Combining Real Options and game theory in incomplete markets.

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Further Developments in Quantitative Finance
Edinburgh, July 11, 2007

Successes and Limitations of Real Options

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- In general, competition erodes the value of flexibility.

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- Finally, competition is generally introduced using game theory.
- Surprisingly, game theory is almost exclusively combined with real options under the hypothesis of risk-neutrality !

Related literature

- Real options and games: Smit and Ankum (1993), Dixit and Pindyck (1994), Grenadier (1996), Kulatikaka and Perotti (1998), Smit and Trigeorgis (2001), Imai and Watanabe (2006).

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- Indifference pricing: Henderson and Hobson (2001), Musiela and Zariphopoulou (2004), Rogers and Scheinkman (2007).

A one–period investment model

- Consider a two–factor market where the discounted prices for the project V and a correlated traded asset S follow:

$$(S_T, V_T) = \begin{cases} (uS_0, hV_0) & \text{with probability } p_1, \\ (uS_0, \ell V_0) & \text{with probability } p_2, \\ (dS_0, hV_0) & \text{with probability } p_3, \\ (dS_0, \ell V_0) & \text{with probability } p_4, \end{cases} \tag{1}$$

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  where $0 < d < 1 < u$ and $0 < \ell < 1 < h$, for positive initial values S_0, V_0 and historical probabilities p_1, p_2, p_3, p_4.
- Let the risk preferences be specified through an exponential utility $U(x) = -e^{-\gamma x}$.
- An investment opportunity is model as an option with discounted payoff $C_t = (V - e^{-rt}I)^+$, for $t = 0, T$.

European Indifference Price

- The indifference price for the option to invest in the final period as the amount π that solves the equation

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- Denoting the two possible pay-offs at the terminal time by C_h and C_ℓ, the European indifference price is explicitly given by

$$\pi = g(C_h, C_\ell) \quad (3)$$

where, for fixed parameters $(u, d, p_1, p_2, p_3, p_4)$ the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined as

$$\begin{aligned} g(x_1, x_2) &= \frac{q}{\gamma} \log\left(\frac{p_1 + p_2}{p_1 e^{-\gamma x_1} + p_2 e^{-\gamma x_2}}\right) \\ &\quad + \frac{1-q}{\gamma} \log\left(\frac{p_3 + p_4}{p_3 e^{-\gamma x_1} + p_4 e^{-\gamma x_2}}\right), \end{aligned} \quad (4)$$

with

$$q = \frac{1-d}{u-d}.$$

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- That is, from the point of view of this agent, the value at time zero for the opportunity to invest in the project either at $t = 0$ or $t = T$ is given by

$$C_0 = \max\{(V_0 - I)^+, g((hV_0 - e^{-rT}I)^+, (\ell V_0 - e^{-rT}I)^+)\}.$$

A multi–period model

- Consider now a continuous-time two–factor market of the form

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\begin{aligned}
dS_t &= (\mu_1 - r)S_t dt + \sigma_1 S_t dW \\
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- We want to approximate this market by a discrete–time processes (S_n, V_n) following the one–period dynamics (1).
- This leads to the following choice of parameters:

$$
\begin{aligned}
u &= e^{\sigma_1\sqrt{\Delta t}}, \qquad h = e^{\sigma_2\sqrt{\Delta t}}, \\
d &= e^{-\sigma_1\sqrt{\Delta t}}, \qquad \ell = e^{-\sigma_2\sqrt{\Delta t}}, \\
p_1 + p_2 &= \frac{e^{(\mu_1 - r)\Delta t} - d}{u - d}, \quad p_1 + p_3 = \frac{e^{(\mu_2 - r)\Delta t} - \ell}{h - \ell} \\
\rho\sigma_1\sigma_2\Delta t &= (u-d)(h-\ell)[p_1 p_4 - p_2 p_3],
\end{aligned}
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supplemented by the condition $p_1 + p_2 + p_3 + p_4 = 1$.

Numerical Experiments - Act I

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- Given these parameters, the CAPM equilibrium expected rate of return on the project for a given correlation ρ is

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- The difference $\delta = \bar{\mu}_2 - \mu_2$ is the below–equilibrium rate–of–return shortfall and plays the role of a dividend rate paid by the project, which we fix at $\delta = 0.04$.

