Real options in incomplete markets

The reflected BSDE approach

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Successes and Limitations of Real Options

- According to a recent survey, 26% of CFOs in North America "always or almost always" consider the value of real options in projects.
- This is due to familiarity with the option valuation paradigm in financial markets and its lessons.
- But most of the literature in Real Options is based on one or both of the following assumptions: (1) infinite time horizon and (2) a perfectly correlated spanning asset.
- Though some problems have long time horizons (30 years or more), most strategic decisions involve much shorter times.
- The vast majority of underlying projects are not perfectly correlated to any asset traded in financial markets.

Alternatives

- The use of well–known numerical methods (e.g binomial trees or finite differences) allows to consider finite–time horizons.
- As for the spanning asset assumption, the absence of perfect correlation with a financial asset leads to an incomplete market.
- Replication arguments can no longer be applied to value managerial opportunities.
- Instead, one needs to rely on risk preferences.
- The most widespread way to do this in the strategic decision making literature is to introduce a risk adjusted discount factor, which replaces the risk–free rate, and use dynamic programming.
- This approach lacks the intuitive understanding of opportunities as options.

Utility-based methods

- We treat an investment opportunity as an option on a non–traded asset and price it using the framework of indifference pricing.
- For investments with a fixed exercise date (European option), this problem was treated, for instance, in Hobson and Henderson (2002).
- For early exercise investment (American option), the problem was solved in Herderson (2005) for the case of infinite time horizon.
- A different utility–based framework (not using indifference pricing), was treated in Hugonnier and Morellec (2004), using the effect of shareholders control on the wealth of a risk averse manager.
- For finite time horizons, a different version of the problem was solved Porchet, Touzi and Warin (2008) using the reflected BSDEs approach introduced in complete markets by Hamadène and Jeanblanc (2007).

A gentle introduction to BSDEs in Finance

- Given a terminal random variable $\xi \in \mathcal{F}_T$ and a generator function $f(t, y, z)$, a solution of a backward SDE is a pair of adapted processes (Y, Z) satisfying

$$Y_t = \xi - \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s' dW_s, \tag{1}$$

  or equivalently

$$dY_t = f(t, Y_t, Z_t)dt + Z_t' dW_t \tag{2}$$

$$Y_T = \xi \tag{3}$$

- **Theorem** (Pardoux/Peng 1990): If ξ is square-integrable and f is uniformly Lipschitz, then the BSDE has a unique square-integrable solution.

First example: pricing and hedging in a complete market

- Consider the market

$$dB_t = B_t r_r dt, \tag{4}$$

$$dS_t^i = S_t^i \left[\mu_t dt + \sum_{j=1}^{n} \sigma_t^{ij} dW_t^j \right] \tag{5}$$

- Given a claim $\xi \geq 0$, we look for a portfolio (V, π) satisfying

$$dX_t = r_t X_t dt + \pi_t' \sigma (dW_t + \lambda_t dt) \tag{6}$$

$$X_T = \xi \tag{7}$$

where $\mu_t - r 1_d = \sigma \lambda_t$

- We see that this corresponds to a linear BSDE with

$$Y_t = X_t \tag{8}$$

$$Z_t = \sigma' \pi_t \tag{9}$$

$$f(t, Y_t, Z_t) = r Y_t + \lambda_t' Z_t \tag{10}$$

The Markovian Case

- For given (t,x), let $S_s^{t,x}$ be the solution of the forward SDE

$$S_s = x + \int_t^s \mu(u, S_u)du + \int_t^s \sigma(u, S_u)dWu, \quad t \leq s \leq T \quad (11)$$

- Consider than the associated BSDE

$$Y_s = \Phi(S_T^{t,x}) - \int_s^T f(u, S_u^{t,x}, Y_u, Z_u)du - \int_s^T Z_u' dW_u \quad (12)$$

- When the coefficients satisfy certain Lipschitz and growth conditions, it can be shown that the solution can be written as $Y_s^{t,x} = u(s, S^{t,x})$ and $Z_s^{t,x} = \sigma' v(s, S_s^{t,x})$ for deterministic Borel functions $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$.
- Under additional regularity conditions on f and Φ (such as uniform continuity in x), it can be shown that the function $u(t,x) = Y_t^{t,x}$ is a viscosity solution of the PDE

$$u_t + \mathcal{L}u - f(t, x, u, \sigma' u_x) = 0, \quad (13)$$

  where $\mathcal{L}$ is the generator of S_t.

