Combining Real Options and game theory in incomplete markets.

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Further Developments in Quantitative Finance
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Successes and Limitations of Real Options

- Real options accurately describe the value of flexibility in decision making under uncertainty.

According to a recent survey, 26% of CFOs in North America "always or almost always" consider the value of real options in projects. This is due to familiarity with the option valuation paradigm in financial markets and its lessons. But most of the literature in Real Options is based on different combinations of the following unrealistic assumptions: (1) infinite time horizon, (2) perfectly correlated spanning asset, (3) absence of competition. Though some problems have long time horizons (30 years or more), most strategic decisions involve much shorter times. The vast majority of underlying projects are not perfectly correlated to any asset traded in financial markets. In general, competition erodes the value of flexibility.
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- Replication arguments can no longer be applied to value managerial opportunities.

- The most widespread alternative to replication in the decision-making literature is to introduce a risk-adjusted rate of return, which replaces the risk–free rate, and use dynamic programming.

- This approach lacks the intuitive understanding of opportunities as options.

- Finally, competition is generally introduced using game theory.

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Related literature


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A one–period investment model

Consider a two–factor market where the discounted prices for the project $V$ and a correlated traded asset $S$ follow:

$$(S_T, V_T) = \begin{cases} 
(uS_0, hV_0) & \text{with probability } p_1, \\
(uS_0, \ell V_0) & \text{with probability } p_2, \\
(dS_0, hV_0) & \text{with probability } p_3, \\
(dS_0, \ell V_0) & \text{with probability } p_4, 
\end{cases}$$

where $0 < d < 1 < u$ and $0 < \ell < 1 < h$, for positive initial values $S_0, V_0$ and historical probabilities $p_1, p_2, p_3, p_4$. 

Let the risk preferences be specified through an exponential utility $U(x) = -e^{-\gamma x}$. 

An investment opportunity is modeled as an option with discounted payoff $C_t = (V_t - e^{-rt}I)^+$, for $t = 0, T$. 

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European Indifference Price

- The **indifference price** for the option to invest in the final period as the amount $\pi$ that solves the equation

$$\max_{H_\pi} E[U(x + H(S_T - S_0))] = \max_{H_\pi} E[U(x - \pi + H(S_T - S_0))]$$

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- Denoting the two possible pay-offs at the terminal time by $C_h$ and $C_\ell$, the European indifference price is explicitly given by

$$\pi = g(C_h, C_\ell) \quad (3)$$

where, for fixed parameters $(u, d, p_1, p_2, p_3, p_4)$ the function $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is defined as

$$g(x_1, x_2) = \frac{q}{\gamma} \log \left( \frac{p_1 + p_2}{p_1 e^{-\gamma x_1} + p_2 e^{-\gamma x_2}} \right) \quad (4)$$

$$+ \frac{1 - q}{\gamma} \log \left( \frac{p_3 + p_4}{p_3 e^{-\gamma x_1} + p_4 e^{-\gamma x_2}} \right),$$

with

$$q = \frac{1 - d}{u - d}.$$
Early exercise

- When investment at time $t = 0$ is allowed, it is clear that immediate exercise of this option will occur whenever its exercise value $(V_0 - I)^+$ is larger than its continuation value $\pi^C$. 
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- That is, from the point of view of this agent, the value at time zero for the opportunity to invest in the project either at $t = 0$ or $t = T$ is given by

$$C_0 = \max\{(V_0 - I)^+, g((hV_0 - e^{-rT}I)^+, (\ell V_0 - e^{-rT}I)^+)\}.$$
A multi–period model

Consider now a continuous-time two–factor market of the form

\[ dS_t = (\mu_1 - r) S_t dt + \sigma_1 S_t dW \]
\[ dV_t = (\mu_2 - r) V_t dt + \sigma_2 V_t (\rho dW + \sqrt{1 - \rho^2} dZ). \]
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We want to approximate this market by a discrete–time processes \((S_n, V_n)\) following the one–period dynamics (1).
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We want to approximate this market by a discrete–time processes \((S_n, V_n)\) following the one–period dynamics (1).