Known Thresholds

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- For our parameters, the adjustment to market risks is accounted by CAPM and this threshold coincides with $V^*_{DP} = 2$

Dependence on Correlation and Risk Aversion

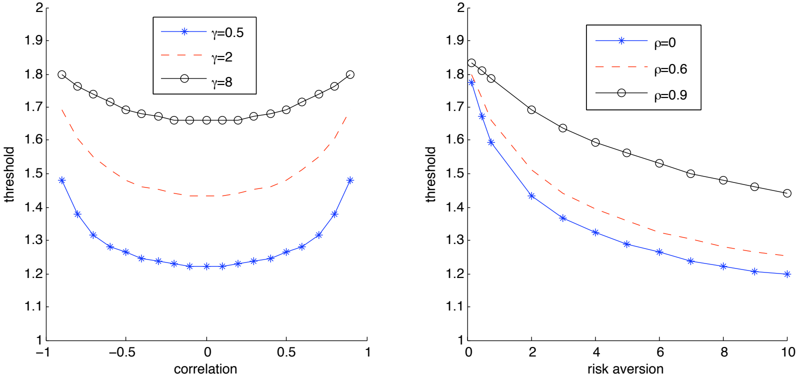

Figure: Exercise threshold as a function of correlation and risk aversion.

Dependence on Volatility and Dividend Rate

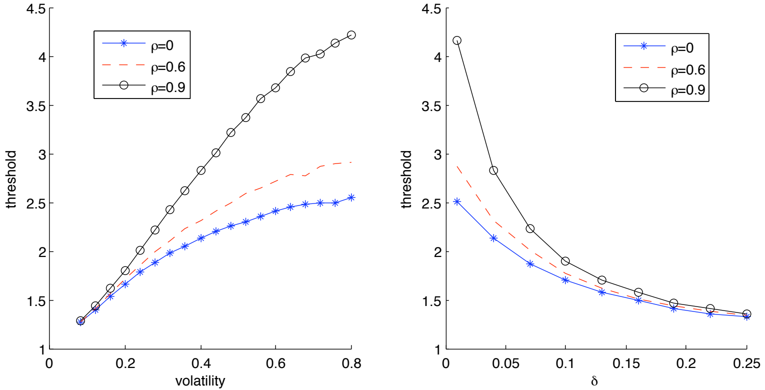

Figure: Exercise threshold as a function of volatility and dividend rate.

Dependence on Time to Maturity

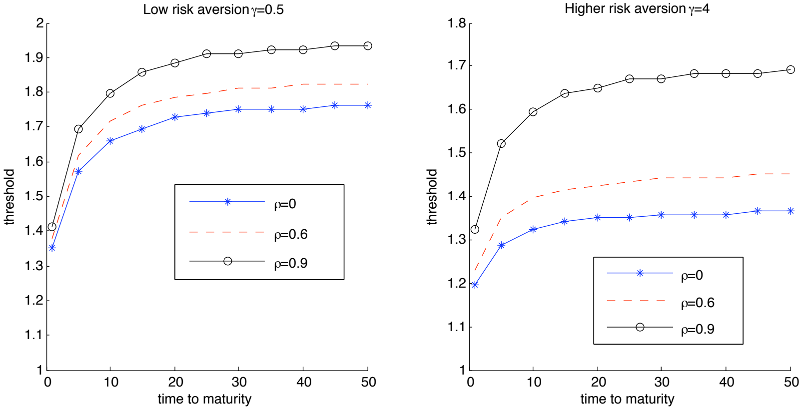

Figure: Exercise threshold as a function of time to maturity.

Values for the Option to Invest

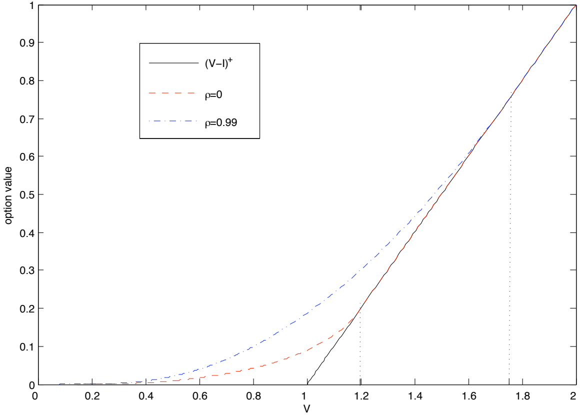

Figure: Option value as a function of underlying project value. The threshold for $\rho = 0$ is 1.1972 and the one for $\rho = 0.99$ is 1.7507.

