment and net profits. In other words, we have that

$$\dot{D} = \kappa(1-\omega-rd)Y - (1-\omega-rd)Y \tag{33}$$

whence it follows that

$$\frac{\dot{d}}{d} = \frac{\dot{D}}{D} - \frac{\dot{Y}}{Y} = \frac{\kappa(1-\omega-rd) - (1-\omega-rd)}{d} - \frac{\kappa(1-\omega-rd)}{\nu} + \delta. \tag{34}$$

Combining (8), (32), (34) we arrive at the following three-dimensional system of autonomous differential equations:

$$\begin{aligned}
\dot{\omega} &= \omega\left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda} &= \lambda\left[\frac{\kappa(1-\omega-rd)}{\nu} - \alpha - \beta - \delta\right] \\
\dot{d} &= d\left[r - \frac{\kappa(1-\omega-rd)}{\nu} + \delta\right] + \kappa(1-\omega-rd) - (1-\omega)
\end{aligned} \tag{35}$$

For the analysis that follows, we assume henceforth that the rate of new investment in (30) is a continuously differentiable function κ satisfying

$$\kappa'(\pi) > 0 \text{ on } (-\infty, \infty) \tag{36}$$

$$\lim_{\pi\to-\infty} \kappa(\pi) = \kappa_0 < \nu(\alpha+\beta+\delta) < \lim_{\pi\to+\infty} \kappa(\pi) \tag{37}$$

$$\lim_{\pi\to-\infty} \pi^2\kappa'(\pi) = 0. \tag{38}$$

3.1 Equilibria in the Keen model

We see that

$$(\overline{\omega}_0, \overline{\lambda}_0, \overline{d}_0) = (0, 0, \overline{d}_0), \tag{39}$$

where $\overline{d}_0$ is any solution of the equation

$$d\left[r - \frac{\kappa(1-rd)}{\nu} + \delta\right] + \kappa(1-rd) - 1 = 0, \tag{40}$$

is an equilibrium point for (35). Equilibria of the form (39) are economically meaningless, and we expect them to be unstable in the same way that $(\overline{\omega}_0, \overline{\lambda}_0) = (0, 0)$ is saddle point in the original Goodwin model.

For a more meaningful equilibrium, observe that it follows from (37) that $\nu(\alpha+\beta+\delta)$ is in the image of κ so that we can define

$$\overline{\pi}_1 = \kappa^{-1}(\nu(\alpha+\beta+\delta)) \tag{41}$$

and verify by direct substitution that the point

$$\begin{aligned} \overline{\omega}_1 &= 1 - \overline{\pi}_1 - r\frac{\nu(\alpha+\beta+\delta) - \overline{\pi}_1}{\alpha+\beta} \\ \overline{\lambda}_1 &= \Phi^{-1}(\alpha) \\ \overline{d}_1 &= \frac{\nu(\alpha+\beta+\delta) - \overline{\pi}_1}{\alpha+\beta} \end{aligned} \tag{42}$$

satisfies the relation

$$1 - \overline{\omega}_1 - r\overline{d}_1 = \overline{\pi}_1 \tag{43}$$

and is an equilibrium for (35). This equilibrium corresponds to a finite level of debt and strictly positive employment rate and is therefore economically desirable, so in the next section we shall investigate conditions guaranteeing that it is locally stable. As with the Goodwin model, it is interesting to note that the growth rate of the economy at this equilibrium point is given by

$$g(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1) = \frac{\kappa(1-\overline{\omega}_1 - r\overline{d}_1)}{\nu} - \delta = \alpha + \beta. \tag{44}$$

We can obtain yet another set of equilibrium points by setting $\omega = 0$ and

$$1 - rd = \overline{\pi}_1 = \kappa^{-1}(\nu(\alpha+\beta+\delta)) \tag{45}$$

so that $\dot{\omega} = \dot{\lambda} = 0$ in (35) regardless of the value of λ. However, to have $\dot{d} = 0$ as well we must have $d = \overline{d}_1$ as before. But this can only be satisfied simultaneously with (45) if the model parameters satisfy the following very specific condition

$$1 - r\frac{\nu(\alpha+\beta+\delta) - \kappa^{-1}(\nu(\alpha+\beta+\delta))}{\alpha+\beta} = \kappa^{-1}(\nu(\alpha+\beta+\delta)). \tag{46}$$

Provided (46) holds, we have that points on the line $(0, \lambda, \overline{d}_1)$ are equilibria for (35) for any value $0 < \lambda < 1$. We see that equilibria of this form are not only economically meaningless, but are also structurally unstable, since a small change in the model parameters leading to a violation of (46) makes them disappear.

Finally, if we rewrite the system with the change of variables $u = 1/d$, we obtain

$$\begin{aligned}
\dot{\omega} &= \omega\left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda} &= \lambda\left[\frac{\kappa(1-\omega-r/u)}{\nu} - \alpha - \beta - \delta\right] \\
\dot{u} &= u\left[\frac{\kappa(1-\omega-r/u)}{\nu} - r - \delta\right] - u^2\left[\kappa(1-\omega-r/u) - (1-\omega)\right].
\end{aligned} \tag{47}$$

We now see that $(0, 0, 0)$ is an equilibrium of (47) corresponding to the point

$$(\overline{\omega}_2, \overline{\lambda}_2, \overline{d}_2) = (0, 0, +\infty) \tag{48}$$

for the original system. This equilibrium for (47) corresponds to the economically undesirable but nevertheless important situation of a collapse in wages and employment when the economy as a whole becomes overwhelmed by debt, rendering of paramount importance to investigate its local stability. Observe that condition (37) guarantees that

$$\kappa(1-\omega-r/u) \to \kappa_0 \tag{49}$$

as $\omega \to 0$ and $u \to 0^+$, so the vector field for (47) remains finite on trajectories approaching $(0, 0, 0)$ along positive values of u.