This leads to the following choice of parameters:

\[
u = e^{\sigma_1 \sqrt{\Delta t}}, \quad h = e^{\sigma_2 \sqrt{\Delta t}},
\]

\[
d = e^{-\sigma_1 \sqrt{\Delta t}}, \quad \ell = e^{-\sigma_2 \sqrt{\Delta t}},
\]

\[
p_1 + p_2 = \frac{e^{(\mu_1 - r) \Delta t} - d}{u - d}, \quad p_1 + p_3 = \frac{e^{(\mu_2 - r) \Delta t} - \ell}{h - \ell}
\]

\[
\rho \sigma_1 \sigma_2 \Delta t \quad = \quad (u - d)(h - \ell)[p_1 p_4 - p_2 p_3],
\]

supplemented by the condition \(p_1 + p_2 + p_3 + p_4 = 1\).
Numerical Experiments - Act I

- We now investigate how the exercise threshold varies with the different model parameters.

\[ \bar{\mu}^2 = r + \rho \left( \mu_1 - r \sigma_1 \right) / \sigma_2. \] (5)

The difference \( \delta = \bar{\mu}^2 - \mu^2 \) is the below–equilibrium rate–of–return shortfall and plays the role of a dividend rate paid by the project, which we fix at \( \delta = 0.04 \).
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The fixed parameters are

\[
\begin{align*}
I &= 1, \quad r = 0.04, \quad T = 10 \\
\mu_1 &= 0.115, \quad \sigma_1 = 0.25, \quad S_0 = 1 \\
\sigma_2 &= 0.2, \quad V_0 = 1
\end{align*}
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Given these parameters, the CAPM equilibrium expected rate of return on the project for a given correlation \(\rho\) is

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Known Thresholds

- In the limit $\rho \rightarrow \pm 1$ (complete market), the closed–form expression for the investment threshold obtained in the case $T = \infty$ gives $V_{DP}^* = 2$.

- This should be contrasted with the NPV criterion (that is, invest whenever the net present value for the project is positive) which in this case gives $V_{NPV}^* = 1$.

- The limit $\gamma \rightarrow 0$ in our model corresponds to the McDonald and Siegel (1986) threshold, obtained by assuming that investors are averse to market risk but neutral towards idiosyncratic risk.

- For our parameters, the adjustment to market risks is accounted by CAPM and this threshold coincides with $V_{DP}^* = 2$. 
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Dependence on Correlation and Risk Aversion

Figure: Exercise threshold as a function of correlation and risk aversion.
Figure: Exercise threshold as a function of volatility and dividend rate.
Dependence on Time to Maturity

**Figure:** Exercise threshold as a function of time to maturity.
**Values for the Option to Invest**

![Graph](image)

**Figure:** Option value as a function of underlying project value. The threshold for $\rho = 0$ is 1.1972 and the one for $\rho = 0.99$ is 1.7507.
Suspension, Reactivation and Scrapping

Let us denote the value of an idle project by $F^0$, an active project by $F^1$ and a mothballed project by $F^M$. 

We obtain its value on the grid using the recursion formula:

$$F_k(i,j) = \max \{ \text{continuation value, possible exercise values} \}.$$ 

As before, the decisions to invest, mothball, reactivate and scrap are triggered by the price thresholds $P_S < P_M < P_R < P_H$. 