Second example: utility maximization

- Now let $r_t = 0$ and consider the market

$$dS_t^i = S_t^i \left[\mu_t^i dt + \sum_{j=1}^{n} \sigma_t^{ij} dW_t^j \right], \quad i = 1, \ldots, d \leq n. \tag{14}$$

where μ_t^i, σ_t^{ij} are predictable uniformly bounded, σ_t is uniformly elliptic and let λ_t be a solution of

$$\sigma_t \lambda_t = \mu_t. \tag{15}$$

- As before, the wealth in a self–financing portfolio satisfies

$$X_t^\pi = x + \int_0^t \pi_s' \sigma_s (dW_s + \lambda_s ds) \tag{16}$$

- We are then interested in the optimization problem

$$u(x) := \sup_{\pi \in \mathcal{A}} E \left[-e^{-\gamma(X_T^\pi + B)} \right] \tag{17}$$

Second example (continued): supermartingales

- To solve (17), we follow Hu/Imkeller/Muller (2004) and look for a family of processes R^π such that
  - $R^\pi_T = U(X^\pi_T + B)$
  - $R^\pi_0 = R_0$ for all $\pi \in \mathcal{A}$.
  - R^π_t is a supermartingale for all $\pi \in \mathcal{A}$.
  - There exists a $\pi^* \in \mathcal{A}$ such that $R^{\pi^*}_t$ is a martingale.
- To construct such family we set

$$R^\pi_t := -e^{-\gamma(X^\pi_t + Y^B_t)}, \tag{18}$$

- Here (Y^B, Z) is a solution of the BSDE

$$Y^B_t = B - \int_t^T f(s, Z_s)ds - \int_t^T Z'_s dW_s, \tag{19}$$

for a function f to be determined.

Second example (continued): the generator

- To determine f, we write R_t^π as the product of a local martingale and a decreasing process.
- Using the definitions of X^π and Y_t we find

$$R_t^\pi = -e^{\gamma(x-Y_0)} e^{-\gamma\left[\int_0^t (\pi_s'\sigma_s + Z_s')dW + \int_0^t (\pi_s'\sigma_s\lambda + f(s,Y_s,Z_s)ds)\right]}$$
$$= -e^{\gamma(x-Y_0)} e^{-\gamma\int_0^t (\pi_s'\sigma_s + Z_s')dW - \frac{1}{2}\int_0^t \gamma^2\|\pi_s'\sigma_s + Z_s'\|^2 ds} e^{\int_0^t v(s,\pi_s,Z_s)ds},$$

where $v(t,\pi,z) = -\gamma\pi'\sigma_t\lambda_t - \gamma f(t,z) + \frac{1}{2}\gamma^2\|\pi'\sigma_t + z'\|^2$.
- We therefore seek for f such that $v(t,\pi_t,Z_t) \geq 0$ for all $\pi_t \in \mathcal{A}$ and $v(t,\pi_t^*,Z_t) = 0$ for some $\pi_t^* \in \mathcal{A}$.
- Rearranging terms in v, we see that it suffices to take

$$f(t,z) = z\lambda_t - \frac{1}{2\gamma}\|\lambda_t\|^2 \tag{20}$$
$$\pi_t^*\sigma_t = \frac{\lambda_t}{\gamma} - Z_t \tag{21}$$

- This can be extended for the case of constrained portfolios.