3.2 Local stability in the Keen model

Denoting $\pi = 1 - \omega - rd$, the Jacobian for (35) is

$$J(\omega, \lambda, d) = \begin{bmatrix} \Phi(\lambda) - \alpha & \omega\Phi'(\lambda) & 0 \\ -\frac{\lambda\kappa'(\pi)}{\nu} & \frac{\kappa(\pi)-\nu(\alpha+\beta+\delta)}{\nu} & -\frac{r\lambda\kappa'(\pi)}{\nu} \\ \frac{(d-\nu)\kappa'(\pi)+\nu}{\nu} & 0 & \frac{\nu(r+\delta)-\kappa(\pi)+r(d-\nu)\kappa'(\pi)}{\nu} \end{bmatrix} \tag{50}$$

At the equilibrium point $(0, 0, \overline{d}_0)$ this reduces to the lower triangular matrix

$$J(0, 0, \overline{d}_0) = \begin{bmatrix} \Phi(0) - \alpha & 0 & 0 \\ 0 & \frac{\kappa(\overline{\pi}_0)-\nu(\alpha+\beta+\delta)}{\nu} & 0 \\ \frac{(\overline{d}_0-\nu)\kappa'(\overline{\pi}_0)+\nu}{\nu} & 0 & \frac{\nu(r+\delta)-\kappa(\overline{\pi}_0)+r(\overline{d}_0-\nu)\kappa'(\overline{\pi}_0)}{\nu} \end{bmatrix} \tag{51}$$

where $\overline{\pi}_0 = 1 - r\overline{d}_0$. Its real eigenvalues are given by the diagonal entries, and it is hard to determine their sign a priori since $\overline{d}_0$ is given as the solution of Eq. (40). Although they can be readily determined once specific parameters are chosen, we observe that these equilibrium points are likely to be unstable, since a sufficiently large value of $\overline{\pi}_0$ makes the second diagonal term above positive, whereas a sufficiently small value of $\overline{\pi}_0$ (and correspondingly large value of $\overline{d}_0$) makes the third diagonal term above positive.

At the equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1)$ the Jacobian takes the interesting form

$$J(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1) = \begin{bmatrix} 0 & K_0 & 0 \\ -K_1 & 0 & -rK_1 \\ K_2 & 0 & rK_2 - (\alpha+\beta) \end{bmatrix} \tag{52}$$

where

$$
\begin{aligned}
K_0 &= \overline{\omega}_1 \Phi'(\overline{\lambda}_1) > 0 \\
K_1 &= \frac{\overline{\lambda}_1 \kappa'(\overline{\pi}_1)}{\nu} > 0 \\
K_2 &= \frac{(\overline{d}_1 - \nu)\kappa'(\overline{\pi}_1) + \nu}{\nu}
\end{aligned}
\tag{53}
$$

Therefore, the characteristic polynomial for the matrix in (52) is

$$
p_3(y) = y^3 + [(\alpha+\beta) - rK_2]y^2 + K_0K_1 y + K_0K_1(\alpha+\beta). \tag{54}
$$

According to the Routh-Hurwitz criterion, a necessary and sufficient condition for all the roots of a cubic polynomial of the form

$$
p(y) = a_3y^3 + a_2y^2 + a_1y + a_0 \tag{55}
$$

to have negative real parts is

$$
a_n > 0, \forall n \text{ and } a_2a_1 > a_3a_0. \tag{56}
$$

Our characteristic polynomial already has three of its coefficients positive; therefore all we need is

$$
(\alpha+\beta) > rK_2 \tag{57}
$$

and

$$
((\alpha+\beta) - rK_2)K_0K_1 > K_0K_1(\alpha+\beta) \tag{58}
$$

Since we are already assuming that $\alpha > 0$ and $\beta > 0$, we see that the equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1)$ is stable if and only if $rK_2 < 0$, which is equivalent to

$$
r\left[\frac{\kappa'(\overline{\pi}_1)}{\nu}\big(\overline{\pi}_1 - \kappa(\overline{\pi}_1) + \nu(\alpha+\beta)\big) - (\alpha+\beta)\right] > 0. \tag{59}
$$

Because the real interest rate r can have any sign, condition (59) needs to be checked in each implementation of the model. Observe, however, that once the sign of r is chosen, the remaining terms are all independent of the magnitude of the interest rate. For the most common situation of $r > 0$, condition (59) imposes interesting constraints on the investment function κ. The intuition behind (30) is that capitalists invest more than their net profits during a boom, indicating that $\pi < \kappa(\pi)$ for high enough π. But (59) states that for $\overline{\pi}_1$ to correspond to a stable equilibrium we must have $\kappa(\overline{\pi}_1) - \overline{\pi}_1 < \nu(\alpha+\beta)$, suggesting an upper bound for how much more than the net profit should be invested, while at the same time $\kappa'(\overline{\pi}_1)$ needs to be sufficiently large, leading to a rapid ramp-up of investment for net profits beyond $\overline{\pi}_1$.

As we mentioned in the previous section, equilibria of the form $(0, \lambda, \overline{d}_1)$ depend on a very specific choice of parameters satisfying (46), making them structurally unstable. Therefore we are not going to discuss them any further, except by verifying that they do not arise for the parameters used in the numerical example implemented later.