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Then

\begin{align*}
F^0 &= \text{option to invest at cost } I \\
F^1 &= \text{cash flow } + \text{option to mothball at cost } E_M \\
F^M &= \text{cash flow } + \text{option to reactivate at cost } R \\
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Numerical Experiments - Act II

- We calculate these thresholds by keeping track of three simultaneous grids of option values.
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\sigma_2 &= 0.2, \quad V_0 = 1 \\
r &= 0.05, \quad \delta = 0.05, \quad T = 30 \\
l &= 2, \quad R = 0.79, \quad E_M = E_S = 0 \\
C &= 1, \quad m = 0.01 \\
\rho &= 0.9, \quad \gamma = 0.1
\end{align*}
\]
Figure: Exercise thresholds as functions of mothballing sunk cost.
Dependence on Mothballing Running Cost

Thresholds Vs Mothballing Running Cost, Increment Size: 0.1

Figure: Exercise thresholds as functions of mothballing running cost.
Figure: Exercise thresholds as functions of correlation.
Dependence on Risk Aversion

Figure: Exercise thresholds as functions of risk aversion.

Thresholds Vs Risk Aversion, Increment Size: 0.5, Rho fixed at 0.6
Combining options and games

- For a systematic application of both real options and game theory in strategic decisions, we consider the following rules:
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In this way, option valuation and game theoretical equilibrium become dynamically related in a decision tree.
Symmetric Innovation Race - SIR (Smit/Trigeorgis 04)

- Consider an innovation race for a new electronic technology between firms $A$ and $B$.
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If both firms invest, we assume that they share this value equally, whereas if only one firm invests immediately, it receives the total market value, while the other receives nothing.
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Suppose that, in a complete market, the value of option to invest is $42 million.
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Since this is larger than the NPV, a monopolistic investor would wait, therefore owning an option worth $42$ million.
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Therefore, if both firms wait, they each own an option worth $21$ million.
This symmetric innovation race can therefore be summarize as:

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<thead>
<tr>
<th></th>
<th>Invest</th>
<th>Wait</th>
</tr>
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<tbody>
<tr>
<td>Invest</td>
<td>(13,13)</td>
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This is the business analogue of the Prisoner's dilemma, since the second row and second column are strictly dominated respectively by the first row and first column.

Therefore, the only NE is (Invest, Invest)!

As with the PD, an analysis of this game in extensive–form, regardless of the order the players move (or even using information sets for simultaneous moves), would lead to exactly the same solution.

In this example, the unique NE is also stable with respect to changes in correlation and risk aversion.
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\end{array}
\]

- This is the business analogue of the Prisoner’s dilemma, since the second row and second column are strictly dominated respectively by the first row and first column.

- Therefore, the only NE is (Invest,Invest)!

- As with the PD, an analysis of this game in extensive-form, regardless of the order the players move (or even using information sets for simultaneous moves), would lead to exactly the same solution.
Solution of the SIR game

- This symmetric innovation race can therefore be summarized as:

\[
\begin{array}{c|cc}
   & \text{Invest} & \text{Wait} \\
\hline
\text{Invest} & (13,13) & (26,0) \\
\text{Wait} & (0,26) & (21,21) \\
\end{array}
\]

- This is the business analogue of the Prisoner’s dilemma, since the second row and second column are strictly dominated respectively by the first row and first column.

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- As with the PD, an analysis of this game in extensive–form, regardless of the order the players move (or even using information sets for simultaneous moves), would lead to exactly the same solution.

- In this example, the unique NE is also stable with respect to changes in correlation and risk aversion.
Two–stage competitive R&D

- Consider two firms contemplating investment on a project with $V_0 = 100$ and equal probabilities to move up to $V^u = 200$ and down to $V^d = 50$. 

Suppose now that firm A can do an R&D investment at cost $I_0 = 25$ at time $t_0$, whereas at time $t_1$ the firms can equally share the follow–on cost $I_1 = 80$.

We will assume that the technology resulting from the R&D investment is proprietary, so that the market share of firm A after the R&D phase is $s = 3/5$.

Moreover, we assume that the market value continues to evolve from time $t_1$ to time $t_2$ following the same dynamics, that is, at time $t_2$ the possible market values in these two–period tree are $V_{uu} = 400$, $V_{ud} = 100$, $V_{dd} = 25$. 