Reflected BSDEs

- Given a terminal condition ξ, a generator function $f(t, y, z)$ and an obstacle C_t with $C_T \leq \xi$, a solution of a reflected BSDE is a triple (Y_t, Z_t, A_t) satisfying
  1. $Y_t = \xi - \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s' dW_s + (A_T - A_t)$,
  2. $Y_t \geq C_t$
  3. A_t is continuous, increasing, $A_0 = 0$, and $\int_0^T (Y_t - C_t) dA_t = 0$.
- **Proposition** (El Karoui et al - 1997): Under further square–integrability conditions on (Y_t, Z_t, A_t) we have that

$$Y_t = \operatorname*{ess\,sup}_{\tau} E\left[-\int_t^\tau f(s, Y_s, Z_s)ds + C_\tau 1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}} | \mathcal{F}_t \right]$$

The obstacle problem for PDEs

- Consider again the solution $S_s^{t,x}$ for the forward SDE (11) and let

$$\xi = \Phi(S_T^{t,x})$$
$$C_s = g(s, S_s^{t,x})$$
$$f(s, y, z) = f(s, S_s^{t,x}, y, z)$$

- Then, under certain continuity, integrability and growth conditions for Φ, g, f, it can be shown that the function $u(t,x) = Y_t^{t,x}$ is a viscosity solution of the obstacle problem

$$\min[-u_t - \mathcal{L}u - f(t, x, u, \sigma' u_x), u(t,x) - h(t,x)] = 0$$
$$u(T, x) = \Phi(x)$$

Third example: American options in a complete market

- Let $dS_t = rS_t dt + \sigma S_t dW_t^Q$.
- It is well-known that the price of an American put option on S_t is given by the Snell envelope

$$P_t = \operatorname*{ess\,sup}_{\tau} E^Q[e^{-r(\tau-t)}(K-S_\tau)^+|\mathcal{F}_t].$$

- We can see that this corresponds to a reflected BSDE with

$$Y_t = e^{-rt}P_t, \qquad f(t,y,z) = 0$$
$$\xi = e^{-rT}(K-S_T)^+, \quad C_t = e^{-rt}(K-S_t)^+$$

- Moreover, setting $u(t,S_t) = e^{-rt}P_t$, we have that

$$\max[u_t + \mathcal{L}u, e^{-rt}(K-x)^+ - u(t,x)] = 0$$
$$u(T,x) = e^{-rT}(K-S_T)^+$$

The option to invest in an incomplete market

- Again let $r_t = 0$ and a two–factor model where discounted prices are given by

$$dS_t = \mu_1 S_t dt + \sigma_1 S_t dW_t^1$$
$$dV_t = \mu_2 V_t dt + \sigma_2 V_t(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2)$$

- In our previous notation this corresponds to

$$\sigma = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2 \rho & \sigma_2 \sqrt{1-\rho^2} \end{pmatrix}, \quad \lambda = \begin{pmatrix} \mu_1/\sigma_1 \\ \frac{1}{\sqrt{1-\rho^2}}[\mu_2/\sigma_2 - \rho\mu_1/\sigma_1] \end{pmatrix}$$

- Here S_t represents the price of a traded asset, whereas V_t is the current value of a project.
- We then model investment in the project as an American call option on V with strike price equals to the sunk cost, which is assumed to grow at rate r_t for simplicity.

Preferences

- Consider then an agent trying to solve the Merton problem
$$u^0(t,x) = \sup_{\pi} \mathbb{E}[-e^{-\gamma X_T^\pi} | X_t = x]$$
- Here π_t is the amount invested in the stock at time t and
$$dX_t = \pi_t \frac{dS_t}{S_t} = \pi_t \sigma (dW_t^1 + \lambda_1 ds).$$
- We denote the solution to this Merton problem by
$$M(t,x) = -e^{-\gamma x} e^{-\frac{\mu^2}{2\sigma^2}(T-t)}.$$
- Finally, consider the modified problem
$$u(t,x,v) = \sup_{\pi,\tau} \mathbb{E}[M(\tau, X_\tau^\pi + (V_\tau - I)^+) | X_t = x, V_t = v].$$
- The indifference price for the option to invest in the project is the value p satisfying
$$u^0(x) = u(x-p, v)$$

System of reflected BSDEs

- From our previous example $u^0(x) = -e^{-\gamma(x+Y_0^1)}$ where

$$Y_t^1 = -\int_t^T f^1(Z_t^1)dt - \int_t^T Z_t^1 \cdot dW_t, \qquad (22)$$

for $f^1(z_1, z_2) = z_1\lambda_1 - \frac{\lambda_1^2}{2\gamma}$.