Finally, regarding the point $(\overline{\omega}_2, \overline{\lambda}_2, \overline{d}_2) = (0, 0, +\infty)$, observe that the Jacobian for the modified system (47) is

$$
J(\omega,\lambda,u) = \begin{bmatrix}
\Phi(\lambda) - \alpha & \omega\Phi'(\lambda) & 0 \\
-\frac{\lambda\kappa'(\pi)}{\nu} & \frac{\kappa(\pi) - \nu(\alpha+\beta+\delta)}{\nu} & -\frac{r\lambda\kappa'(\pi)}{u^2\nu} \\
\frac{(\nu u^2 - u)\kappa'(\pi) - \nu u^2}{\nu} & 0 & \frac{\kappa(\pi)(1-2u) + r\kappa'(\pi)(1/u - 1) + 2u\nu(1-\omega) - \nu(r+\delta)}{\nu}
\end{bmatrix}
$$

where $\pi = 1 - \omega - r/u$. At the equilibrium $(\overline{\omega}_2, \overline{\lambda}_2, \overline{u}_2) = (0, 0, 0)$, conditions (37) and (38) ensure that this reduces to the diagonal matrix

$$J(0,0,0) = \begin{bmatrix} \Phi(0) - \alpha & 0 & 0 \\ 0 & \frac{\kappa_0 - \nu(\alpha+\beta+\delta)}{\nu} & 0 \\ 0 & 0 & \frac{\kappa_0 - \nu(r+\delta)}{\nu} \end{bmatrix}, \tag{60}$$

from which all real eigenvalues can be readily obtained. Observe that the first two eigenvalues are negative by virtue of conditions (16) and (36) on the functions Φ and κ, so this equilibrium is stable if and only if

$$\frac{\kappa_0}{\nu} - \delta < r. \tag{61}$$

Recalling expression (31), we see that this equilibrium is stable if and only if the real interest rate exceeds the growth rate of the economy at infinite levels of debt and zero wages.

It is interesting to note that assumptions (16) and (36) were made in order to guarantee the existence of the economically desirable equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1)$, but perversely contribute to the stability of the undesirable point $(\overline{\omega}_2, \overline{\lambda}_2, \overline{d}_2) = (0, 0, +\infty)$. Moreover, in view of (36), we see that a sufficient condition for (61) to hold is

$$\alpha + \beta < r. \tag{62}$$

Recalling (44) we conclude that a sufficient condition for $(\overline{\omega}_2, \overline{\lambda}_2, \overline{u}_2) = (0, 0, 0)$ to be a locally stable equilibrium for (47) is that the real interest rate r exceeds the growth rate of the economy at the equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1)$, which resembles the condition derived by Tirole [15] for the absence of rational bubbles in an overlapping generation model, corresponding to an "efficient" economy.

3.3 Example

Choosing the fundamental economic constants to be the same as in (24) with the addition of

$$r = 0.03, \tag{63}$$

taking the Phillips curve as in (25) and (26), and, following [10], taking an investment function κ of the form

$$\kappa(x) = \kappa_0 + \kappa_1 e^{\kappa_2 x}, \tag{64}$$

for constants

$$\kappa_0 = -0.0065, \quad \kappa_1 = e^{-5}, \quad \kappa_2 = 20, \tag{65}$$

it follows that conditions (36)–(38) are satisfied.

Observe first that the real solutions for Eq. (40) in this case are

$$\overline{d}_0 = \begin{cases} 2.999995114 \\ 30.76701645 \end{cases}. \tag{66}$$

The eigenvalues for $J(0, 0, \overline{d}_0)$ for these two points are

$$\begin{aligned} &(-0.065, 1.801221163 \times 10^5, -1.801226593 \times 10^5) \\ &(-0.065, -0.04669227782, 0.2061977943), \end{aligned}$$

confirming that the equilibrium $(0, 0, \overline{d}_0)$ is unstable in either case, as expected.

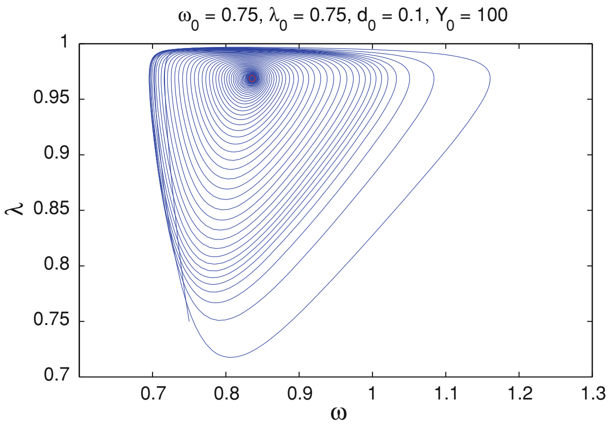

Fig. 3 Phase portrait of employment and wages converging to a stable equilibrium with finite debt in the Keen model

Moving to the economically meaningful equilibrium, we obtain

$$\overline{\pi}_1 = \frac{1}{\kappa_2} \log\left(\frac{\nu(\alpha+\beta+\delta)-\kappa_0}{\kappa_1}\right) = 0.1618, \tag{67}$$

which leads to equilibrium values

$$(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1) = (0.8361, 0.9686, 0.0702). \tag{68}$$

Moreover, we find that

$$\frac{\kappa'(\overline{\pi}_1)}{\nu}\big(\overline{\pi}_1 - \kappa(\overline{\pi}_1) + \nu(\alpha+\beta)\big) - (\alpha+\beta) = 0.105738666 > 0, \tag{69}$$

so that (59) is satisfied and this equilibrium is locally stable. When the initial conditions are chosen sufficiently close to the equilibrium values, we observe the convergent behaviour shown in the phase portrait for employment and wages in Fig. 3. The oscillatory behaviour of all variables can be seen in Fig. 4, where we also show the growing output Y as a function of time.