Two–stage competitive R&D

Consider two firms contemplating investment on a project with $V_0 = 100$ and equal probabilities to move up to $V^u = 200$ and down to $V^d = 50$.

We take $u = 3/2$, $h = 2$, $p_1 = p_4 = 127/256$, $p_2 = p_3 = 1/256$, $\gamma = 0.1$, $r = 0$. 

Suppose now that firm $A$ can do an R&D investment at cost $I_0 = 25$ at time $t_0$, whereas at time $t_1$ the firms can equally share the follow–on cost $I_1 = 80$.

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$$V^{uu} = 400, \quad V^{ud} = 100, \quad V^{dd} = 25.$$
Analyzing the R&D game

- If demand is high at time $t_1$ ($V^u = 200$), we have:

<table>
<thead>
<tr>
<th></th>
<th>Invest</th>
<th>Wait</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (leader) Invest</td>
<td>(80,40)</td>
<td>(120,0)</td>
</tr>
<tr>
<td>A (leader) Wait</td>
<td>(0,120)</td>
<td>(42,22)</td>
</tr>
<tr>
<td>B (follower) Invest</td>
<td></td>
<td></td>
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- If demands is low at time $t_1$ ($V^d = 60$), we have:

<table>
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<tbody>
<tr>
<td>A (leader) Invest</td>
<td>(-10,-20)</td>
<td>(-30,0)</td>
</tr>
<tr>
<td>A (leader) Wait</td>
<td>(0,-30)</td>
<td>(8,0)</td>
</tr>
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- Then $C_A = -I_0 + g(80,8) = -25 + 30 = 5 > 0$,

- whereas $C_B = g(40,0) = 15$.

- Therefore the R&D investment is recommended for A.

- For comparison, the complete market results are $C_A = 10$ and $C_B = 7$. 
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A multi-period investment game

- Consider two firms $L$ and $F$ each operating a project with an option to re-invest at cost $I$ and increase cash-flow according to an uncertain demand

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dW.$$
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Suppose that the option to re-invest has maturity $T$, let $t_m$, $m = 0, \ldots, M$ be a partition of the interval $[0, T]$ and denote by $(x_L(t_m), x_F(t_m)) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ the possible states of the firms after a decision has been at time $t_m$. 

Let $D_{x_i}(t_m)$ denote the cash-flow per unit of demand of firm $i$. Assume that $D_{10} > D_{11} > D_{00} > D_{01}$.

We say that there is FMA is $(D_{10} - D_{00}) > (D_{11} - D_{01})$ and that there is SMA otherwise.
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- We say that there is FMA is $(D_{10} - D_{00}) > (D_{11} - D_{01})$ and that there is SMA otherwise.
Derivation of project values (1)

Let $V_i^{(x(t_{m-1}), x_j(t_{m-1}))}(t_m, y)$ denote the project value for firm $i$ at time $t_m$ and demand level $y$. 
Derivation of project values (1)

- Let \( V_{i}^{(x_{i}(t_{m-1}),x_{j}(t_{m-1}))(t_{m},y)} \) denote the project value for firm \( i \) at time \( t_{m} \) and demand level \( y \).
- Denote by \( v_{i}^{(x_{i}(t_{m}),x_{j}(t_{m}))(t_{m},y)} \) the continuation values:

\[
\begin{align*}
v_{i}^{(1,1)}(t_{m}, y) &= D_{11}y\Delta t + \frac{g(V_{i}^{(1,1)}(t_{m+1}, y^{u}), (V_{i}^{(1,1)}(t_{m+1}, y^{d}))}{e^{r\Delta t}} \\
v_{L}^{(1,0)}(t_{m}, y) &= D_{10}y\Delta t + \frac{g(V_{L}^{(1,0)}(t_{m+1}, y^{u}), (V_{L}^{(1,0)}(t_{m+1}, y^{d}))}{e^{r\Delta t}} \\
v_{L}^{(0,1)}(t_{m}, y) &= D_{01}y\Delta t + \frac{g(V_{L}^{(0,1)}(t_{m+1}, y^{u}), (V_{L}^{(0,1)}(t_{m+1}, y^{d}))}{e^{r\Delta t}} \\
v_{F}^{(1,0)}(t_{m}, y) &= D_{01}y\Delta t + \frac{g(V_{F}^{(1,0)}(t_{m+1}, y^{u}), (V_{F}^{(1,0)}(t_{m+1}, y^{d}))}{e^{r\Delta t}} \\
v_{F}^{(0,1)}(t_{m}, y) &= D_{10}y\Delta t + \frac{g(V_{F}^{(0,1)}(t_{m+1}, y^{u}), (V_{F}^{(0,1)}(t_{m+1}, y^{d}))}{e^{r\Delta t}} \\
v_{i}^{(0,0)}(t_{m}, y) &= D_{00}y\Delta t + \frac{g(V_{i}^{(0,0)}(t_{m+1}, y^{u}), (V_{i}^{(0,0)}(t_{m+1}, y^{d}))}{e^{r\Delta t}}
\end{align*}
\]
For fully invested firms, the project values are simply given by

\[ V_i^{(1,1)}(t_m, y) = v_i^{(1,1)}(t_m, y). \]
Derivation of project values (2)

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\[ V_i^{(1,1)}(t_m, y) = v_i^{(1,1)}(t_m, y). \]

- Now consider the project value for firm \( F \) when \( L \) has already invested and \( F \) hasn’t:

\[ V_F^{(1,0)}(t_m, y) = \max \{ v_F^{(1,1)}(t_m, y) - I, v_F^{(1,0)}(t_m, y) \}. \]
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\[ V_F^{(1,0)}(t_m, y) = \max\{v_F^{(1,1)}(t_m, y) - l, v_F^{(1,0)}(t_m, y)\}. \]

Similarly, the project value for \( L \) when \( F \) has invested and \( L \) hasn’t is

\[ V_L^{(0,1)}(t_m, y) = \max\{v_L^{(1,1)}(t_m, y) - l, v_L^{(0,1)}(t_m, y)\}. \]
Next consider the project value for $L$ when it has already invest and $F$ hasn’t:

$$V_{L}^{(1,0)}(t_m, y) = \begin{cases} v_{L}^{(1,1)}(t_m, y) & \text{if } v_{F}^{(1,1)}(t_m, y) - I > v_{F}^{(1,0)}(t_m, y), \\ v_{L}^{(1,0)}(t_m, y) & \text{otherwise}. \end{cases}$$
Next consider the project value for $L$ when it has already invest and $F$ hasn’t:

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Similarly, the project value for $F$ when it has already invest and $L$ hasn’t is

$$V_{F}^{(0,1)}(t_{m}, y) = \begin{cases} v_{F}^{(1,1)}(t_{m}, y) & \text{if } v_{L}^{(1,1)}(t_{m}, y) - I > v_{L}^{(0,1)}(t_{m}, y), \\ v_{F}^{(0,0)}(t_{m}, y) & \text{otherwise.} \end{cases}$$
Finally, the project values $V_{i}^{(0,0)}$ are obtained as a Nash equilibrium, since both firms still have the option to invest.
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The pay-off matrix for the game is

<table>
<thead>
<tr>
<th></th>
<th>Firm F</th>
<th></th>
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</thead>
<tbody>
<tr>
<td><strong>Invest</strong></td>
<td>($v_L^{(1,1)} - l$, $v_F^{(1,1)} - l$)</td>
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<td></td>
</tr>
<tr>
<td><strong>Wait</strong></td>
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</tr>
<tr>
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FMA: dependence on risk aversion.

Figure: Project values in FMA case for different risk aversions.
FMA: dependence on correlation.

Figure: Project values in FMA case as function of correlation.
Figure: Project values in SMA case for different risk aversions.
Figure: Project values in SMA case as function of correlation.
Figure: Project values for FMA and SMA.