- Similarly, we will show that $u(x, v) = -e^{-\gamma(x+Y_0^2)}$ where

$$Y_t^2 = (V_T - I)^+ - \int_t^T f^2(Z_t^2)dt - \int_t^T Z_t^2 \cdot dW_t + (A_T - A_t)$$
$$Y_t^2 \geq (V_t - I)^+ + Y_t^1$$
$$A_0 = 0, \quad \int_0^T (Y_t^2 - (V_t - I)^+ - Y_t^1)dA_t = 0.$$

for $f^2(z_1, z_2) = \frac{\gamma}{2}\left(\frac{\lambda_2}{\gamma} - z_2\right)^2 + z \cdot \lambda - \frac{\|\lambda\|^2}{2\gamma}$.

Sketch of the proof

- For this choices, it follows that $R_t^\pi = -e^{\gamma(X_t^\pi + Y_t^2)}$ is a supermartingale for any π.
- Now let $0 \leq \tau T$ be an arbitrary stopping time, $\pi \in \mathcal{A}_{[0,\tau]}$ and $\bar{\pi} \in \mathcal{A}(\tau, T]$. From the dynamic principle satisfied by $Y^1 + t$ it follows that

$$\mathbb{E}\left[-e^{-\gamma\left(X_\tau^\pi + (V_\tau - I)^+ + \int_\tau^T \bar{\pi}\frac{dS}{S}\right)}\right] \leq -e^{-\gamma\left(X_\tau^\pi + (V_\tau - I)^+ + Y_\tau^1\right)}$$

- We then have

$$\begin{aligned}\mathbb{E}\left[-e^{-\gamma\left(X_\tau^\pi + (V_\tau - I)^+ + Y_\tau^1\right)}\right] &\leq \mathbb{E}\left[-e^{-\gamma\left(X_\tau^\pi + Y_\tau^2\right)}\right] \\ &\leq -e^{-\gamma\left(x + Y_0^2\right)}\end{aligned}$$

- We obtain equalities by setting

$$\tau^* = \inf\{0 \leq t \leq T : Y_t^2 = (V_t - I)^+ + Y_t^1\}$$

$$\pi_t^*\sigma = \begin{cases} \lambda_1/\gamma - Z_{1,t}^2 & 0 \leq t \leq \tau^* \\ \lambda_1/\gamma - Z_{1,t}^1 & \tau < t \leq T \end{cases}$$

The indifference price process

- From the definition it is then clear that $p = Y_0^2 - Y_0^1$.
- Moreover, we have that the process $p_t := Y_t^2 - Y_t^1$ satisfies the reflected BSDE

$$p_t = (V_T - I)^+ - \int_t^T f(Z_t)dt - \int_t^T Z_t \cdot dW_t + (A_T - A_t)$$

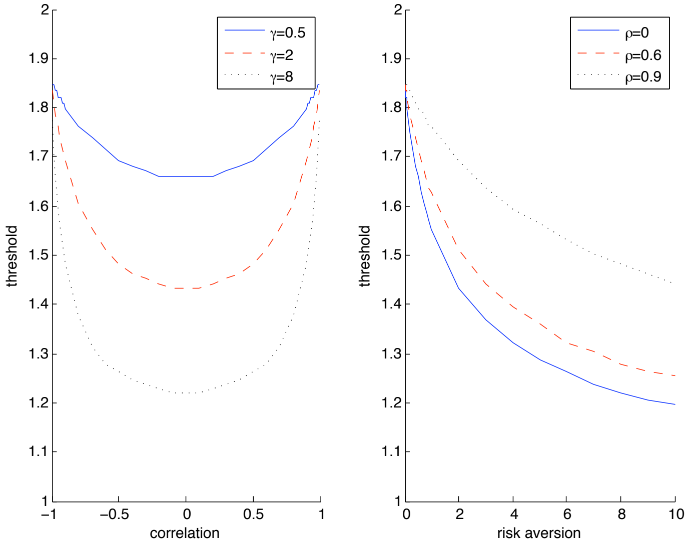
$$p_t \geq (V_t - I)^+, \quad A_0 = 0, \quad \int_0^T (p_t - (V_t - I)^+)dA_t = 0,$$

  where $f(z_1, z_2) = z_1\lambda_1 + \frac{\gamma}{2}(z_2)^2$
- We can then characterize the indifference price as the initial value of the viscosity solution of an obstacle problem and calculate it numerically.