We notice that condition (46) is violated by our model parameters, so we do not need to consider the structurally unstable equilibria of the form $(0, \lambda, \overline{d}_1)$. Moving on to the equilibrium with infinite debt, we observe that

$$\frac{\kappa_0}{\nu} - \delta - r = -0.0421666 < 0, \tag{70}$$

so that (61) is satisfied and $(\overline{\omega}_2, \overline{\lambda}_2, \overline{d}_2) = (0, 0, +\infty)$ corresponds to a stable equilibrium of (47). Therefore we expect to observe ever increasing debt levels when the initial conditions are sufficiently far from the equilibrium values $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1)$. This is depicted in Fig. 5, where we can see both wages and employment collapsing to zero while debt explodes to infinity. We also show the output Y which increases to very high levels propelled by the increasing debt before starting an inexorable descent.

While it is difficult to determine the basin of convergence for the equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1)$ analytically, we plot in Fig. 6 the set of initial conditions for which we observed con-

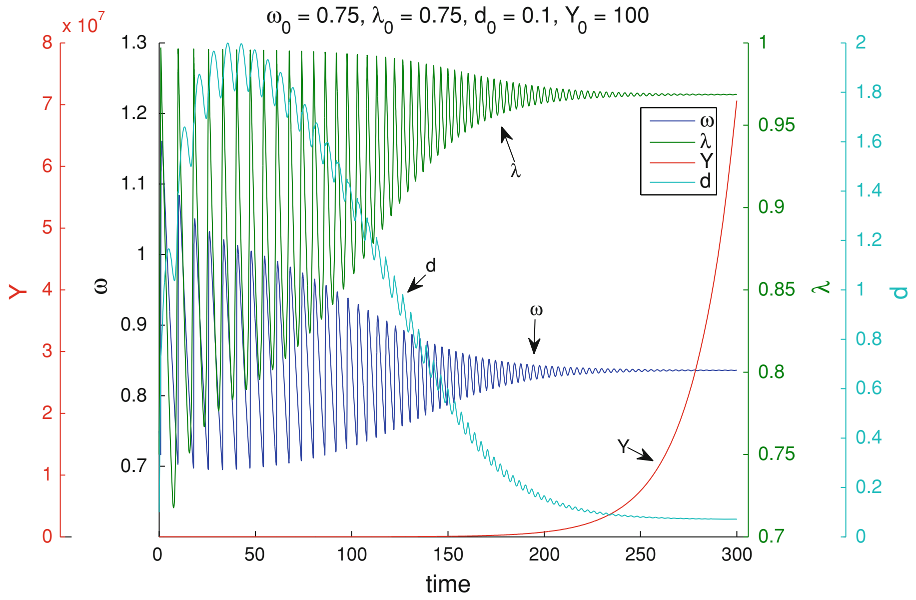

Fig. 4 Employment, wages, debt and output as functions of time converging to a stable equilibrium with finite debt in the Keen model

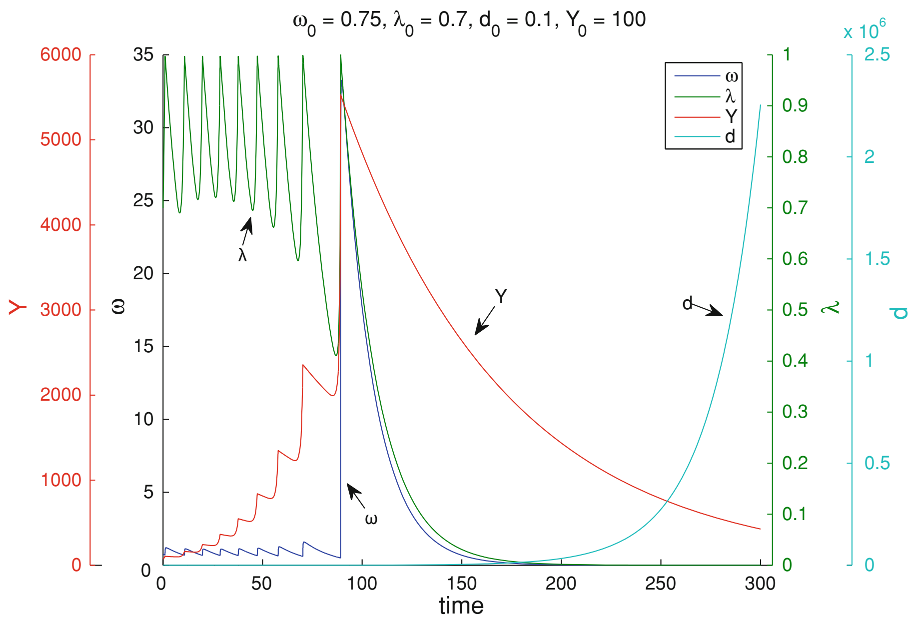

Fig. 5 Employment, wages, debt and output as functions of time converging to a stable equilibrium with infinite debt in the Keen model

vergence to this equilibrium numerically. As expected, the set of initial values for wages and employment leading to convergence becomes smaller as the initial value for debt increases.

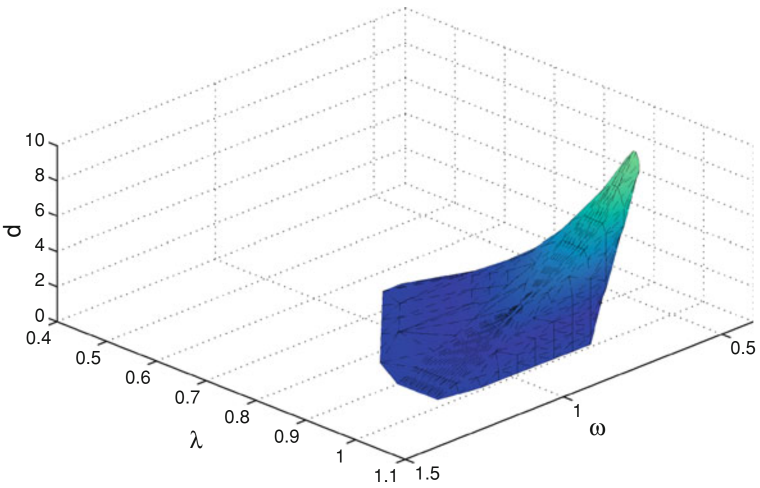

Fig. 6 Basin of convergence for the Keen model for wages, employment and private debt