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F^1 &= \text{cash flow } + \text{option to mothball at cost } E_M \\
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- As before, the decisions to invest, mothball, reactivate and scrap are triggered by the price thresholds $P_S < P_M < P_R < P_H$.

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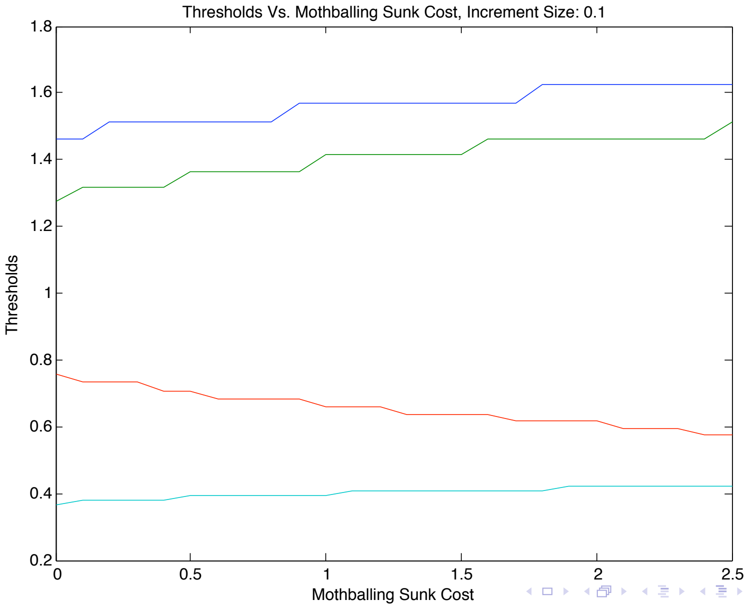
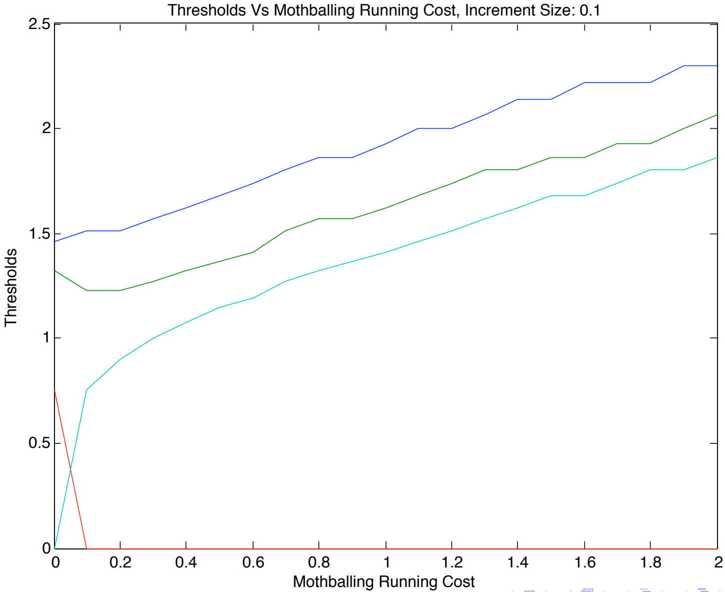
\mu_1 &= 0.12, \quad \sigma_1 = 0.2, \quad S_0 = 1 \\
\sigma_2 &= 0.2, \quad V_0 = 1 \\
r &= 0.05, \quad \delta = 0.05, \quad T = 30 \\
I &= 2, \quad R = 0.79, \quad E_M = E_S = 0 \\
C &= 1, \quad m = 0.01 \\
\rho &= 0.9, \quad \gamma = 0.1
\end{aligned}
$$