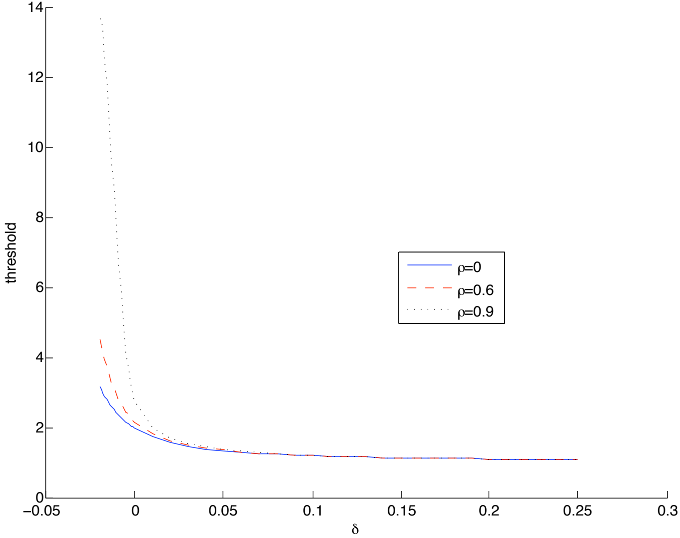
Sensitivities of indifference price

- Using comparison results for solutions of reflected BSDEs we can deduce the following properties for both the indifference price and the investment threshold.
- If $|\rho_1| \leq |\rho_2|$ then $p(\rho_1) \leq p(\rho_2)$.
- If $\gamma_1 \leq \gamma_2$ then $p(\gamma_1) \geq p(\gamma_2)$.
- Define $\delta := \bar{\mu}_2 - \mu_2$, where $\bar{\mu}_2$ is the equilibrium rate for a financial asset with volatility σ_2.
- If $-\frac{\sigma_2^2}{2} \leq \delta_1 \leq \delta_2$ then $p(\delta_1) \geq p(\delta_2)$.
- p is an increasing function of σ_2 for $\delta > 0$, but it is decreasing in σ_2 when $\delta < 0$.

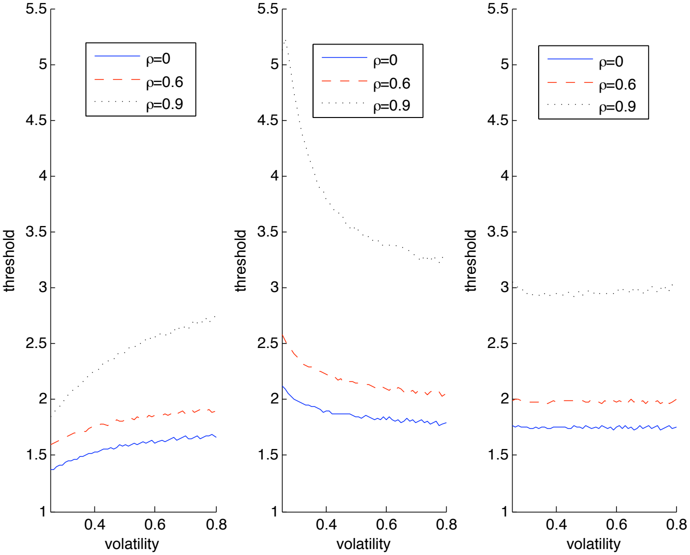
Dependence with Correlation and Risk Aversion