4 Ponzi financing

Despite the existence of an equilibrium with infinite debt, the model of the previous section does not capture Minsky's [13] well known conclusion that "stability—or tranquility—in a world with a cyclical past and capitalist financial institutions is destabilizing". A second extension proposed by Keen [11] consists of using debt-financing for purely speculative purposes. This is achieved by rewriting the debt evolution Eq. (33) as

$$\dot{D} = \kappa(1-\omega-rd)Y - (1-\omega-rd)Y + P, \tag{71}$$

where P corresponds to Ponzi speculation and evolves according to

$$\dot{P} = \Psi\big(g(\omega,d)\big)Y \tag{72}$$

where Ψ is an increasing function of the growth rate of the economy g defined in (31). Although Keen [11] shows that this Ponzi term introduces the desired destabilizing effect, the interpretation of the corresponding model is complicated by the fact that (72) allows for the possibility of P to become negative, which precludes it to be seen as the price of an asset that can be disposed when it becomes worthless. We therefore propose to replace (72) with

$$\dot{P} = \Psi\big(g(\omega,d)\big)P \tag{73}$$

and proceed with the analysis of the modified model.

Denoting the ratio of speculation to economic output by $p = P/Y$, it is easy to see that the additional speculative term leads to the following four-dimensional system:

$$\begin{aligned}
\dot{\omega} &= \omega\left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda} &= \lambda\left[\frac{\kappa(1-\omega-rd)}{\nu} - \alpha - \beta - \delta\right] \\
\dot{d} &= d\left[r - \frac{\kappa(1-\omega-rd)}{\nu} + \delta\right] + \kappa(1-\omega-rd) - (1-\omega) + p \\
\dot{p} &= p\left[\Psi\left(\frac{\kappa(1-\omega-rd)}{\nu} - \delta\right) - \frac{\kappa(1-\omega-rd)}{\nu} + \delta\right]
\end{aligned} \tag{74}$$

4.1 Equilibria with Ponzi speculation

This four-dimensional system has many possible equilibria, so we will restrict ourselves to the economically relevant ones. We start by observing that defining $\overline{\pi}_1$ as in (41) and $\overline{\omega}_1$,$\overline{\lambda}_1$ and $\overline{d}_1$ as in (42) leads to the equilibrium point $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1, 0)$ for (74) corresponding to an economy without Ponzi speculation, as expected.

As before, to investigate the behaviour for large values of debt we use the change of variable $u = 1/d$ leading to the modified system

$$\begin{aligned}
\dot{\omega} &= \omega\left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda} &= \lambda\left[\frac{\kappa(1-\omega-r/u)}{\nu} - \alpha - \beta - \delta\right] \\
\dot{u} &= u\left[\frac{\kappa(1-\omega-r/u)}{\nu} - r - \delta\right] - u^2\left[\kappa(1-\omega-r/u) - (1-\omega) + p\right] \\
\dot{p} &= p\left[\Psi\left(\frac{\kappa(1-\omega-r/u)}{\nu} - \delta\right) - \frac{\kappa(1-\omega-r/u)}{\nu} + \delta\right].
\end{aligned} \tag{75}$$

We then observe that $(\overline{\omega}_2, \overline{\lambda}_2, \overline{u}_2, \overline{p}_2) = (0, 0, 0, 0)$ is an equilibrium of (75) corresponding to collapsing wages and employment, exploding debt, and no Ponzi speculation.

In addition, we might wish to investigate the behaviour for large values of speculation through the further change of variables $x = 1/p$, which leads to

$$\begin{aligned}
\dot{\omega} &= \omega\left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda} &= \lambda\left[\frac{\kappa(1-\omega-r/u)}{\nu} - \alpha - \beta - \delta\right] \\
\dot{u} &= u\left[\frac{\kappa(1-\omega-r/u)}{\nu} - r - \delta\right] - u^2\left[\kappa(1-\omega-r/u) - (1-\omega) + 1/x\right] \\
\dot{x} &= x\left[-\Psi\left(\frac{\kappa(1-\omega-r/u)}{\nu} - \delta\right) + \frac{\kappa(1-\omega-r/u)}{\nu} - \delta\right]
\end{aligned} \tag{76}$$

Because of the term u^2/x above, it is not clear whether $(0, 0, 0, 0)$ is an equilibrium of (76). To remedy this, we introduce the ratio $v = \frac{p}{d} = \frac{u}{x}$ as a new variable, leading to the system

$$\begin{aligned}
\dot{\omega} &= \omega\left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda} &= \lambda\left[\frac{\kappa(1-\omega-r/(vx))}{\nu} - \alpha - \beta - \delta\right] \\
\dot{v} &= v\left[\Psi\left(\frac{\kappa(1-\omega-r/(vx))}{\nu} - \delta\right) - r\right] - v^2\left[x\kappa(1-\omega-r/(vx)) - x(1-\omega) + 1\right] \\
\dot{x} &= x\left[-\Psi\left(\frac{\kappa(1-\omega-r/(vx))}{\nu} - \delta\right) + \frac{\kappa(1-\omega-r/(vx))}{\nu} - \delta\right]
\end{aligned} \tag{77}$$

We now see that both $(0, 0, 0, 0)$ and $\left(0, 0, \Psi\left(\frac{\kappa_0}{\nu} - \delta\right) - r, 0\right)$ are equilibria for (77) since condition (37) guarantees that

$$\kappa(1-\omega-r/(vx)) \to \kappa_0 \tag{78}$$

as $\omega \to 0$ and $vx \to 0^+$, so that the vector field for (77) remains finite on trajectories approaching it along positive values of $u = vx$ as before. In terms of the original system (74) this corresponds to the points

$$(\overline{\omega}_3, \overline{\lambda}_3, \overline{d}_3, \overline{p}_3) = (0, 0, +\infty, \pm\infty), \tag{79}$$

where wages and employment collapse, while debt and Ponzi speculation become infinitely large, but with debt increasing faster than Ponzi speculation since either $\upsilon \to 0^+$ or $\upsilon \to \Psi\left(\frac{\kappa_0}{\upsilon} - \delta\right) - r$.