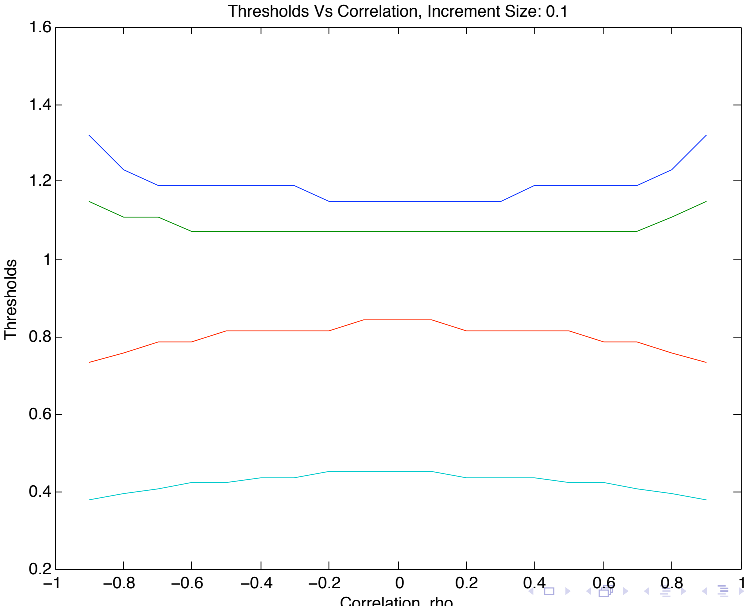
Dependence on Mothballing Sunk Cost



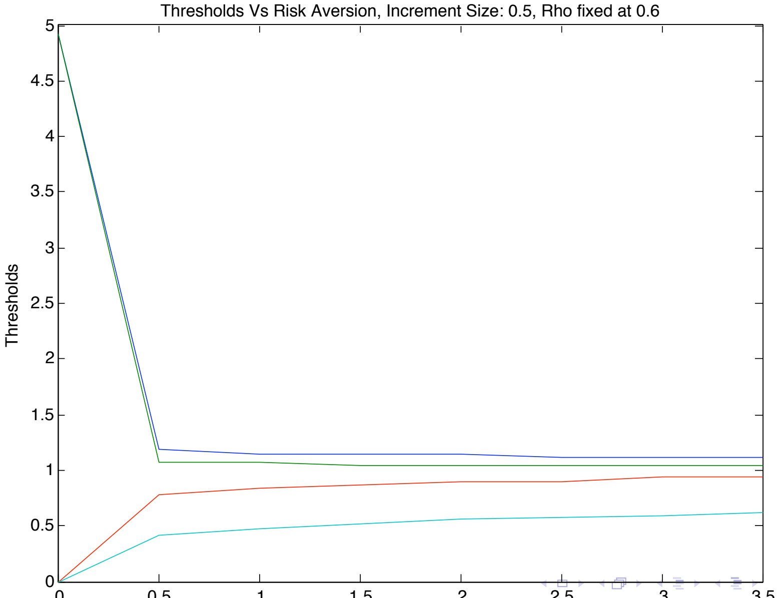
Dependence on Mothballing Running Cost



Dependence on Correlation



Dependence on Risk Aversion



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  2. Once the solution for a given game is found on a decision node, its value becomes the pay-off for an option at that node.
- In this way, option valuation and game theoretical equilibrium become dynamically related in a decision tree.

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- Suppose that, in a complete market, the value of option to invest is \$42 million.
- Since this is larger than the NPV, a monopolistic investor would wait, therefore owning an option worth \$42 million.
- Therefore, if both firms wait, they each own an option worth \$21 million.

Solution of the SIR game

- This symmetric innovation race can therefore be summarize as

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- As with the PD, an analysis of this game in extensive–form, regardless of the order the players move (or even using information sets for simultaneous moves), would lead to exactly the same solution.
- In this example, the unique NE is also stable with respect to changes in correlation and risk aversion.

Two–stage competitive R&D

- Consider two firms contemplating investment on a project with $V_0 = 100$ and equal probabilities to move up to $V^u = 200$ and down to $V^d = 50$.

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- We take $u = 3/2$, $h = 2$, $p_1 = p_4 = 127/256$, $p_2 = p_3 = 1/256$, $\gamma = 0.1$, $r = 0$.

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- Consider two firms contemplating investment on a project with $V_0 = 100$ and equal probabilities to move up to $V^u = 200$ and down to $V^d = 50$.
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- We will assume that the technology resulting from the R&D investment is proprietary, so that the market share of firm A after the R&D phase is $s = 3/5$.
- Moreover, we assume that the market value continues to evolve from time t_1 to time t_2 following the same dynamics, that is, at time t_2 the possible market values in these two–period tree are

$$V^{uu} = 400, \quad V^{ud} = 100, \quad V^{dd} = 25.$$

Analyzing the R&D game

- If demand is high at time t_1 ($V^u = 200$), we have:

		B (follower) Invest	B (follower) Wait
A (leader)	Invest	**(80,40)**	(120,0)
	Wait	(0,120)	(42,22)

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- For comparison, the complete market results are $C_A = 10$ and $C_B = 7$.