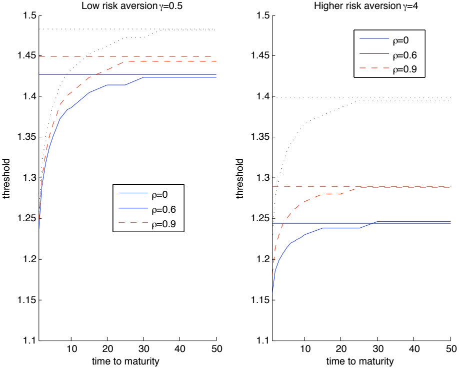
Dependence with Dividend Rate



Dependence with Volatility



Dependence with Time to Maturity



Depreciation

- Instead of the project value itself, we can model the output cash–flow rate

$$dP_t = \mu_2 P_t dt + \sigma_2 P_t(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)$$

- If the project has fixed lifetime $\bar{T}$ from moment of investment, then

$$V(P_t) = E\left[\int_0^{\bar{T}} e^{-\bar{\mu}_2 t} P_s ds\right] = \frac{P_t}{\delta}[1 - e^{-\delta \bar{T}}]$$

- If the project expires at an exponentially distributed time τ, then

$$V(P_t) = E\left[\int_0^{\tau} e^{-\bar{\mu}_2 t} P_s ds\right] = \frac{P_t}{\lambda + \delta}$$

The abandonment option

- The previous framework ignores the possibility of negative cash flows arising from the active project, for instance, when operating costs exceed the revenue.
- For a constant operating cost rate C (and no depreciation), we have that

$$V(P_t) = E\left[\int_t^{\infty} e^{-\bar{\mu}_2 s} P_s ds\right] - \int_t^{\infty} e^{-rs} C ds = \frac{P_t}{\delta} - \frac{C}{r}.$$

- We now suppose that the active project can be abandoned for a fixed cost E and later restarted at a fixed cost I.
- Notice that E can be somewhat negative if there is some scrap value to the project, as long as $-I < E < 0$.
- How can we value the combine entry/exit options ?

Investment strategies and stopping times

- An entry/exit strategy in this setting is a process

$$\xi_t = \sum_{n \geq 1} \mathbf{1}_{\{\tau_{2n-1} \leq t < \tau_{2n}\}}$$

  where $\tau_0 = 0$, τ_{2n-1} are investment times and τ_{2n} are abandonment time.

- For a given ξ, we consider the wealth process

$$\begin{aligned} dX_t^{\pi,\xi} &= \pi_t \sigma (dW_t^1 + \lambda_1 dt), \quad \tau_k \leq t < \tau_{k+1} \\ X_{\tau_{2n-1}}^{\pi,\xi} &= X_{\tau_{2n-1}^-}^{\pi,\xi} + V(P_{\tau_{2n-1}}) - I \\ X_{\tau_{2n}}^{\pi,\xi} &= X_{\tau_{2n}^-}^{\pi,\xi} - E \end{aligned}$$

Utility valuation

- We can then show that

$$u(t,x,P) = \sup_{\pi,\xi} E\left[-e^{-\gamma X^{\pi,\xi}} | X_t^{\pi,\xi} = x\right] = -e^{x+Y_0^2},$$

- Here Y_0^2 is the solution of the following system of reflected BSDE

$$Y_t^1 = \max(V_T, -E) - \int_t^T f^1(Z_t^1)dt - \int_t^T Z_t^1 \cdot dW_t + (A_T^1 - A_t^1)$$

$$Y_t^2 = \max(V_T - I, 0) - \int_t^T f^2(Z_t^2)dt - \int_t^T Z_t^2 \cdot dW_t + (A_T^2 - A_t^2)$$

$$Y_t^2 \geq Y_t^1 + (V(P_t) - I)^+, \qquad Y_t^1 \geq Y_t^2 - E$$

$$A_0^1 = 0, \quad \int_0^T (Y_t^1 - Y_t^1 + E)dA_t^1 = 0$$

$$A_0^2 = 0, \quad \int_0^T (Y_t^2 - (V(P_t) - I)^+ - Y_t^1)dA_t^2 = 0$$