4.2 Modified local stability

Denoting $\pi = 1 - \omega - rd$, the Jacobian for (74) is

$$J(\omega, \lambda, d, p) =$$
$$\begin{bmatrix} \Phi(\lambda) - \alpha & \omega\Phi'(\lambda) & 0 & 0 \\ -\frac{\lambda\kappa'(\pi)}{\nu} & \frac{\kappa(\pi)-\nu(\alpha+\beta+\delta)}{\nu} & -\frac{r\lambda\kappa'(\pi)}{\nu} & 0 \\ \frac{(d-\nu)\kappa'(\pi)+\nu}{\nu} & 0 & \frac{\nu(r+\delta)-\kappa(\pi)+r(d-\nu)\kappa'(\pi)}{\nu} & 1 \\ \frac{p\kappa'(\pi)}{\nu}\left[1-\Psi'\left(\frac{\kappa(\pi)}{\nu}-\delta\right)\right] & 0 & \frac{pr\kappa'(\pi)}{\nu}\left[1-\Psi'\left(\frac{\kappa(\pi)}{\nu}-\delta\right)\right] & \Psi\left(\frac{\kappa(\pi)}{\nu}-\delta\right)-\frac{\kappa(\pi)}{\nu}+\delta \end{bmatrix}$$

At the equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1, 0)$, this matrix becomes:

$$J(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1, 0) = \begin{bmatrix} 0 & K_0 & 0 & 0 \\ -K_1 & 0 & -rK_1 & 0 \\ K_2 & 0 & rK_2 - (\alpha+\beta) & 1 \\ 0 & 0 & 0 & K_3 \end{bmatrix} \tag{80}$$

where K_0, K_1, K_2 are defined in (53) and

$$K_3 = \Psi(\alpha+\beta) - (\alpha+\beta) \tag{81}$$

Therefore, the characteristic polynomial for this matrix is

$$p_4(y) = p_3(y)(K_3 - y) \tag{82}$$

where p_3 is the third-degree polynomial defined in (54). Recalling the conditions for the roots of p_3 to be negative, if we assume that (59) holds we see that the equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1, 0)$ is stable if and only if $K_3 < 0$, which is equivalent to

$$\Psi(\alpha+\beta) < \alpha+\beta. \tag{83}$$

In other words, the presence of Ponzi speculation can destabilize an otherwise stable equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1)$ provided its growth rate exceeds the growth rate of the economy at equilibrium.

Similarly, using conditions (37) and (38) we find that the Jacobian of the modified system (75) at the point $(\omega, \lambda, u, p) = (0, 0, 0, 0)$ reduces to

$$J(0,0,0,0) = \begin{bmatrix} \Phi(0) - \alpha & 0 & 0 & 0 \\ 0 & \frac{\kappa_0-\nu(\alpha+\beta+\delta)}{\nu} & 0 & 0 \\ 0 & 0 & \frac{\kappa_0-\nu(r+\delta)}{\nu} & 0 \\ 0 & 0 & 0 & \Psi\left(\frac{\kappa_0}{\nu}-\delta\right) - \left(\frac{\kappa_0}{\nu}-\delta\right) \end{bmatrix}$$

Denoting the growth rate of the economy at infinite debt by

$$g_0 = \frac{\kappa_0}{\nu} - \delta, \tag{84}$$

we see that under (16), (37) and (61) the point $(\overline{\omega}_2, \overline{\lambda}_2, \overline{d}_2, \overline{p}_2) = (0, 0, +\infty, 0)$ corresponds to a stable equilibrium of (75) if and only if

$$\Psi(g_0) < g_0. \tag{85}$$

That is, we see again that the presence of Ponzi can destabilize an otherwise stable equilibrium $(\overline{\omega}_2, \overline{\lambda}_2, \overline{d}_2)$ provided its own growth rate when the economy is deeply depressed is large enough. But since such equilibrium is economically undesirable we might say that this particular destabilizing effect is rather welcome.

Before rejoicing in the benign consequence of Ponzi speculation in this peculiar equilibrium with zero worker income, zero employment and infinite debt, observe that the Jacobian of the modified system (77) at the point $(\omega, \lambda, v, x) = (0, 0, 0, 0)$ reduces to

$$J(0,0,0,0) = \begin{bmatrix} \Phi(0)-\alpha & 0 & 0 & 0 \\ 0 & \frac{\kappa_0-\nu(\alpha+\beta+\delta)}{\nu} & 0 & 0 \\ 0 & 0 & \Psi(g_0)-r & 0 \\ 0 & 0 & 0 & -\Psi(g_0)+g_0 \end{bmatrix} \tag{86}$$

whereas at $(\omega, \lambda, v, x) = (0, 0, \Psi(g_0) - r, 0)$ we have

$$J(0,0,\Psi(g_0)-r,0) = \begin{bmatrix} \Phi(0)-\alpha & 0 & 0 & 0 \\ 0 & \frac{\kappa_0-\nu(\alpha+\beta+\delta)}{\nu} & 0 & 0 \\ 0 & 0 & -\Psi(g_0)+r & -(\Psi(g_0)-r)^2(\kappa_0-1) \\ 0 & 0 & 0 & -\Psi(g_0)+g_0 \end{bmatrix} \tag{87}$$