A multi-period investment game

- Consider two firms L and F each operating a project with an option to re-invest at cost I and increase cash–flow according to an uncertain demand

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- Let $D_{x_i(t_m)x_j(t_m)}$ denote the cash–flow per unit of demand of firm i.
- Assume that $D_{10} > D_{11} > D_{00} > D_{01}$.
- We say that there is FMA is $(D_{10} - D_{00}) > (D_{11} - D_{01})$ and that there is SMA otherwise.

Derivation of project values (1)

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$$
\begin{aligned}
v_i^{(1,1)}(t_m, y) &= D_{11} y \Delta t + \frac{g(V_i^{(1,1)}(t_{m+1}, y^u), (V_i^{(1,1)}(t_{m+1}, y^d))}{e^{r\Delta t}} \\
v_L^{(1,0)}(t_m, y) &= D_{10} y \Delta t + \frac{g(V_L^{(1,0)}(t_{m+1}, y^u), (V_L^{(1,0)}(t_{m+1}, y^d))}{e^{r\Delta t}} \\
v_L^{(0,1)}(t_m, y) &= D_{01} y \Delta t + \frac{g(V_L^{(0,1)}(t_{m+1}, y^u), (V_L^{(0,1)}(t_{m+1}, y^d))}{e^{r\Delta t}} \\
v_F^{(1,0)}(t_m, y) &= D_{01} y \Delta t + \frac{g(V_F^{(1,0)}(t_{m+1}, y^u), (V_F^{(1,0)}(t_{m+1}, y^d))}{e^{r\Delta t}} \\
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v_i^{(0,0)}(t_m, y) &= D_{00} y \Delta t + \frac{g(V_i^{(0,0)}(t_{m+1}, y^u), (V_i^{(0,0)}(t_{m+1}, y^d))}{e^{r\Delta t}}
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$$V_i^{(1,1)}(t_m, y) = v_i^{(1,1)}(t_m, y).$$

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- Next consider the project value for L when it has already invest and F hasn't:

$$V_L^{(1,0)}(t_m, y) = \begin{cases} v_L^{(1,1)}(t_m, y) \text{ if } v_F^{(1,1)}(t_m, y) - I > v_F^{(1,0)}(t_m, y), \\ v_L^{(1,0)}(t_m, y) \text{ otherwise.} \end{cases}$$

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- Similarly, the project value for F when it has already invest and L hasn't is

$$V_F^{(0,1)}(t_m, y) = \begin{cases} v_F^{(1,1)}(t_m, y) \text{ if } v_L^{(1,1)}(t_m, y) - I > v_L^{(0,1)}(t_m, y), \\ v_F^{(0,0)}(t_m, y) \text{ otherwise.} \end{cases}$$

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- The pay-off matrix for the game is

		Firm F	
		Invest	Wait
Firm L	Invest	$(v_L^{(1,1)} - I, v_F^{(1,1)} - I)$	$(v_L^{(1,0)} - I, v_F^{(1,0)})$
	Wait	$(v_L^{(0,1)}, v_F^{(0,1)} - I)$	$(v_L^{(0,0)}, v_F^{(0,0)})$

FMA: dependence on risk aversion.

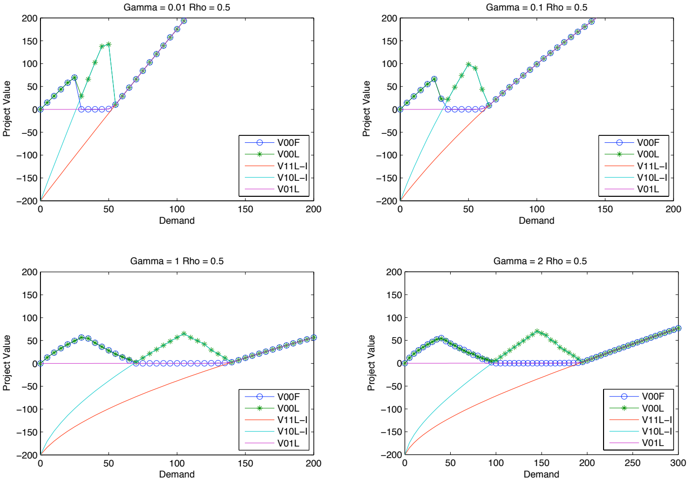

Figure: Project values in FMA case for different risk aversions.