Therefore, under (16) and (37), the point $(\overline{\omega}_3, \overline{\lambda}_3, \overline{d}_3, \overline{p}_3) = (0, 0, +\infty, +\infty)$ corresponds to a stable equilibrium for (77) provided

$$g_0 < \Psi(g_0). \tag{88}$$

Observe that no matter whether $\Psi(g_0) < r$ or $\Psi(g_0) > r$, this equilibrium remains stable. The only difference is that the ratio $v = p/d$ will be attracted to $\Psi(g_0) - r$, instead of 0, once $\Psi(g_0)$ becomes larger than r. In either situation, v remains positive, guaranteeing that both $d \to +\infty$ and $p \to +\infty$.

Notice that if we assume (16), (37) and (61), then the stability of both type of equilibria with infinitely large debt is entirely determined by the growth rate of Ponzi speculation when the economy is at its worst. The possibilities for $(\omega, \lambda, d, p) = (0, 0, +\infty, 0)$ or $(0, 0, +\infty, +\infty)$ are summarized as follows:

(i) if $\Psi(g_0) < g_0$ then the equilibrium with infinite debt and no Ponzi speculation is locally stable, whereas the equilibrium with infinite debt and infinite Ponzi speculation is unstable;

(ii) if $g_0 < \Psi(g_0) < r$, the stability properties of these two equilibria are reversed and $v = p/d \to 0$;

(iii) if $\Psi(g_0) > r$, the stability properties of these the two equilibria are the same as in (ii) but $v = p/d \to \Psi(g_0) - r > 0$.

4.3 Example

We now choose the same economic constants as in (24) and (63), the Phillips curve as in (25) and (26), the investment function as in (64) and (65), and, following [11], the Ponzi speculation function as

$$\Psi(x) = \Psi_0 + \Psi_1 e^{\Psi_2 x} \tag{89}$$

with constants

$$\Psi_0 = -0.25, \quad \Psi_1 = 0.25e^{-0.36}, \quad \Psi_2 = 12. \tag{90}$$

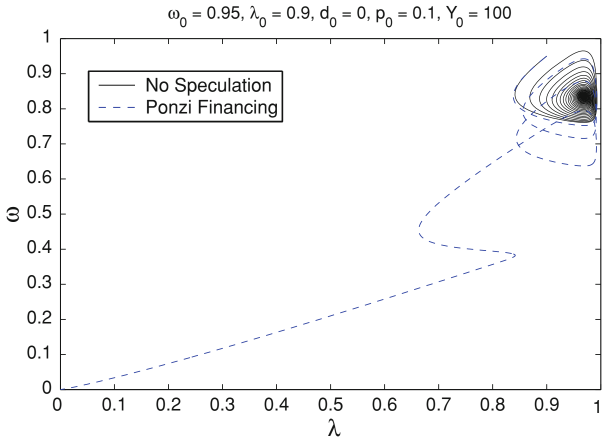

Fig. 7 Phase portraits of employment and wage share in a Keen model with and without Ponzi financing

Observe first that

$$\Psi(\alpha+\beta)-(\alpha+\beta)=0.0043, \tag{91}$$

so that (83) is violated, indicating that the presence of Ponzi speculation turned the previously stable equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1)$ into the unstable equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1, 0)$. On the other hand, we have that

$$g_0=-0.0121666, \quad \Psi(g_0)=-0.0992744071, \tag{92}$$

so that (85) is satisfied while (88) is violated. In other words, the equilibrium with infinite debt and no Ponzi speculation is locally stable, whereas the equilibrium with infinite debt and infinite Ponzi speculation is unstable.

The destabilizing effects of Ponzi speculation can be clearly seen in Figs. 7 and 8, where we compare wages, employment, debt and output with and without Ponzi financing. The phase portraits shown in Fig. 7 show wage share and employment oscillating towards a stable equilibrium when there is no Ponzi speculation, but descending to zero after an initial oscillatory period when Ponzi financing is present. This is accompanied by the explosion in debt levels shown in Fig. 8 in stark contrast with the finite debt level at equilibrium in the absence of Ponzi speculation. We can also see that Ponzi financing fuels a temporary boom before debt overwhelms the economy and makes both output and the speculation ratio collapse.

5 Conclusion

We have analyzed three different models for the dynamics of basic macroeconomic variables. In the two-dimensional Goodwin model the wage share and employment rates display the typical oscillatory behaviour observed in a predator–prey system, with closed trajectories circling around a non-hyperbolic equilibrium. Introducing debt to finance new investment leads to the three-dimensional Keen model exhibiting two distinct equilibria, a good one with finite debt and strictly positive employment and wage share, and a bad one with infinite debt and zero employment and wage share. We have determined that for typical model parameters,

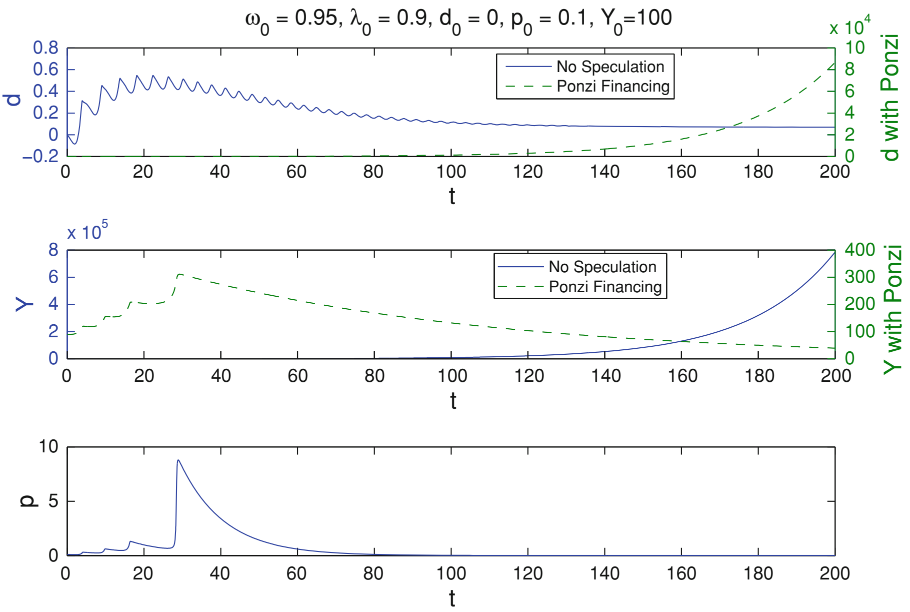

Fig. 8 Debt, output and speculation as functions of time in a Keen model with and without Ponzi financing

both can be locally stable. Introducing a purely speculative asset financed entirely by debt leads to a four-dimensional model for which the finite debt equilibrium can become unstable even when it is stable in the absence of speculation, whereas the equilibria with infinite debt can occur with either infinite or zero speculation.

As we have seen, these simple models are able to generate remarkably rich dynamics, but can still be generalized in a variety of ways. Staying in the realm of deterministic models, one possible extension already considered by Keen [9,11] consists of introducing a government sector with corresponding spending and taxation, increasing the dimensionality of model and the complexity of its outcomes.

In a different direction, instead of treating the Ponzi speculation term as an asset price itself, we could interpret it as a factor driving the intensity of downward jumps in a stochastic asset price model. Under this interpretation, when the economy does well, the Ponzi speculation factor increases and drives the asset price up through a compensator on its drift, while at the same time increasing the probability of a crash. In addition, we can replace the deterministic interest rate by a combination of a base rate plus a risk premium that increases when the asset price crashes. Extended in this way, the model becomes fully stochastic, albeit driven by the same basic structure described in this paper.

Acknowledgments We are indebted to Steve Keen, Tom Hurd, Gail Wolkowicz, and two anonymous referees for helpful comments and suggestions. In addition, we would like to thank the participants at the Workshop on Stochastic Analysis in Finance and Insurance, Oberwolfach (January 2011), the Imperial College Finance Seminar (March 2011), the Stochastic Analysis Seminar, ORFE, Princeton (April 2011), the Mathematics in Finance Conference, Kruger Park (August 2011), and the Fields Institute Quantitative Finance Seminar (October 2011) where this work was presented.

References

1. Aguiar-Conraria, L.: A note on the stability properties of Goodwin's predator–prey model. Rev. Radic. Political Econ. **40**(4), 518–523 (2008)
2. Arrow, K., Hahn, F.: General Competitive Analysis. Mathematical Economics Texts. Holden-Day, San Francisco (1971)
3. Desai, M.: Growth cycles and inflation in a model of the class struggle. J. Econ. Theory **6**(6), 527–545 (1973)
4. Desai, M., Henry, B., Mosley, A., Pemberton, M.: clarification of the Goodwin model of the growth cycle. J. Econ. Dyn. Control **30**(12), 2661–2670 (2006)
5. Goodwin, R.M.: A growth cycle. In: Feinstein, C.H. (ed.) Socialism, Capitalism and Economic Growth, pp. 54–58. Cambridge University Press, Cambridge (1967)
6. Goodwin, R.: Economic evolution, chaotic dynamics and the Marx–Keynes–Schumpeter system. In: Rethinking Economics: Markets, Technology and Economic Evolution, pp. 138–152. Edward Elgar, Cheltenham (1991)
7. Harvie, D.: Testing Goodwin: growth cycles in ten OECD countries. Camb. J. Econ. **24**(3), 349–376 (2000)
8. Hirsch, M., Smale, S., Devaney, R.: Differential Equations, Dynamical Systems, and an Introduction to Chaos. Pure and Applied Mathematics. Academic Press, New York (2004)
9. Keen, S.: Finance and economic breakdown: modeling Minsky's "Financial Instability Hypothesis" . J. Post Keynes. Econ. **17**(4), 607–635 (1995)
10. Keen, S.: The nonlinear economics of debt deflation. In: Barnett, W.A. (ed.) Commerce, Complexity, and Evolution: Topics in Economics, Finance, Marketing, and Management. Proceedings of the Twelfth International Symposium in Economic Theory and Econometrics, pp. 83–110, Cambridge University Press, New York (2000)
11. Keen, S.: Household debt: the final stage in an artificially extended Ponzi bubble. Aust. Econ. Rev. **42**(3), 347–357 (2009)
12. Kindleberger, C.P., Aliber, R.Z.: Manias, Panics, and Crashes: A History of Financial Crises. Palgrave Macmillan, Houndmills (2011)
13. Misnky, H.P.: Can 'It' Happen Again? Essays on Instability and Finance. M.E. Sharpe, Armonk (1982)
14. Solow, R.M.: Goodwin's growth cycle: reminiscence and rumination. In: Velupillai, K. (ed.) Nonlinear and Multisectoral Macrodynamics, Macmillan, London (1990)
15. Tirole, J.: Asset bubbles and overlapping generations. Econometrica **53**(6), 1499–1528 (1985)
16. van der Ploeg, F.: Classical growth cycles. Metroeconomica **37**(2), 221–230 (1985)
17. Veneziani, R., Mohun, S.: Structural stability and Goodwin's growth cycle. Struct. Change Econ. Dyn. **17**(4), 437–451 (2006)