FMA: dependence on correlation.

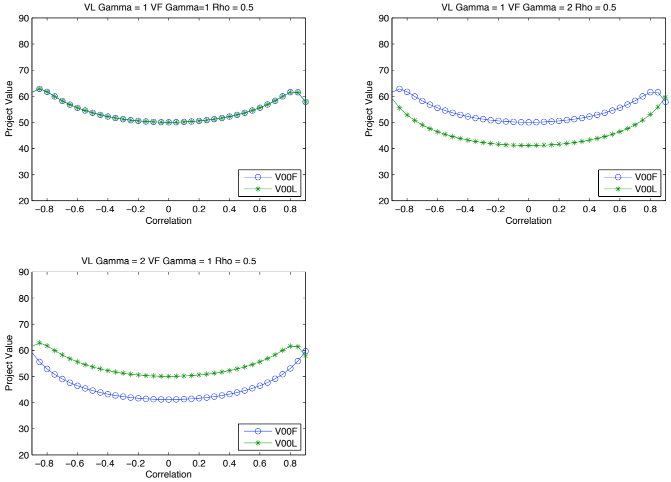

Figure: Project values in FMA case as function of correlation.

SMA: dependence on risk aversion

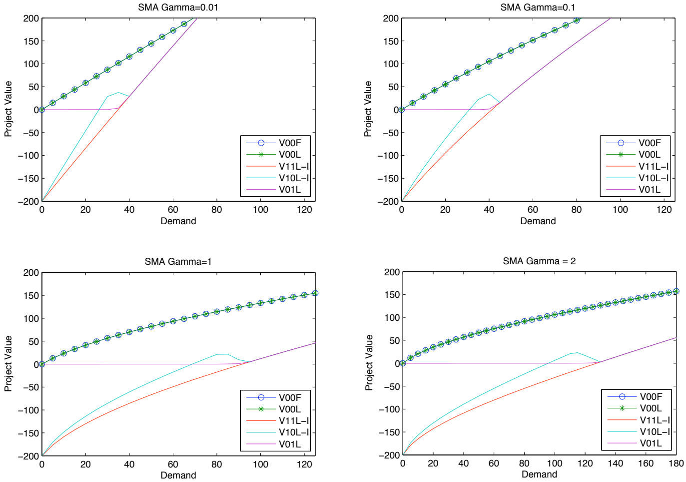

Figure: Project values in SMA case for different risk aversions.

SMA: dependence on correlation.

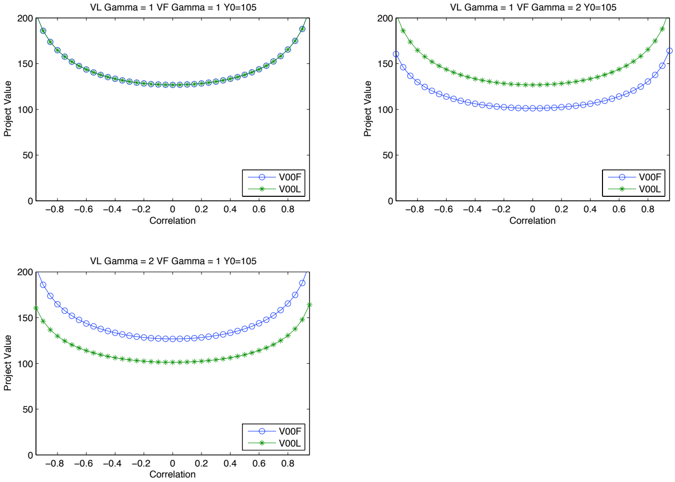

Figure: Project values in SMA case as function of correlation.

SMA x FMA

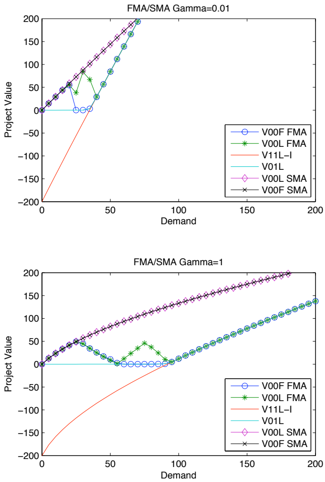

Figure: Project values for FMA and SMA.